

## 16.50 Lecture 33

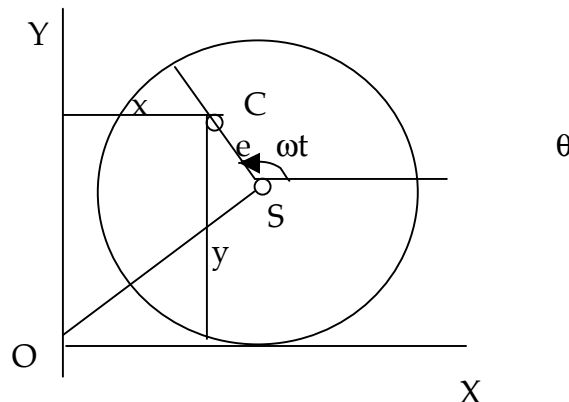
### Subjects: Critical speeds and vibration

#### Critical Speeds

As noted in the last lecture, gas turbine engines are unusual structures in that a large fraction of their mass is rotating at high speed. Thus, vibration and dynamics have always been a serious issue with them. What is intended here is that the physical phenomena of principal importance be conveyed by study of a set of simplified models.

Schematically, we can think of the rotating system as a mass on a flexible shaft, mounted on flexible supports. If most of the mass resides in the rotor and the part of the shaft adjacent to it, we can combine the two flexibilities as two springs in series with a combined stiffness which tends to restore the rotor's center to the axis from whatever direction it is deflected into. More complex models, with additional degrees of freedom, would be needed if the bearings and the parts of the shaft near them had a significant mass. We still retain the simpler formulation, but will allow the rotating mass to be mounted eccentrically, to study the loads this can generate.

The schematic below shows the situation. The center of mass  $C$  has coordinates  $(x,y)$ , and the geometrical shaft center, also assumed to be its elastic center, is at  $S$ . The center of mass is offset by a distance  $e$  from  $S$ , and rotates with the shaft at the angular rate  $\omega$ . There is an elastic restoring force  $-ke$  directed from  $S$  to  $O$ , as well as a viscous damping force, normally provided by an oil squeeze film damper between the shaft and the bearing. This damping force is proportional to the rate of change of that part of  $OS$  that comes from the bearing displacement, but since we ignore the mass of the bearings, that distance is a fixed fraction of  $OS$  itself (depending on the ratio of shaft to bearing stiffness), and we model it as  $-b(de/dt)$ .



We can now write a pair of equations of motion, one for  $x$ , the other for  $y$ :

$$M\ddot{x} = -kx_s - b\dot{x}_s$$

$$M\ddot{y} = -ky_s - b\dot{y}_s$$

where

$$x_s = x - e \cos \omega t; \quad y_s = y - e \sin \omega t$$

It is actually more convenient to work with a complex displacement  $z=x+iy$ , so that a single complex equation is needed:

$$M\ddot{z} = -kz_s - b\dot{z}_s; \quad z_s = z - e \exp(i\omega t)$$

Introducing the “natural frequency”  $\omega_n = \sqrt{k/M}$  and the friction parameter  $n = b/M$ , and eliminating  $z_s$ , we obtain

$$\ddot{z} + n\dot{z} + \omega_n^2 z = e(\omega_n^2 + i n \omega) e^{i\omega t}$$

The general solution to this equation consists of the homogeneous part alone, with two arbitrary constants determined by initial conditions, plus the forced, or “particular” solution, determined by the right hand side. The homogeneous solution is a superposition of damped sines and cosines at the natural frequency  $\omega_n$ . Because of the damping, it will disappear some time after a transient event, leaving only the forced solution, which is proportional to  $\exp(i\omega t)$ , like the right hand side of the equation. We concentrate here on the forced solution only (the steady state). Putting  $z=B \exp(i\omega t)$  and substituting into the governing equation, we can solve for B, and find

$$z = e \frac{\omega_n^2 + n\omega i}{\omega_n^2 - \omega^2 + n\omega i} e^{i\omega t}$$

$$z_s = z - e \exp(i\omega t) = -e \frac{\omega^2}{\omega_n^2 - \omega^2 + n\omega i} e^{i\omega t}$$

The force on the supports is, in complex form,

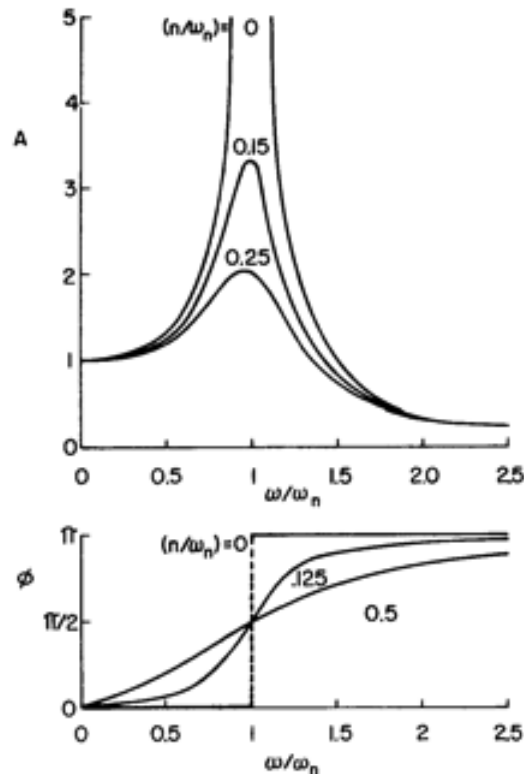
$$F = F_x + iF_y = -kz_s - b\dot{z}_s = M\ddot{z}_s = -M\omega^2 z$$

where the last form can be directly rationalized as the result of “centrifugating” the center of mass. It is, of course, a rotating force (around O), proportional to  $\exp(i\omega t)$ . The amplitude of this force is obtained from the expression for z:

$$|F| = M\omega^2 e \sqrt{\frac{\omega_n^4 + (n\omega_n\omega)^2}{(\omega_n^2 - \omega^2)^2 + (n\omega_n\omega)^2}}$$

and the phase can also be written down, although we omit it here.

The solutions for the amplitude of  $z$  (normalized by the eccentricity  $e$ ) or for  $F$  (normalized by  $M\omega^2e$ ) are shown below, together with the phase information:



From Kerrebrock, Jack L. *Aircraft Engines and Gas Turbines*. 2nd edition. MIT Press, 1992. © Massachusetts Institute of Technology. Used with permission.

Some important points are:

- 1) With no damping,  $z/e \rightarrow \infty$  and  $F/e \rightarrow \infty$  for  $\omega \rightarrow \omega_n$  (resonance condition).
- 2) Damping limits the resonant amplitudes to finite values.
- 3) The system can be run above  $\omega_n$  with a flexible shaft or support, in which case the mass rotates about C for very high speed ( $x$  and  $y$  tend to zero).

Now let us ask how the bearing stiffness and damping influence the forces on the structure.

- a) At “subcritical” speeds, namely,  $\omega^2 \ll \omega_n^2$ ,  $F \rightarrow -M\omega^2e \exp(i\omega t)$ , and the forces die out as the square of the speed, as could be expected. Notice this condition might not mean a very low rotational speed, if the shaft and the bearings are stiff enough, or the rotor is light enough, to produce a very high natural frequency (this is unusual in jet engines, though).

(b) At supercritical speeds,  $\omega^2 \gg \omega_n^2$ , and with relatively weak damping,  $|z| \rightarrow -e \frac{\omega_n^2}{\omega^2} \rightarrow 0$ , i.e., the center of mass tends to remain stationary, with the shaft rotating about it. The force then tends to  $|F| \rightarrow M\omega_n^2 e = ke$ , a constant elastic force towards the center of mass (but a rotating force, that transmits to the supports and causes vibrations). This is the normal condition for most engines at design speed.

Elaborate procedures are followed to reduce eccentricities to a minimum, but with engine wear, they are difficult to maintain near zero. Their effects can be mitigated by (a) Staying away from operation at a speed equal to the natural ("critical") frequency. (b) If normal operation is at supercritical speed, crossing the critical range as quickly as possible, and (c) Providing as much damping as practical with squeeze-film or other dampers.

MIT OpenCourseWare  
<http://ocw.mit.edu>

16.50 Introduction to Propulsion Systems  
Spring 2012

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.