16.410/413
Principles of Autonomy and Decision Making

Lecture 16: Mathematical Programming I

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Assignments

Readings

- Lecture notes
- [IOR] Chapters 2, 3, 9.1-3.
- [PA] Chapter 6.1-2

Shortest Path Problems on Graphs

Input: $\langle V, E, w, s, G \rangle$:

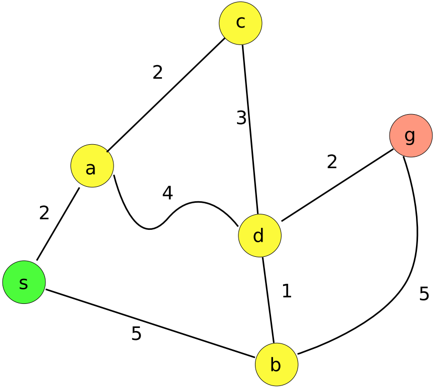
- V: set of vertices (finite, or in some cases countably infinite).
- $E \subseteq V \times V$: set of edges.
- $w : E \to \mathbb{R}_+, e \mapsto w(e)$: a function that associates to each edge a strictly positive weight (cost, length, time, fuel, prob. of detection).
- $S, G \subseteq V$: respectively, start and end sets. Either S or G, or both, contain only one element. For a point-to-point problem, both S and G contain only one element.

Output: $\langle T, W \rangle$

- T is a weighted tree (graph with no cycles) containing one minimum-weight path for each pair of start-goal vertices $(s, g) \in S \times G$.
- $W : S \times G \to \mathbb{R}_+$ is a function that returns, for each pair of start-goal vertices $(s, g) \in S \times G$, the weight $W(s, g)$ of the minimum-weight path from s to g. The weight of a path is the sum of the weights of its edges.

Example: point-to-point shortest path

Find the minimum-weight path from s to g in the graph below:



Solution: a simple path $P = \langle s, a, d, g \rangle$ ($P = \langle s, b, d, g \rangle$ would be acceptable, too), and its weight $W(s, g) = 8$.

Another look at shortest path problems

Cost formulation

- The cost of a path P is the sum of the cost of the edges on the path. *Can we express this as a simple mathematical formula?*
  - Label all the edges in the graph with consecutive integers, e.g., $E = \{e_1, e_2, \ldots, e_{n_E}\}$.
  - Define $w_i = w(e_i)$, for all $i \in 1, \ldots, n_E$.
  - Associate with each edge a variable x_i, such that:

$$x_i = \begin{cases} 1 & \text{if } e_i \in P, \\ 0 & \text{otherwise.} \end{cases}$$

- Then, the cost of a path can be written as:

$$\mathrm{Cost}(P) = \sum_{i=1}^{n_E} w_i x_i.$$

- Notice that the cost is a **linear function** of the unknowns $\{x_i\}$

Another look at shortest path problems (2)

Constraints formulation

- Clearly, if we just wanted to minimize the cost, we would choose $x_i = 0$, for all $i = 1, \ldots, n_E$: this would not be a path connecting the start and goal vertices (in fact, it is the empty path).
- Add these constraints:
  - There must be an edge in P that goes out of the start vertex.
  - There must be an edge in P that goes into the goal vertex.
  - Every (non start/goal) node with an incoming edge must have an outgoing edge
- A neater formulation is obtained by adding a “virtual” edge e_0 from the goal to the start vertex:
  - $x_0 = 1$, i.e., the virtual edge is always chosen.
  - Every node with an incoming edge must have an outgoing edge

Another look at shortest path problems (3)

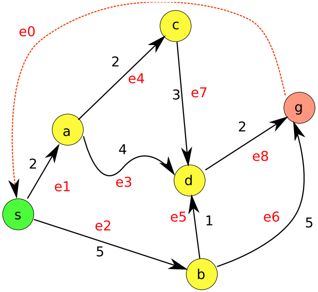
- Summarizing, what we want to do is:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n_E} w_i x_i \\
\text{subject to:} \quad & \sum_{e_i \in \mathrm{In}(s)} x_i - \sum_{e_j \in \mathrm{Out}(s)} x_j = 0, \quad \forall s \in V; \\
& x_i \geq 0, \quad i = 1, \ldots, n_E; \\
& x_0 = 1.
\end{aligned}
$$

- It turns out that the solution of this problem yields the shortest path. (Interestingly, we do not have to set that $x_i \in \{0, 1\}$, this will be automatically satisfied by the optimal solution!)

Another look at shortest path problems (4)

Consider again the following shortest path problem:



$$
\begin{aligned}
\min \quad & 2x_1 + 5x_2 + 4x_3 + 2x_4 + x_5 + 5x_6 + 3x_7 + 2x_8 \\
\text{s.t.:} \quad & x_0 - x_1 - x_2 = 0, (\text{node } s); \\
& x_1 - x_3 - x_4 = 0, (\text{node } a); \\
& x_2 - x_5 - x_6 = 0, (\text{node } b); \\
& x_4 - x_7 = 0, (\text{node } c); \\
& x_3 + x_5 + x_7 - x_8 = 0, (\text{node } c); \\
& x_2 + x_5 - x_0 = 0, (\text{node } g); \\
& x_i \geq 0, \quad i = 1, \ldots, 8; \\
& x_0 = 1.
\end{aligned}
$$

Notice: cost function and constraints are affine ("linear") functions of the unknowns $(x_1, \ldots, x_8)$.

A fire-fighting problem: formulation

Three fires

- Fire 1 needs 1000 units of water;
- Fire 2 needs 2000 units of water;
- Fire 3 needs 3000 units of water.

Two fire-fighting autonomous aircraft

- Aircraft A can deliver 1 unit of water per unit time;
- Aircraft B can deliver 2 units of water per unit time.

Objective

It is desired to extinguish all the fires in minimum time.

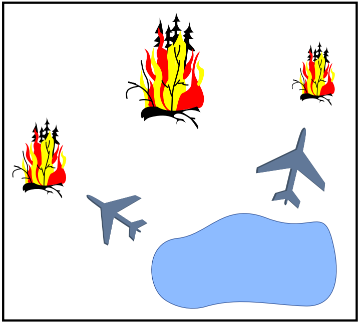


Image by MIT OpenCourseWare.

A fire-fighting problem: formulation (2)

- Let t_{A1}, t_{A2}, t_{A3} the the time vehicle A devotes to fire 1, 2, 3, respectively.
  Definte t_{B1}, t_{B2}, t_{B3} in a similar way, for vehicle B.
- Let T be the total time needed to extinguish all three fires.
- Optimal value (and optimal strategy) found solving the following problem:

$$
\begin{aligned}
\text{min} \quad & T \\
\text{s.t.:} \quad & t_{A1} + 2t_{B1} = 1000, \\
& t_{A2} + 2t_{B2} = 2000, \\
& t_{A3} + 2t_{B3} = 3000, \\
& t_{A1} + t_{A2} + t_{A3} \leq T, \\
& t_{B1} + t_{B2} + t_{B3} \leq T, \\
& t_{A1}, t_{A2}, t_{A3}, t_{B1}, t_{B2}, t_{B3}, T \geq 0.
\end{aligned}
$$

- (if you are curious about the solution, the optimal T is 2000 time units)

Outline

1. Mathematical Programming
2. Linear Programming
3. Geometric Interpretation

Mathematical Programming

- Many (most, maybe all?) problems in engineering can be defined as:
  - A set of constraints defining all candidate ("feasible") solutions, e.g., $g(x) \leq 0$.
  - A cost function defining the "quality" of a solution, e.g., $f(x)$.

- The formalization of a problem in these terms is called a **Mathematical Program**, or **Optimization Problem**.
  (Notice this has nothing to do with "computer programs!")

- The two problems we just discussed are examples of mathematical program. Furthermore, both of them are such that both f and g are affine functions of x. Such problems are called **Linear Programs**.

Outline

1. Mathematical Programming
2. Linear Programming
   - Historical notes
   - Geometric Interpretation
   - Reduction to standard form
3. Geometric Interpretation

Linear Programs

- The **Standard Form** of a linear program is an optimization problem of the form

$$
\begin{aligned}
\max \quad & z = c_1 x_1 + c_2 x_2 + \ldots, c_n x_n, \\
\text{s.t.:} \quad & a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1, \\
& a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2, \\
& \ldots \\
& a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = b_m, \\
& x_1, x_2, \ldots, x_n \geq 0.
\end{aligned}
$$

- In a more compact form, the above can be rewritten as:

$$
\begin{aligned}
\min \quad & z = c^T x, \\
\text{s.t.:} \quad & Ax = b, \\
& x \geq 0.
\end{aligned}
$$

Historical Notes

- Historical contributor: **G. Dantzig** (1914-2005), in the late 1940s. (He was at Stanford University.) Realize many real-world design problems can be formulated as linear programs and solved efficiently. Finds algorithm, the Simplex method, to solve LPs. As of 1997, still best algorithm for most applications.

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- So important for world economy that any new algorithmic development on LPs is likely to make the front page of major newspapers (e.g. NY times, Wall Street Journal). Example: 1979 L. Khachyans adaptation of ellipsoid algorithm, N. Karmarkars new interior-point algorithm.

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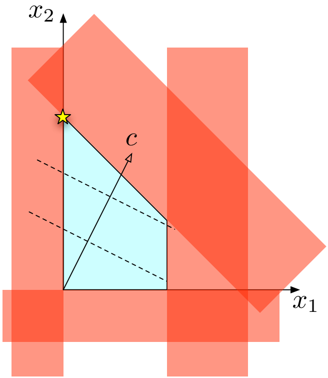
- A remarkably practical and theoretical framework: LPs eat a large chunk of total scientific computational power expended today. It is crucial for economic success of most distribution/transport industries and to manufacturing.
- Now becomes suitable for real-time applications, often as the fundamental tool to solve or approximate much more complex optimization problem.

- Consider the following simple LP:

$$
\begin{aligned}
\text{max} \quad & z = x_1 + 2x_2 = (1, 2) \cdot (x_1, x_2), \\
\text{s.t.:} \quad & x_1 \le 3, \\
& x_1 + x_2 \le 5, \\
& x_1, x_2 \ge 0.
\end{aligned}
$$



- Each inequality constraint defines a hyperplane, and a **feasible** half-space.
- The intersection of all feasible half spaces is called the **feasible region**.
- The feasible region is a (possibly **unbounded**) polyhedron.
- The feasible region could be the empty set: in such case the problem is said **unfeasible**.

Geometric Interpretation (2)

- Consider the following simple LP:

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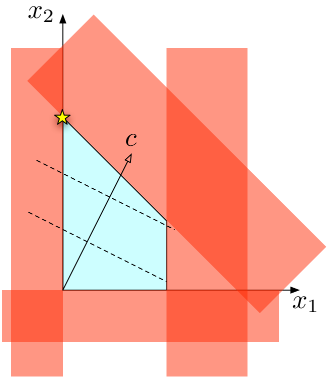

- The "c" vector defines the gradient of the cost.
- Constant-cost loci are planes normal to c.
- Most often, the optimal point is located at a vertex (corner) of the feasible region.
  - If there is a single optimum, it must be a corner of the feasible region.
  - If there are more than one, two of them must be adjacent corners.
  - If a corner does not have any adjacent corner that provides a better solution, then that corner is in fact the optimum.

Converting a LP into standard form

- Convert to maximization problem by flipping the sign of c.
- Turn all "technological" inequality constraints into equalities:
  - **less than** constraints: introduce **slack** variables.

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \Rightarrow \sum_{j=1}^{n} a_{ij}x_j + s_i = b_i, \quad s_i \geq 0.$$

  - **greater than** constraints: introduce **excess** variables.

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i \Rightarrow \sum_{j=1}^{n} a_{ij}x_j - e_i = b_i, \quad e_i \geq 0.$$

- Flip the sign of non-positive variables: $x_i \leq 0 \Rightarrow x_i' = -x_i \geq 0$.
- If a variable does not have sign constraints, use the following trick:

$$x_i \Rightarrow x_i' - x_i'', \quad x_i', x_i'' \geq 0.$$

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Geometric Interpretation

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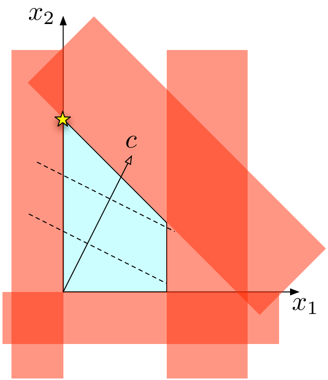

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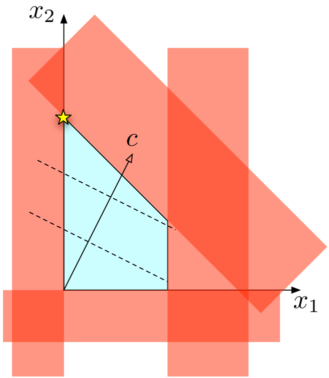

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A naïve algorithm (1)

- Recall the standard form:

$$\begin{aligned} \min \quad & z = c^T x \\ \text{s.t.:} \quad & Ax = b, \\ & x \geq 0. \end{aligned}$$

- Corners of the feasible regions (also called **basic feasible solutions**) are solutions of $Ax = b$ (m equations in n unknowns, $n > m$), obtained setting $n - m$ variables to zero, and solving for the others (basic variables), ensuring that all variables are non-negative.

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$$\begin{array}{ll} \min & z = c^T x \\ \text{s.t.:} & Ax = b, \\ & x \geq 0. \end{array} \quad \left(\text{or, really:} \quad \begin{array}{ll} \min & z = c_y^T y + c_s^T s \\ \text{s.t.:} & A_y y + Is = b, \\ & y, s \geq 0. \end{array} \right)$$

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- This amounts to:
  - picking n_y inequality constraints, (notice that $n = n_y + n_s = n_y + m$).
  - making them active (or binding),
  - finding the (unique) point where all these hyperplanes meet.
  - If all the variables are non-negative, this point is in fact a vertex of the feasible region.

A naïve algorithm (2)

- One could possibly generate all basic feasible solutions, and then check the value of the cost function, finding the optimum by enumeration.

- **Problem**: how many candidates?

$$\binom{n}{n-m} = \frac{n!}{m!(n-m)!}.$$

  - for a "small" problem with $n = 10$, $m = 3$, we get 120 candidates.
  - this number grows very quickly, the typical size of realistic LPs is such that n,m are often in the range of several hundreds, or even thousands.

- Much more clever algorithms exist: stay tuned.

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