

## 16.50 Lecture 1

### Subjects: Rocket Equation; Gravity Loss; Optimum Acceleration.

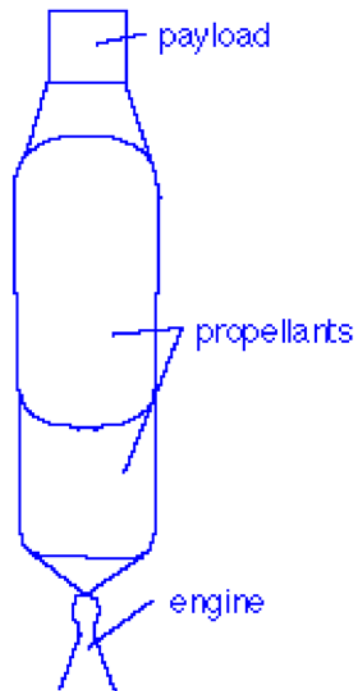
#### 1) Rocket Equation

A rocket is a propulsive device that produces a thrust force  $F$  on a vehicle by ejecting mass a high relative velocity  $c$ . This force is simply equal to the rate of momentum outflow from a control volume that encloses the vehicle. If  $\dot{m}$  is the mass expulsion rate, we then find

$$F = \dot{m} c$$

This is a particularly useful result because  $c$  and  $\dot{m}$  are insensitive to the operating environment of the rocket. In more detail,  $c$  is not necessarily the velocity of the expelled mass as it leaves the rocket's exit plane, because the pressure field may continue accelerating this mass for some distance. In that case,  $c$  represents the limiting "far field" velocity of the ejected mass.

For a rocket-powered vehicle in a vertical launch from the Earth,



$$m \frac{dv}{dt} = -c \frac{dm}{dt} - mg \rightarrow dv = -c \frac{dm}{m} - g dt$$

$$v = -\int_0^t c(h) \frac{dm}{m} - \int_0^t g(h) dt; \quad h = \int_0^t v dt$$

As indicated, both  $c$  and  $g$  are functions of  $h$ . In the case of  $c$ , the dependence is through the atmospheric pressure,  $p_0$ , an effect we will explore in detail later. The atmospheric pressure varies with weather and with temperature, but a rule of thumb is:

$$\frac{p_0}{p_0(r_E)} = e^{-\frac{r-r_E}{6.68}} \quad r - r_E = \text{km}$$

In fact the pressure variation is much more important in the context of aircraft engines, whose thrust is essentially proportional to the atmospheric density, other factors being equal.

The gravitational attraction varies as  $g(r) = g_E (r_E/r)^2$ , where  $r_E$  is the Earth's radius,  $r = r_E + h$  and  $g_E$  is the gravitational acceleration at the Earth's surface. Since rockets typically operate for a short time only,  $g$  may be taken here as a constant to a good approximation.

If as a first approximation we take  $c$  and  $g$  constant we can integrate the rocket equation to get:

$$v - v_0 = -c \ln \frac{m(t)}{m_0} - gt$$

and this is the usual form of the Rocket Equation. Remember that it applies to a vertical launch. For other flight paths the last term changes. Also remember that we have neglected atmospheric drag forces (this last approximation is valid for large rockets, but may not be for smaller vehicles, which have more area per unit volume).

## 2) Gravity Loss & Optimum Acceleration

As we will see, it is helpful to express this relation as one for the mass ratio in terms of the velocity change, since it is the latter that is set by the mission, e.g. launch to LEO. But this leaves  $gt$  to be determined. If we make the additional assumption that  $\dot{m} = \text{constant}$ , then

$$m_0 - m(t) = \dot{m} t = \frac{F}{c} t = \frac{m_0 a_0}{c} t$$

where  $a_0$  is the "initial acceleration", defined as the thrust divided by the initial mass. Note that  $a_0 > g$  for achieving takeoff.

So now we can eliminate  $t$ , to find

$$\frac{v - v_0}{c} = -\ln \frac{m(t)}{m_0} - \frac{g}{a_0} \left[ 1 - \frac{m(t)}{m_0} \right]$$

This is an (implicit) equation for  $\frac{m(t)}{m_0}$  in terms of  $\frac{v-v_0}{c}$ , with the initial acceleration  $a_0$  as a parameter.

The last term is often termed the "gravity loss", since it is the reduction in velocity increment that the vehicle suffers due to the (downward) acceleration of gravity. As  $a_0$  increases, the gravity loss decreases, since the time over which gravity acts is reduced. So why not use a very large  $a_0$  to get the largest possible  $\frac{v-v_0}{c}$  for a given  $\frac{m(t)}{m_0}$ ? To see why we must examine the effect of the acceleration on the mass of the propulsion system. We divide the mass of the rocket into parts:

$$m_0 = m_{struct} + m_{eng} + m_{pay} + m_{prop}$$

When all the propellant has been expended, at the time  $t_b$ ,  $m(t_b) = m_{struct} + m_{eng} + m_{pay}$ , or as a fraction of initial mass

$$\frac{m(t_b)}{m_0} = \frac{m_0 - m_{prop}}{m_0} = 1 - \frac{m_{prop}}{m_0} \equiv 1 - m'_{prop}$$

Here the prime on the propellant fraction indicates that it is divided by  $m_0$ , i.e. is the fraction of the initial mass represented by the propellant.

Clearly, the smaller  $\frac{m(t)}{m_0}$  or the closer  $m'_{prop}$  is to 1, the larger the  $\frac{v-v_0}{c}$  that can be achieved. But the limit is set by:

$$\frac{m(t_b)}{m_0} = 1 - m'_{prop} = m'_{struct} + m'_{eng} + m'_{pay}$$

We want to design the components of our vehicle so that  $m'_{struct}$  and  $m'_{eng}$  are as small as possible, given the loads and available technology.

The engine mass is proportional to the thrust it develops; define

$$\alpha = \frac{\text{Engine Weight}}{\text{Thrust}} = \frac{m_{eng}g}{F} = \frac{m_{eng}}{m_0(a_0/g)} = \frac{m'_{eng}}{n}$$

where we call for short  $a_0/g=n$ . Notice that this is the number of "g's" due to thrust only, while the more commonly used g-number (the specific downwards force felt by a pilot) includes gravity and would be  $n+1$ .

We now have  $m'(t_b) = m'_{pay} + m'_{str} + n\alpha$ , and the rocket equation becomes

$$v_b - v_0 = -c \ln(m'_{pay} + m'_{str} + n\alpha) - \left( \frac{1 - m'_{pay} - m'_{str}}{n} - \alpha \right)$$

We notice here that the logarithmic term (the ideal velocity increment) decreases with  $n$  (heavy engine), but the last parenthesis (the gravity loss) decreases with  $n$  (reduced burn time). We can optimize the choice of initial acceleration by differentiating wrt. to  $n$ :

$$-\frac{\alpha}{m'_{pay} + m'_{str} + n\alpha} + \frac{1 - m'_{pay} - m'_{str}}{n^2} = 0$$

which is a quadratic equation with the solution

$$n_{opt} = \frac{1 - m'_{pay} - m'_{str}}{2} + \sqrt{\left(\frac{1 - m'_{pay} - m'_{str}}{2}\right)^2 + \frac{(m'_{pay} + m'_{str})(1 - m'_{pay} - m'_{str})}{\alpha}}$$

Some reasonable estimates are:

$$m'_{struct} \approx 0.1 \quad ; \quad \alpha = 0.02$$

For these values, Fig. 1 shows how the actual velocity increment varies with  $n = a_0/g$  for a range of payload fractions. The optima are clearly visible, and can be obtained from the graph or calculated from our quadratic solution. These  $n_{opt}$  values, and the associated maximum velocity increments, are shown in Fig. 2. Interestingly,  $n_{opt}$  varies little, and stays in the range from 3 to 4, familiar to astronauts. The velocity increment is usually set by the mission requirement, e.g. to get to LEO, and Fig. 2 illustrates how small the payload fraction becomes if this required velocity increment exceeds the jet velocity  $c$  (in fact, for this example,  $m'_0 = 0$  for  $(v - v_0) \approx 1.5$ )

This optimization has considered only the effects of the "gravity loss" and the engine weight on the choice of  $\frac{a_0}{g}$ . If we include other effects, such as drag, we may get a different answer. A large value of  $n$  implies reaching high speeds at low altitudes, where air density is still high, and for small rockets this would generate large drag losses.

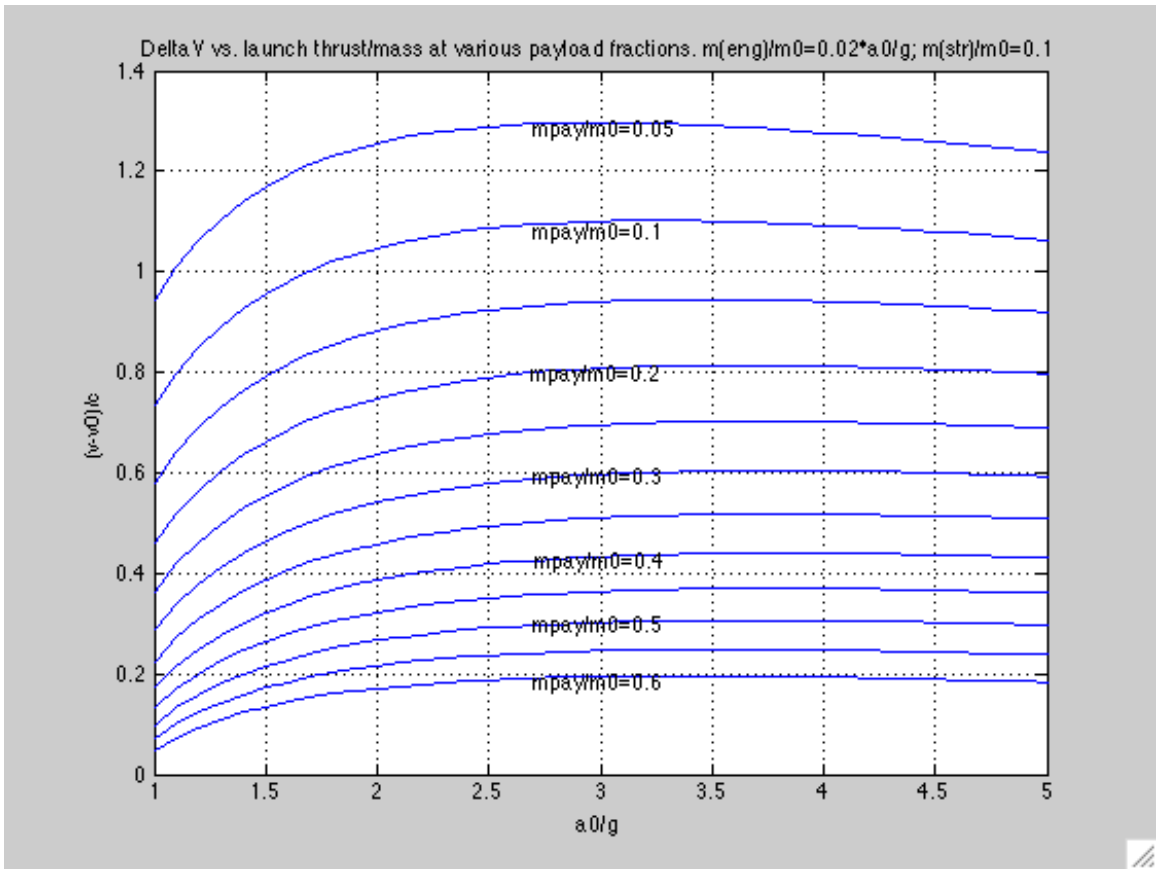


Fig. 1

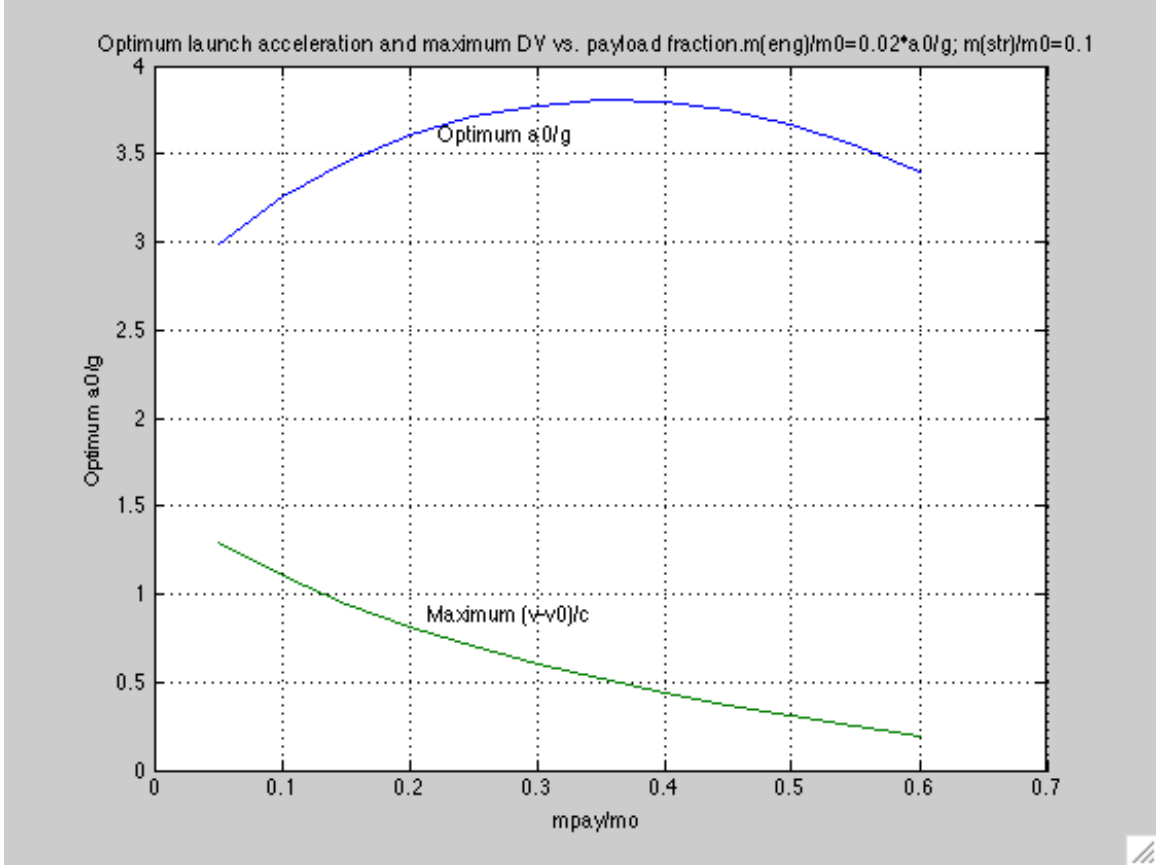


Fig. 2

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