16.346 Astrodynamics
Fall 2008

Using the Lagrange Coefficients for the Boundary-Value Problem

$$\begin{aligned} \mathbf{r} &= F\,\mathbf{r}_0 + G\,\mathbf{v}_0 \\ \mathbf{v} &= F_t\,\mathbf{r}_0 + G_t\,\mathbf{v}_0 \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} \mathbf{r}_2 &= F\,\mathbf{r}_1 + G\,\mathbf{v}_1 \\ \mathbf{v}_2 &= F_t\,\mathbf{r}_1 + G_t\,\mathbf{v}_1 \end{aligned}$$

Terminal Velocity Components along Skewed Axes Thore Godal (1960) #6.1

From the terminal position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, find the velocity vectors $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\begin{bmatrix} \mathbf{r}_2 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} F & G \\ F_t & G_t \end{bmatrix} \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{v}_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{v}_1 \end{bmatrix} = \begin{bmatrix} G_t & -G \\ -F_t & F \end{bmatrix} \begin{bmatrix} \mathbf{r}_2 \\ \mathbf{v}_2 \end{bmatrix}$$

where

$$F = 1 - \frac{r_2}{p}(1-\cos\theta) \qquad G = \frac{r_1 r_2}{\sqrt{\mu p}}\sin\theta \qquad G_t = 1 - \frac{r_1}{p}(1-\cos\theta)$$

Then

$$\mathbf{v}_1 = \frac{1}{G}(\mathbf{r}_2 - F\mathbf{r}_1) = \frac{\sqrt{\mu p}}{r_1 r_2 \sin\theta}\left[(\mathbf{r}_2 - \mathbf{r}_1) + \frac{r_2}{p}(1-\cos\theta)\,\mathbf{r}_1\right]$$

$$\mathbf{v}_2 = \frac{1}{G}(G_t\mathbf{r}_2 - \mathbf{r}_1) = \frac{\sqrt{\mu p}}{r_1 r_2 \sin\theta}\left[(\mathbf{r}_2 - \mathbf{r}_1) - \frac{r_1}{p}(1-\cos\theta)\,\mathbf{r}_2\right]$$

Next, define the unit vectors: $\mathbf{i}_{r_1} = \dfrac{\mathbf{r}_1}{r_1} \qquad \mathbf{i}_{r_2} = \dfrac{\mathbf{r}_2}{r_2} \qquad \mathbf{i}_c = \dfrac{\mathbf{r}_2 - \mathbf{r}_1}{c}$ so that

$$\boxed{\begin{aligned} \mathbf{v}_1 &= v_c\,\mathbf{i}_c + v_\rho\,\mathbf{i}_{r_1} \\ \mathbf{v}_2 &= v_c\,\mathbf{i}_c - v_\rho\,\mathbf{i}_{r_2} \end{aligned}} \qquad \text{where} \qquad v_c = \frac{c\sqrt{\mu p}}{r_1 r_2 \sin\theta} \quad \text{and} \quad v_\rho = \sqrt{\frac{\mu}{p}}\,\frac{1-\cos\theta}{\sin\theta}$$

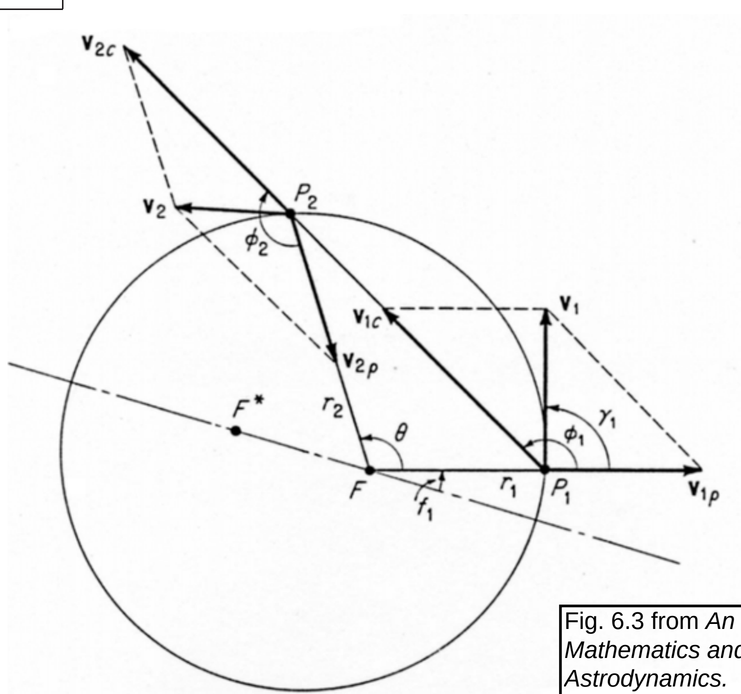


Fig. 6.3 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.

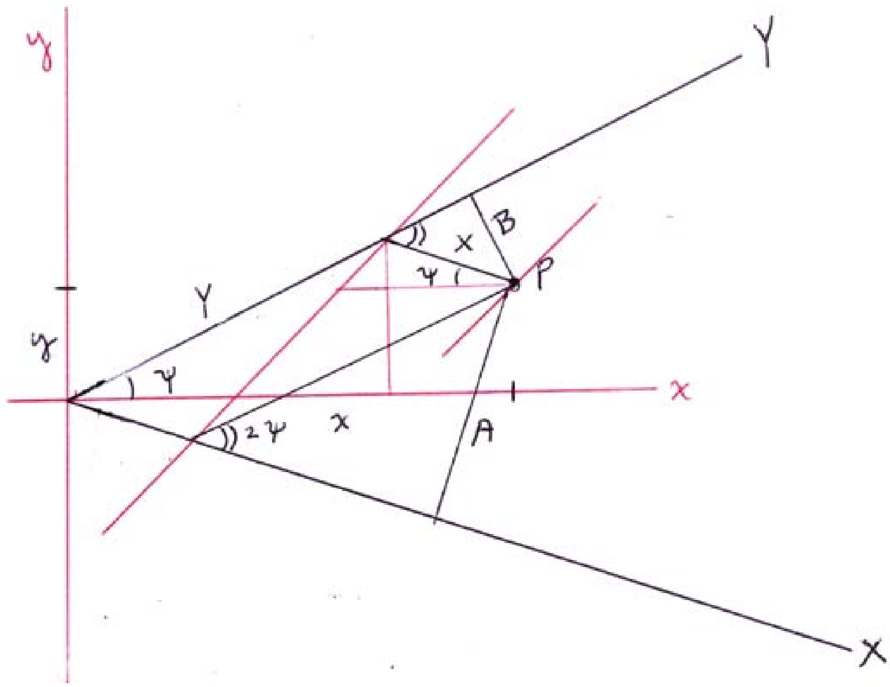
Properties of Skewed-Axes Velocity Components

Product: $\boxed{v_c v_\rho = \dfrac{\mu c}{2r_1 r_2}\sec^2 \tfrac{1}{2}\theta = \tfrac{1}{4}\, v_m^2 \sec^2 \tfrac{1}{2}\phi = \text{constant}}$ (*independent of the orbit*)

Ratio: $\dfrac{v_c}{v_\rho} = \dfrac{cp}{r_1 r_2(1-\cos\theta)}$ or $\boxed{\dfrac{p}{p_m} = \dfrac{v_c}{v_\rho}}$

Minimum-energy orbit parameter: $v_c = v_\rho \implies \boxed{p_m = \dfrac{r_1 r_2}{c}(1-\cos\theta)}$

Euler's Equation of the Hyperbola in Asymptotic Coordinates 1748



- **In Cartesian Coordinates** x, y

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1} \quad \text{with} \quad e = \sec\psi \; (= \sec \tfrac{1}{2}\phi) \quad \Longrightarrow \quad x^2 - y^2\cot^2\psi = a^2$$

- **In Asymptotic Coordinates** $X\,,\,Y$: $\quad x = (Y+X)\cos\psi \qquad y = (Y-X)\sin\psi$

$$(Y+X)^2\cos^2\psi - (Y-X)^2\sin^2\psi\cot^2\psi = a^2$$
$$(Y+X)^2 - (Y-X)^2 = a^2\sec^2\psi = a^2e^2$$
$$\boxed{XY = \tfrac{1}{4}a^2e^2 = \tfrac{1}{4}(a^2+b^2)}$$

- **In Vertical Projection Coordinates** A, B: $\quad A = Y\sin 2\psi \qquad B = X\sin 2\psi$

$$AB = 4XY\sin^2\psi\cos^2\psi = \frac{4}{e^2}XY\sin^2\psi = a^2\sin^2\psi = \frac{a^2}{e^2}(e^2-1) = \frac{b^2}{e^2} \qquad \boxed{AB = \frac{b^2}{e^2}}$$

Euler's Tangent to the Hyperbola — Page 171

Define α as the angle of intersection of the tangent at point P of the hyperbola with the x axis. Then the slope of the tangent is

$$\tan\alpha = \frac{dy}{dx} = \frac{b^2}{a^2}\times\frac{x}{y} = \frac{x}{y}\tan^2\psi = \frac{Y+X}{Y-X}\tan\psi$$

Now $\alpha+\psi$ is the angle between the tangent to the hyperbola and the asymptote. Then

$$\tan(\alpha+\psi) = \frac{\tan\alpha+\tan\psi}{1-\tan\alpha\tan\psi} = \frac{Y\sin 2\psi}{Y\cos 2\psi - X} = \frac{A\sin 2\psi}{A\cos 2\psi - B}$$

which is the slope of the diagonal of the parallelogram whose sides are X and Y. (Also, Y and X can be replaced by A and B.)

Hyperbolic Locus of Velocity Vectors for the Boundary-Value Problem

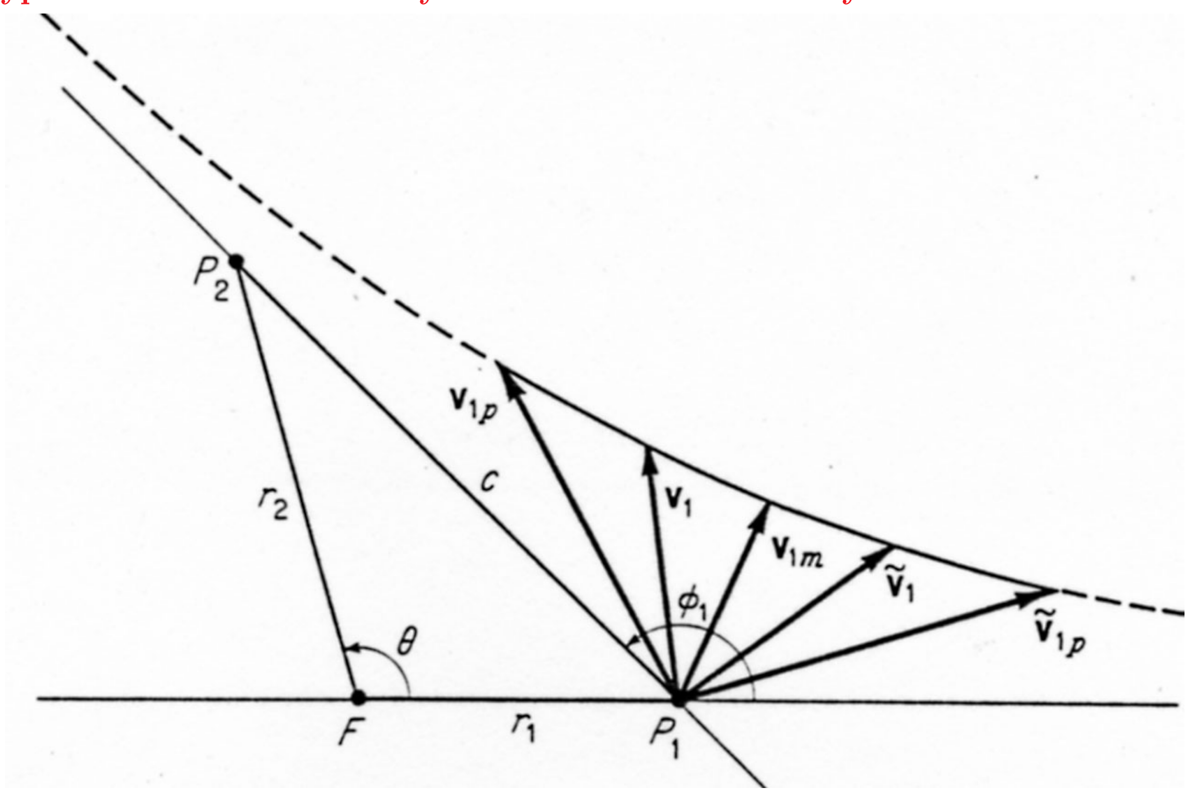


Fig. 6.4 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.

At the terminal P_1 we have

$$A = v_1 \sin\gamma_1$$
$$B = v_1 \sin(\phi_1 - \gamma_1)$$

so that

$$\frac{v_c}{v_\rho} = \frac{A}{B} = \frac{\sin\gamma_1}{\sin(\phi_1 - \gamma_1)} = \frac{p}{p_m} = \frac{c\sin\gamma_1}{r_1\sin\gamma_1 + r_2\sin(\theta - \gamma_1)}$$

For the last step, replace ϕ_1 by θ using the law of sines for the triangle $\Delta P_1 F P_2$

Similarly, at the terminal P_2,

$$A = v_2 \sin(\pi - \gamma_2)$$
$$B = v_2 \sin(\phi_2 + \gamma_2 - \pi)$$

$$\frac{v_c}{v_\rho} = \frac{A}{B} = -\frac{\sin\gamma_2}{\sin(\phi_2 + \gamma_2)} = \boxed{\frac{p}{p_m} = \frac{c\sin\gamma_2}{r_2\sin\gamma_2 - r_1\sin(\theta + \gamma_2)}}$$