16.512, Rocket Propulsion
Prof. Manuel Martinez-Sanchez
Lecture 6: Heat Conduction: Thermal Stresses

Effect of Solid or Liquid Particles in Nozzle Flow

An issue in highly aluminized solid rocket motors.

$$2Al + \frac{3}{2}O_2 \rightarrow Al_2O_3$$

m.p. $2072^\circ C$, b.p. $2980^\circ C$

In modern formulations, with $\sim 20\%$ Al by mass, the Al_2O_3 mass fraction of the exhaust can be 35-40%. This material does not expand, so there must be a loss in exit velocity, hence in I_{sp}.

Assume mass flows $\dot{m}_g$ (gas) $\dot{m}_s$ (solids), non-converting.

The momentum equation is

$$\dot{m}_g\, du_g + \dot{m}_s\, du_s + Adp = 0$$

Call ρ_s the (mass of solids)/(volume) (not the density of the solid, theory)

$$\rho_g\, u_g\, du_g + \rho_s\, u_s\, du_s + dp = 0$$

Define a mass flux function

$$x = \frac{\dot{m}_s}{\dot{m}_g + \dot{m}_s} = \frac{\rho_s u_s}{\rho_g u_g + \rho_s u_s}$$

$$\Rightarrow \rho_g u_g \left(du_g + \frac{x}{1-x} du_s \right) + dp = 0$$

$$u_g du_g = -\frac{dp}{\rho_g} - \frac{x}{1-x} u_g du_s$$

The energy equation is similarly,

$$\left(1-x\right)\left(c_{pg} dT_g + u_g du_g\right) + x\left(c_s dT_s + u_s du_s\right) = 0$$

Substitute here $u_g du_g$ from above:

$$c_{pg} dT_g - \frac{dp}{\rho_g} - \frac{x}{1-x} u_g du_s + \frac{x}{1-x}\left(c_s dT_s + u_s du_s\right) = 0$$

$$\boxed{\frac{dp}{\rho_g} = c_{pg} dT_g + \frac{x}{1-x}\left[c_s dT_s + \left(u_s - u_g\right) du_s\right]}$$

with no particles (x=0), this gives $R_g T \frac{dp}{P} = c_p dT \rightarrow \frac{P}{P_0} = \left(\frac{T}{T_0}\right)^{\gamma/\gamma - 1}$

With particles, we need to know the history of the <u>velocity slip</u> $u_s - u_g$ and of the <u>temperature slip</u> $T_s - T_g$. This is a difficult problem, requiring detailed modeling of the motion and heating/cooling of the particle. But we can look at the extreme cases easily.

(a) <u>Very Small Particles</u> → good contact. For sub-micro particles (not a bad representation of reality), we can say that

$u_s \simeq u_g = u$,
$T_s \simeq T_g = T$. Then

$$\frac{dp}{\rho_g} = \left(c_{pg} + \frac{x}{1-x} c_s\right) dT$$

Note that the mean specific heat (c_{pg} and c_s are per unit mass) is

$$\left.\begin{aligned} \bar{c}_p &= \left(1-x\right) c_{pg} + x c_s \\ \text{and also} \quad \bar{c}_v &= \left(1-x\right) c_{vg} + x c_s \end{aligned}\right\} \quad \bar{R}_g = \bar{c}_p - \bar{c}_v = \left(1-x\right)\left(c_{pg} - c_{vg}\right) = \left(1-x\right) R_g$$

So that $\frac{dp}{\rho_g} = \frac{\bar{c}_p}{1-x} dT$

$$R_g T \frac{dp}{P} = \frac{\bar{c}_p}{1-x} dT \rightarrow \frac{dp}{P} = \frac{\bar{c}_p}{R_g} \frac{dT}{T}$$

and defining an effective $\bar{\gamma}$ by the usual $\bar{\gamma} = \frac{\bar{c}_p}{\bar{c}_v}$,

$$\boxed{\frac{P}{P_0} = \left(\frac{T}{T_0}\right)^{\frac{\bar{\gamma}}{\bar{\gamma} - 1}}}$$

The equation of motion is now

$$\left(\rho_g + \rho_s\right) udu + dp = 0$$

Or

$$\frac{\rho_g}{1-x} udu + dp = 0$$

$$\frac{\rho_g}{1-x} = \frac{P}{R_g T\left(1-x\right)} = \frac{P}{\overline{R}_g T}$$

$$\boxed{\frac{P}{\overline{R}_g T} udu + dp = 0}$$

From the two boxed equations we see that <u>everything</u> from here can proceed as if the gas were simple, but with molecular mass

$$\overline{M} = \frac{M_g}{1-x}$$

(or $\overline{R}_g = \left(1-x\right) R_g$),

and with

$$\overline{c}_p = \left(1-x\right) c_{pg} + x\, c_s\,.$$

For example,

$$\boxed{u_e = \sqrt{2\frac{\overline{\gamma}}{\overline{\gamma}-1}\overline{R}_g T_c\left[1-\left(\frac{P_e}{P_c}\right)^{\frac{\overline{\gamma}-1}{\overline{\gamma}}}\right]}}$$

T_c, P_c in chamber etc.

For sensitivity analysis it may be of interest to linearize this for $x \ll 1$. The algebra is tedious, but one gets,

$$\boxed{\frac{u_e}{u_{e_0}} \cong 1 - \frac{1}{2} \times \left\{1 - c\left[1 + \frac{\left(1-\eta_0\right)\ln\left(1-\eta_0\right)}{\eta_0}\right]\right\}}$$

with $c = \frac{c_{ps}}{c_{pg}}$, $\eta_0 = 1 - \left(\frac{P_e}{P_c}\right)^{\frac{\gamma-1}{\gamma}}$

and, of course,

$$u_{e_0} = \sqrt{2\frac{\gamma}{\gamma-1}R_g T_c\left[1-\left(\frac{P_e}{P_c}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

We see from this that if $c \lesssim 1$ ($c_{ps} \lesssim c_{pg}$, which is common), then $u_e < u_{e_0}$ (and vice-versa).

For a numerical example, look at Problem 2 (attached)

(b) <u>Very Large Particles</u> Hard to quantify, but probably for diameter $\gtrsim 100\,\mu$m or so, the particles have too much inertia (and thermal inertia) to follow the gas acceleration and cooling. We then have

$du_s \ll du_g$; $T_s \simeq T_c$ ($\cong T_g$ at chamber)

or $du_s \simeq 0$; $dT_s \simeq 0$

Returning to the $\frac{dp}{\rho_g}$ equation, it now looks <u>as if there were no particles</u>:

$$\frac{dp}{\rho_g} = c_{pg}dT_g$$

(i.e. , particles just do not participate in the dynamics or in the thermal balances). So, we still have $\frac{P}{P_0} = \left(\frac{T}{T_0}\right)^{\gamma/\gamma-1}$. This does not mean zero performance effect, though. We do not get the full <u>gas</u> exit velocity

$$u_e = \sqrt{2\frac{\gamma}{\gamma-1}R_g T_c\left[1-\left(\frac{P_e}{P_c}\right)^{\gamma-1/\gamma}\right]}$$

but the particulates do not contribute to thrust, because they exit at $u_s \ll u_e$:

$$\boxed{g\ I_{sp} = \frac{\dot{m}_g u_e + \cancel{\dot{m}_s u_s}}{\dot{m}_g + \dot{m}_s} = (1-x)g\ I_{sp_0}}$$

This is actually more loss than in the small particle case (about twice as much, depending on c).

From the example, this is a serious loss in solid rockets.

Criterion for Slip

$$\overset{\frac{4}{3}\cancel{\pi}R_p^{\overset{2}{\cancel{3}}}\rho_s}{m_p}\frac{du_p}{dt} = 6\cancel{\pi}\mu_g\cancel{R_p}\left(u_g - u_p\right) \quad \Big\| \quad \frac{2}{9}\frac{R_p^2\rho_s}{\mu_g}\frac{du_p}{dt} = u_g - u_p \qquad \tau_R = \frac{2}{9}\frac{\rho_s R_p^2}{\mu_g}$$

$$\frac{du_p}{dt} = \frac{u_g - u_p}{\tau_R} \quad \text{call } u_g - u_p = s \qquad u_p = u_g - s$$

$$\frac{du_g}{dt} - \frac{ds}{dt} = \frac{s}{\tau_R} \qquad\qquad \frac{ds}{dt} + \frac{s}{\tau_R} = \frac{du_g}{dt}$$

Say τ_R and $\frac{du_g}{dt} = a_g$ are constant $\rightarrow s \simeq a_g\tau_R + Ce^{-\frac{t}{\tau_R}}$

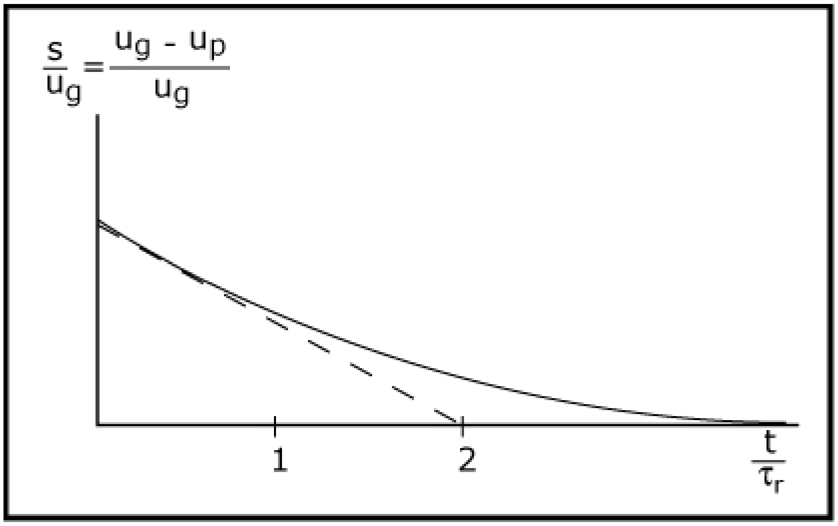
$$s(0) = 0 \rightarrow C = -a_g\tau_R$$

$$= \frac{1 - \left(1 - \varepsilon + \frac{\varepsilon^2}{2}\dots\right)}{\varepsilon} = 1 - \frac{\varepsilon}{2} + \dots$$

$$\boxed{s = a_g\tau_R\left(1 - e^{-\frac{t}{\tau_R}}\right)}$$

and $u_g = a_g t$

$$\begin{tikzcd} & 1-\frac{1}{2}\frac{t}{\tau_R}\dots \\ \frac{s}{u_g} = \frac{\tau_R}{t}\left(1 - e^{-\frac{t}{\tau_R}}\right) \arrow[ur, dash, "t \ll \tau_R"] \arrow[dr, dash, "t \gg \tau_R"'] & \\ & \frac{1}{t/\tau_R} \end{tikzcd}$$



So, small slip for $t \gg \tau_R$

$$\frac{L}{u} \gg \tau_R$$

$$\frac{L}{u_g} \gg \frac{2}{9}\frac{\rho_s R_p^2}{\mu_g}$$

$$\boxed{R_p \ll \sqrt{\frac{9}{2}\frac{\mu_g L}{\rho_s u_g}}}$$

Say

$\mu_g \sim 3 \times 10^{-5}$ Kg / m / s

$L \sim 0.3$ m

$\rho_s \sim 3 \times 10^{3}$ Kg / m^3

$u_g \sim 1.5 \times 10^{3}$ m / s

$$R_p \ll \sqrt{4.5\ \frac{\cancel{3} \times 10^{-5} \times 0.3}{\cancel{3} \times 10^{3} \times 1.5 \times 10^{3}}} = \sqrt{0.9 \times 10^{-11}} = 3 \times 10^{-6}\,\text{m} = 3\mu\text{m}$$

So, $\left| \begin{array}{l} R_p \ll 3\mu m \to \text{no slip} \\ R_p \gg 3\mu m \to \text{full lag} \end{array} \right.$

Problems

Problem 2

As noted in class, the effect of carrying a mass fraction x of fine solid particles in the expanding gas in a rocket nozzle can be accounted for by using an average specific heat ratio

$$\bar{\gamma} = \frac{(1-x)c_{pg} + xc_s}{(1-x)c_{vg} + xc_s}$$

and an average molecular mass

$$\overline{M} = \frac{M_g}{1-x}$$

For Al_2O_3 the high temperature specific heat is $c_s = 1260\ J/Kg/K$.
Consider a solid rocket with $\gamma = 1.17\ (1.25), P_e/P_c = 0.01, \overline{M}_g = 18\,g\,/\,\text{mol}$.
For a 20% aluminum loading in the propellant, x is of the order of 37%.
Calculate the matched specific impulse of the rocket and compare to what it would be for the same $T_c = 3300\,K$, but with no particles.

Problem 2

Specific heat of clean gas

$$c_{pg} = \frac{r}{r-1}\frac{R}{M} = \frac{\overset{1.25}{1.17}}{\underset{0.25}{0.17}}\ \frac{8.314}{0.018} = \underset{2309}{3180} \ \ J/Kg/K$$

$$c_{vg} = \frac{c_{pg}}{r} = \overset{\frac{2309}{1.25}}{\frac{3180}{1.17}} = \underset{1845}{2718} \ \ J/Kg/K$$

The specific heat of the solid (or liquid) Al_2O_3 is $c_s = 1260\,J/Kg/K$.

The average specific heat ratio is then

$$\bar{r} = \frac{(1-x)c_{pg} + x\,c_s}{(1-x)c_{vg} + x\,c_s} = \frac{(1-0.37)\times \overset{2309}{3180} + 0.37\times 1260}{(1-0.37)\times \underset{1845}{2718} + 0.37\times 1260} = \underset{1.1795}{1.1336}$$

And the average molecular mass ($M_s \simeq \infty$) is

$$\overline{M} = \frac{M_g}{1-x} = \frac{18}{1-0.37} = 28.57 \ \ g/mol$$

The exit speed for $\frac{P_e}{P_c} = 0.01$ and $T_c = 3300\,K$ is then

$$u_e = \sqrt{2\frac{\bar{\gamma}}{\bar{\gamma}-1}\frac{Q}{\overline{M}}T_c\left[1-\left(\frac{P_e}{P_c}\right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}}\right]} = \underset{2521}{2613} \ \ m/sec$$

NOTE: Alternatively, and easier to do, you can use

$$u_e = \sqrt{2\,\overline{C}_p\,T_c\left[1-\left(\frac{P_e}{P_c}\right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}}\right]}$$

with $\overline{C}_p = (1-0.37)3180 + 0.37\times 1260 = 2469$ J/Kg/K

(so $\overline{M}$ is not really needed)

As a check,

$$\frac{\bar{\gamma}}{\bar{\gamma}-1}\ \frac{R}{\bar{M}} = \frac{1.1336}{0.1336}\ \frac{8.314}{0.02857} = 2469\ \text{J/Kg/K}$$

as it should.

Under pressure-matched conditions, there is no exit pressure contribution to thrust or I_{sp}, and hence

$$\boxed{I_{sp} = \frac{2613}{9.81} = \begin{matrix}266.3\ \text{sec}\\ 257.3\ \text{s}\end{matrix}}$$

Without particulate but with the same P_c, P_e and T_c, we would obtain

$$u_{e_0} = \sqrt{\frac{2\gamma}{\gamma-1}\ \frac{Q}{M_g}\ T_c\left[1-\left(\frac{P_e}{P_c}\right)^{\frac{\gamma-1}{\gamma}}\right]} = \begin{matrix}3199\ \text{m / sec}\\ 3029\end{matrix}$$

and $\boxed{I_{sp_0} = \dfrac{3199}{9.81} = \begin{matrix}326.1\ \text{sec}\\ 309.1\end{matrix}}$

There is therefore a loss of $\left(1-\dfrac{266.3}{326.1}\right)\times 100 = \overset{16.8\%}{18.3\%}$ in I_{gp}

$1-\dfrac{257.3}{309.1}$

It is interesting to test the accuracy of the <u>linear</u> approximation given in class for small x:

$$\frac{u_e}{u_{e_0}} \simeq 1-\frac{x}{2}\left[1-\frac{c}{c_{pg}}\left(1+\frac{(1-\eta_0)\ln(1-\eta_0)}{\eta_0}\right)\right];\ \eta_0 = 1-\left(\frac{P_e}{P_c}\right)^{\frac{\gamma-1}{\gamma}}$$

We find $\eta_0 = 0.4878$, and then $\dfrac{u_e}{u_{e_0}} = 0.837$ (16.3% loss)

(not too different, despite large x)

NOTE: Alternatively, and easier to do, you can use

$$u_e = \sqrt{2\bar{C}_p\ T_c\left[1-\left(\frac{P_e}{P_c}\right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}}\right]}$$

with $\bar{c}_p = (1 - 0.37)3180 + 0.37 \times 1260 = 2469$ J/Kg/K

(So $\bar{M}$ is not really needed)

As a check, $\frac{\bar{\gamma}}{\bar{\gamma}-1}\frac{R}{\bar{M}} = \frac{1.1336}{0.1336}\frac{8.314}{0.02857} = 2469$ J/Kg/K as it should.