16.50 Lecture 20

Subject: Introduction to Component Matching and Off-Design Operation

At this point it is well to reflect on which of the many parameters we have introduced (like M_2, τ_c, τ_t, ϑ_t, f, etc.) are free for the pilot to control, and what the inter-relationships are that determine the others. This connectivity is in part mechanical, like the shaft power balance (Eq. 9 of Lecture 18), but it also comes via flow continuity among components. This topic is usually relegated to the very end of the study of engine components, where it is introduced under the rubric of "Component Matching" (Lecture 31 in our NOTES). We find it advantageous to move most of it forward to this point.

The price to pay for the insight to be gained is the need to introduce one assumption at this point (to be justified later). This is the assumption that the stators leading to the turbine (the "turbine nozzles" are choked. This means the mass flow rate can be written as

$$\dot{m} = \dot{m}_4 = \Gamma(\gamma)\frac{P_{t4}A_4}{\sqrt{R\,T_{t4}}} \qquad \left(\Gamma = \sqrt{\gamma}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\right) \tag{1}$$

Where A_4 is the effective flow area of these nozzles. But in addition, we have already shown in Lect. 18 that the exhaust nozzle is also choked. Passing the same flow through two choked apertures in series imposes very strong constraints on the flow conditions. The flow can be expressed at the throat as

$$\dot{m} = \dot{m}_7 = \Gamma(\gamma)\frac{P_{t7}A_7}{\sqrt{R\,T_{t7}}} \tag{2}$$

and equating (1) and (2) ,

$$\frac{P_{t7}}{P_{t4}}\sqrt{\frac{T_{t4}}{T_{t7}}} = \frac{A_4}{A_7} \tag{3}$$

For a non-afterburning turbojet, $P_{t7} \cong P_{t5}$ and $T_{t7} = T_{t5}$

$$\frac{\pi_t}{\sqrt{\tau_t}} = \frac{A_4}{A_7} \tag{4}$$

and if the turbine is ideal, $\pi_t = \tau_t^{\frac{\gamma}{\gamma-1}}$, and we obtain

$$\boxed{\tau_t = \left(\frac{A_4}{A_7}\right)^{\frac{2(\gamma-1)}{\gamma+1}}} \tag{5}$$

and then $$\boxed{\pi_t = \left(\frac{A_4}{A_7}\right)^{\frac{2\gamma}{\gamma+1}}} \qquad (6)$$

This is a strong result: as long as both, the turbine nozzles and the exhaust throat remain choked, <u>the turbine maintains the same pressure and temperature ratios</u> (same operating point), regardless of fuel flow, Mach number, altitude, etc. We can now trace the variability of other quantities:

(1) <u>Compressor ratios</u>. In terms of $\vartheta = T_{t4}/T_{to}$, Eq. (9) of Lecture 18 gives

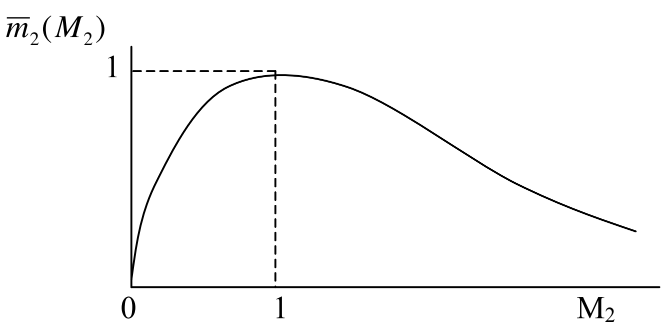
$$\boxed{\tau_c = 1 + \vartheta(1-\tau_t)} \quad ; \qquad \pi_c = \tau_c^{\gamma/\gamma-1} \qquad (7)$$

Thus τ_c and π_c do vary, but <u>only as a function of the single quantity ϑ:</u> $\tau_c = \tau_c(\vartheta)$ for a given engine.

(2) <u>Mach number at compressor inlet (M_2).</u> The flow at compressor inlet is generally subsonic, so we express the flow rate there as

$$\dot{m} = \dot{m}_2 = \Gamma \frac{P_{t2} A_2}{\sqrt{R\,T_{t2}}} \overline{m}_2(M_2) \;\; ; \;\; \overline{m}_2(M_2) = M_2 \left(\frac{\frac{\gamma+1}{2}}{1 + \frac{\gamma-1}{2} M_2^2} \right)^{\frac{\gamma+1}{2(\gamma-1)}} \qquad (8)$$

The dimensionless flow function $\overline{m}_2(M_2)$ increases to a maximum of 1 when $M_2 = 1$, then decreases again. Equating (8) to (1), we see that



$$\overline{m}_2(M_2) = \frac{P_{t4}}{P_{t2}} \sqrt{\frac{T_{t2}}{T_{t4}}} \frac{A_4}{A_2}$$

For an ideal combustor, $P_{t4} = P_{t3}$, and so, using $\pi_c = \tau_c^{\gamma/\gamma-1}$, $T_{t2} = T_{to}$,

$$\boxed{\overline{m}_2(M_2) = \frac{\tau_c^{\gamma/\gamma-1}}{\sqrt{\vartheta}} \frac{A_4}{A_2}} \qquad (9)$$

Since $\tau_c = \tau_c(\vartheta)$, we see now that $M_2 = M_2(\vartheta)$ as well (the supersonic solution for M_2 given $\overline{m}_2$ can be disregarded).

(3) Dimensionless air flow. Returning to (8), we see that the dimensionless mass flow

$$\overline{m}_2 \equiv \frac{\dot{m}}{\left(\Gamma \frac{P_{to} A_2}{\sqrt{R\, T_{to}}}\right)} \tag{10}$$

(flow rate as a fraction of what the compressor would pass if its inlet were choked), is once more a unique function of ϑ. This is very useful for scaling from one operating condition to another.

(4) Fuel/air ratio. The combustor heat balance is

$$\dot{m}_f h = \dot{m}\, c_p (T_{t4} - T_{t3}) = \dot{m}\, c_p T_{t2} (\vartheta - \tau_c) \tag{11}$$

and using $T_{t2} = T_{to}$ and $f \equiv \dot{m}_f / \dot{m}$,

$$\boxed{\frac{fh}{c_p T_{to}} = \vartheta - \tau_c(\vartheta)} \tag{12}$$

so the quantity f / T_{to} is another function of ϑ alone. But notice that f itself does depend on M_o at a fixed T_o.

(5) Throat pressure (normalized)

$$\frac{P_7}{P_{to}} = \frac{P_{t2}}{P_{to}} \frac{P_{t3}}{P_{t2}} \frac{P_{t4}}{P_{t3}} \frac{P_{t5}}{P_{t4}} \frac{P_{t1}}{P_{t5}} \frac{P_7}{P_{t7}}$$

and $\frac{P_7}{Pt_7} = \frac{1}{\left(1 + \frac{\gamma - 1}{2} \times 1\right)^{\gamma/\gamma-1}} = \left(\frac{2}{\gamma+1}\right)^{\gamma/\gamma-1}$. Also $\frac{P_{t3}}{P_{t2}} = \pi_c = \tau_c^{\gamma/\gamma-1}$, $\frac{P_{t5}}{P_{t4}} = \pi_t = \tau_t^{\gamma/\gamma-1}$;

$$\frac{P_7}{P_{to}} = \left(\frac{2}{\gamma+1} \tau_t \tau_c(\vartheta)\right)^{\frac{\gamma}{\gamma-1}} \tag{13}$$

which is yet another function of ϑ alone.

(6) Thrust (matched nozzle). We already have Eq. (10) of Lecture 18, but it is sometimes better to normalize thrust by the total free-stream pressure on the compressor inlet, $P_{to}A_2$,

which is known from flight conditions. If $P_e = P_o$ (variable nozzle, or just design point for a fixed nozzle),

$$\boxed{\varphi_2 \equiv \frac{F}{P_{to}A_2}} = \frac{\dot{m}\left(u_e - u_o\right)}{P_{to}A_2} \tag{14}$$

$$\varphi_2 = \overline{m}_2 \Gamma \frac{P_{to}A_2}{\sqrt{RT_{to}}} \frac{u_e - u_o}{P_{to}A_2}$$

For u_e, we go back to Lecture 18, Eq. (7):

$$u_e = a_o \sqrt{\frac{2}{\gamma - 1}\left(\vartheta_o \tau_c \tau_t - 1\right)\frac{\vartheta}{\tau_c}}$$

and $\dfrac{a_o}{\sqrt{RT_{to}}} = \sqrt{\dfrac{\gamma R\, T_o}{R\, T_{to}}} = \sqrt{\dfrac{\gamma}{\vartheta_o}}$. All together then,

$$\boxed{\varphi_2 = \Gamma \overline{m}_2 \sqrt{\frac{\gamma}{\vartheta_o}}\left[\sqrt{\frac{2\vartheta}{\gamma - 1}\frac{\left(\vartheta_o \tau_c \tau_t - 1\right)}{\tau_c}} - M_o\right]} \tag{15}$$

Here the quantities $\overline{m}_2$ and τ_c depend on ϑ only, but we can see that the Mach number M_o appears explicitly (as M_o and as $\vartheta_o = 1 + \frac{\gamma - 1}{2}M_o^2$), so the normalized thrust φ_2 depends on both ϑ <u>and</u> M_o.

(7) <u>Thrust (truncated sonic nozzle)</u>. We now have $m_e = m_7 = 1$, but $P_e = P_7 > P_o$, so

$$\begin{aligned}\varphi_2 = \frac{F}{P_{to}A_2} &= \frac{\dot{m}\left(u_e - u_o\right) + \left(P_e - P_o\right)A_e}{P_{to}A_2} \\ &= \Gamma \overline{m}_2 \frac{u_e - u_o}{\sqrt{R\, T_{to}}} + \left(\frac{P_e}{P_{to}} - \frac{P_o}{P_{to}}\right)\frac{A_e}{A_2}\end{aligned} \tag{16}$$

and this time $M_e = 1$, so $u_e = \sqrt{\gamma R T_e} = \sqrt{\gamma R \frac{2}{\gamma + 1} T_{t5}} = \sqrt{\frac{2\gamma}{\gamma + 1} R T_{to} \vartheta \tau_t}$

so that $\dfrac{u_e}{\sqrt{R\, T_{to}}} = \sqrt{\dfrac{2\gamma}{\gamma + 1}\vartheta \tau_t}$ <u>depends on ϑ alone</u>. Since we also know that $\overline{m}_2$ and $\dfrac{P_e}{P_{to}}$ are functions of ϑ alone, it makes sense to separate out Eq. (16) in the form

$$\varphi_2 = \left(\Gamma \overline{m}_2 \frac{u_e}{\sqrt{R\, T_{to}}} + \frac{P_e}{P_{to}} \frac{A_e}{A_2} \right) - \Gamma\, \overline{m}_2 \frac{u_o}{\sqrt{R\, T_{to}}} - \frac{P_o}{P_{to}} \frac{A_e}{A_2} \qquad (17)$$

$$\varphi_2 = \underbrace{\left(\overline{m}_2 \Gamma \sqrt{\frac{2\gamma}{\gamma+1} \vartheta \tau_t} + \left(\frac{2}{\gamma+1} \tau_t \tau_c \right)^{\frac{\gamma}{\gamma-1}} \frac{A_e}{A_2} \right)}_{\varphi_2^*(\vartheta)} - \left[\Gamma \overline{m}_2 M_0 \sqrt{\frac{\gamma}{\vartheta_o}} + \frac{1}{\vartheta_o^{\gamma/\gamma-1}} \frac{A_e}{A_2} \right] \qquad (18)$$

Once again, the normalized thrust depends on both, ϑ and M_o, but the structure is fairly simple, and in particular, the portion φ_2^* of φ_2 (neglecting the incoming momentum and the external pressure) is a function of ϑ alone. This portion can be very easily scaled between conditions, and the rest can be subtracted separately.

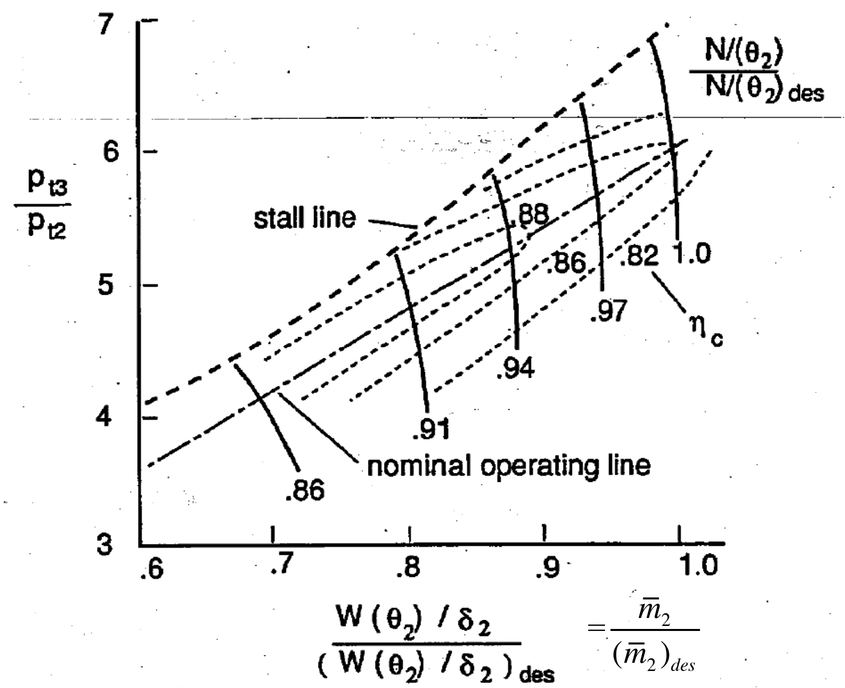
A note on ϑ: the near-constancy of the engine operating point

Two important points in the flight envelope of an aircraft engine are (a) Take-off conditions $\left(M_o \cong 0.25,\ T_o \cong 290K\right)$, and (b) End-of-climb conditions ($M_0 \approx 0.85$, $T_0 \approx 220$K. The total temperatures are $T_{to} = 290\left(1 + 0.2 \times 0.25^2\right) = 294K$ (take-off) and $T_{to} = 220\left(1 + 0.2 \times 0.85^2\right) = 252K$ (end of climb). Suppose the engine is dimensioned for end-of-climb, which is common, and that the peak temperature T_{t4}, which will have to be maintained for many hours of cruise, is selected at a conservative T_{t4} = 1600K. We then have $\vartheta = \frac{1600}{252} = 6.35$ at this condition. If we now decided to maintain ϑ = 6.35 also for take-off, we would need then $T_{t4} = 6.35 \times 294 = 1868K$. While this is too high for long-term operation (creep, corrosion), it may be acceptable for the few minutes per cycle that the engine will be at take-off maximum power.

As a second example, consider a commercial jet in a long cruise. As the fuel is consumed and the weight decreases, so must the lift $L = 1/2 \rho_0 u_0^{\,2} A_w c_L$. Now, the lift coefficient will be kept close to that for optimum L/D, and the Mach number M_0 is unlikely to change much, as it will stay just below the transonic drag peak, and so u_0^2 will be proportional to T_0 due to the speed of sound variation. Together with the density part of lift, we can see that the ambient pressure p_0 must be decreasing in proportion to the airplane's weight, i.e., the plane must be climbing gradually. Turning now to the forward force balance, given a constant L/D, the drag, and hence the engine thrust, must also be decreasing in time in the same proportion as the ambient pressure. Therefore, from Eq. (14), the nondimensional thrust $\varphi_2(\vartheta, M_0)$ will remain constant, and since M_0 does too, the peak temperature ratio θ will also remain constant, and with it all the important ratios like τ_c, M_2, etc.

In other words, ϑ may not vary much among (important) flight conditions, and the engine will be operating at a fixed nondimensional condition (constant compression ratio, nondimensional flow, compressor inlet Mach number, etc.). But of course, the dimensional quantities (flow rate, peak pressure, etc.) will be different, depending on p_o, etc.

(8) The Operating Line in the compressor map. Compressor performance is typically presented as a map of π_c vs. $\overline{m}_2$, with lines of constant normalized rotational speed ϖ and η_c superimposed. The details are the subject of later Lectures, but the general shape is as shown below. (The flow and speed variables are renormalized by the "Design" values):



Kerrebrock, Jack L. (1992). Aircraft Engines and Gas Turbines (2nd Edition). MIT Press, © Massachusetts Institute of Technology. Used with permission.

Actually the "nominal operating line" shown in the figure is not a property of the compressor, but rather of the rest of the engine. We can calculate this line with the information we have now, before deciding what particular compressor to use. From Eqs. (11) and (9),

$$\overline{m}_2 = \frac{A_4}{A_2}\frac{\tau_c^{\gamma/\gamma-1}}{\sqrt{\vartheta}} \tag{19}$$

and from the shaft power balance (Eq. 7),

$$\vartheta = \frac{\tau_c - 1}{1 - \tau_t} \tag{20}$$

where we recall that τ_t is fixed for a fixed geometry. Eliminating ϑ,

$$\overline{m}_2 = \frac{A_4}{A_2}\tau_c^{\gamma/\gamma-1}\sqrt{\frac{1-\tau_t}{\tau_c - 1}} \tag{21}$$

or, in terms of π_c,

$$\boxed{\overline{m}_2 = \frac{A_4}{A_2}\pi_c\sqrt{\frac{1-\tau_t}{\pi_c^{\frac{\gamma-1}{\gamma}} - 1}}} \tag{22}$$

which is the equation for the operating line (written in reverse).

If the compressor is already available, we see from (22) that we can adjust the nozzle area A_4 to place this line in a "good" place on the map, i.e., below the stall line and through the best efficiency points.

Since $\overline{m}_2$ depends on $\vartheta = \frac{T_{t4}}{T_{to}}$, varying T_{t4} moves the operating point along the operating line, and this is what the pilot does with the throttle stick to power the engine up or down. At each selected ϑ, the engine settles to a π_c, a M_2, a (normalized) rotation rate, etc.

Effects of Mach number

If we look at operation of a given engine at different flight Mach numbers, we may try to maintain the same non-dimensional conditions throughout, which, as we have seen, can be done by maintaining for example a constant compressor inlet Mach, M_2. This, in turn guarantees a constant $\vartheta = \frac{T_{t4}}{T_{to}}$, but since now we have a varying Mach number, so that T_{t0} increases with M_0, we may find that the turbine inlet temperature T_{t4} needs to become too high at the higher Mach numbers. For example, T_{t4} would have to be 1.8 times higher at M_0=2.0 than at static conditions, and 2.25 times at M_0=2.5.

A more reasonable assumption is that the ratio $\theta_t = T_{t4}/T_0$ can be maintained the same at all Mach numbers, since at least in the stratosphere, T_0 is almost invariant. The compressor temperature ratio now follows from $\tau_c = 1 + \frac{\vartheta_t(1-\tau_t)}{\vartheta_0}$, where the numerator is a constant; thus, τ_c will be lowered as the Mach number increases, but less strongly than would be required to maintain maximum thrust per unit flow ($\tau_c = \sqrt{\vartheta_t}/\vartheta_0$). The flow parameter $\overline{m}_2$ is now determined by Eq. (9), i.e. compressor-turbine flow matching, and then the compressor-inlet Mach number from Eq. (8). Once these parameters are known, we can use Eq. (15) to calculate the normalized thrust; since we are interested in the effect of Mach number, it makes sense to re-normalize thrust by p_0A_2, or

$$\frac{F}{p_0 A_2} = \varphi_2 \vartheta_0^{\frac{\gamma}{\gamma-1}}.$$

A numerical example

We take now θ_t=7, or T_{t4}=1540K in the stratosphere. The geometry of the engine must have been specified in advance. This means that the turbine temperature ratio (Eq. 5) is a known fixed number. For the example, we select τ_t such as to obtain maximum thrust at M_0=1. From the shaft balance equation,

$$\tau_t = 1 - \frac{\vartheta_0}{\vartheta_t(\tau_c - 1)}$$

and we put now θ_0=1.2 and $\tau_c=\sqrt{7/1.2}=2.2048$ (at M_0=1). This fixes τ_t=0.7935.
Similarly, the area ratio A_4/A_2 must have been fixed, and we select it here so as to obtain at M_0=1 a compressor-face Mach number M_2=0.5, which, from Eq. (8) implies $\overline{m}_2$=0.7464. From Eq. (9) then,

$$\overline{m}_2 = 0.7464\left(\frac{\tau_c}{2.2048}\right)^{3.5}\sqrt{\frac{\vartheta_0}{1.2}}$$

and the rest of the steps are as described above. The table below summarizes the results:

M_0	0	1	2	2.5
θ_0	1	1.2	1.8	2.25
τ_c	2.4458	2.2048	1.8032	1.6426
$\overline{m}_2$	0.9796	0.7464	0.4523	0.3648
M_2	0.8486	0.5	0.2737	0.2172
φ_2	2.9117	1.5531	0.5795	0.3503
$F/(p_0A_2)$	2.9117	2.9399	4.534	5.985

We find that at a fixed altitude the thrust is nearly constant up to Mach 1, then it increases rapidly. Actually the increase is less rapid than this simple model predicts, because of losses in the supersonic flow in the engine inlet.
Finally to this point, we should note that an aircraft normally flies at increasing altitude as the Mach Number increases, so that dynamic pressure $\gamma p_0 M_0^2$ is roughly constant. In this case the change in F between Mach 1 and Mach 2 is actually a thrust reduction.