

Lecture 21

Last time:

$$F(s) = K \frac{(c-s)}{s(d+s)}$$

$$[(s)] = \frac{A(c+s)}{(a+s)(c-s)(b-s)}$$

$$[(s)]_L = \frac{A(c-a)}{(c+a)(b+a)} \equiv \frac{B}{a+s}$$

Plugging in for the optimum compensator:

$$\begin{aligned} H_0(s) &= \frac{[(s)]_L}{F(s)_L F(-s)_L S_{ii}(s)_L} \\ &= \frac{Bs(d+s)(a+s)}{(a+s)K(c+s)S_n(b+s)} \\ &= \frac{Bs(d+s)}{S_n K(c+s)(b+s)} \end{aligned}$$

This means that in the cascade of $H_0(s)$ with $F(s)$, the effect of the RHP zero on the amplitude of the product is cancelled out, but the effect on the phase is not. In fact, the phase lag due to the RHP zero is doubled in the product.

Note:

- Cancellation of K and two LHP poles in $F(s)$
- Zero at $s = c$ (RHP) is not cancelled, but a pole is placed at the symmetric point $s = -c$
- Another pole is added at $s = -b$ which is beyond the signal cut-off by an amount which depends on $\frac{A}{S_n}$
- Also the gain $\frac{B}{S_n}$ depends on $\frac{A}{S_n}$ directly

Also: $[(s)]_R$ has the form $\frac{C}{c-s} + \frac{D}{b-s} = \frac{(e+fs)}{(c-s)(b-s)}$

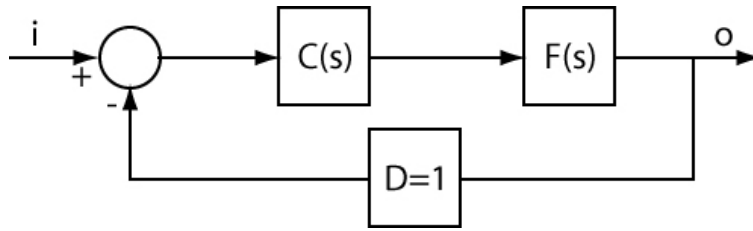
Then

$$\begin{aligned}
 L(s) &= -F(-s)_R F(s)_R S_{ii}(s)_R \left[(s) \right]_R \\
 &= -\frac{K(c-s)(b-s)(e+fs)}{(\varepsilon-s)(d-s)(a-s)(c-s)(b-s)} \\
 &= -\frac{K(e+fs)}{(\varepsilon-s)(d-s)(a-s)}
 \end{aligned}$$

which is analytic in the LHP and varies as $\frac{1}{s^2}$ for large s .

Necessary condition: $L(s)$ must be analytic in LHP and go to zero for large $|s|$ at least as fast as $\frac{1}{s}$.

Now find the loop compensator if the feedback transfer is unity.



$$C = \frac{H_0}{1 + H_0 F D} = \frac{\frac{B}{K S_n} s(d+s)}{s^2 + \left(b + c + \frac{B}{S_n} \right) s + \left(cb - \frac{Bc}{S_n} \right)}$$

Cancellation of the pole at the origin leaves the system with an uncontrollable mode corresponding to that pole. This is not good since that normal mode does not attenuate. As a practical matter, it might be better to move that zero away from the origin a bit.

That also means the system will not have unit input-output sensitivity.

Also note that the in-the-loop compensator need not be stable. It depends on the parameter values.

End of Quiz 2 material.

Estimation

We wish to estimate the values of a set of parameters which may be

- static
- dynamic

based on all available information

- measurements
- prior knowledge
- physical constraints

The measurements may be

- direct
- inferential (e.g. airspeed indicator – infers airspeed from dynamic pressure measurement)
- of varying quality.

Discussion

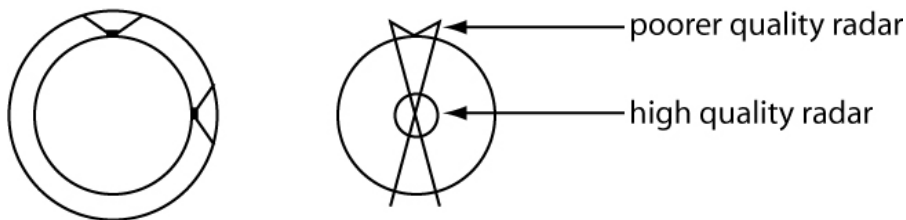
Problem formulation

- static or dynamic parameters
- static affords some simplifications but consider the whole problem: what is to be done with the estimates?

All information should be used

- better quality information weighted more heavily than poorer

Value of measurements depends both on noise and sensitivity to least certain parameters.



Suppose you have a radar looking at satellites overflying your position. Having a second location 90 degrees around the world would clarify parameters that are not were characterized by a single sensor.

Include the knowledge you have prior to the current set of measurements.

Physical constraints add information – use all known.

Differential equations relating dynamic parameters
Physical characteristics of environment

Suppose you are given a set of data and asked to smooth it. Unless you know the physics of the situation you cannot opt for one scheme over another (e.g., elevation data for a satellite overpass).

To determine where to point your telescope, you can use prior data to calculate the constants that describe the satellite's orbit, but it may be easier to use the satellite's current position and velocity to estimate its future position.

Direct measurements are convenient, but not always possible e.g., temperature of a star. Inferential measurements depend on physical constraints relating measured to estimated quantities. There may be uncertainty in these relationships, which should be modeled somehow.

A basic principle: should formulate the complete estimate problem at once – especially if any non-linear relations would be involved in deriving the desired quantities.

Example estimation problems

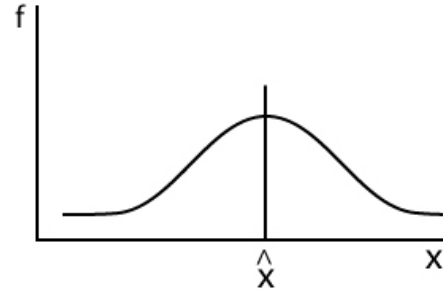
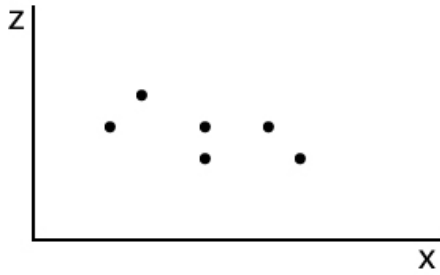
Example: Estimate x scalar constant

A deterministic quantity, x , is being observed directly – these observations are not necessarily each of equal quality.

The observations:

$$z_k = x + n_k$$

The noises are independent, unbiased, normal random variables. They may have different variances σ_k^2 . The conditional probability density function for z_k , conditioned on a given value of x , is just the density function for n_k centered around x .



Classical Maximum Likelihood estimate:

The \hat{x} maximizes $f(z_1, \dots, z_N | x)$

$$f(z_k | x) = f_n(z_k - x) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\left(\frac{(z_k - x)^2}{2\sigma_k^2}\right)}$$

In scalar form,

$$f(z_1, \dots, z_N | x) = \frac{1}{(2\pi)^{\frac{N}{2}} \sigma_1 \dots \sigma_N} e^{-\left[\frac{1}{2} \sum_{k=1}^N \left(\frac{z_k - x}{\sigma_k}\right)^2\right]}$$

The maximum value of f is

$$\min \sum_{k=1}^N \left(\frac{z_k - x}{\sigma_k}\right)^2 = \min \sum_{k=1}^N \frac{(z_k - x)^2}{\sigma_k^2}$$

The numerator $(z_k - x)^2$ is the measurement residual.

$$\frac{\partial}{\partial x} () = -2 \sum_{k=1}^N \frac{z_k - x}{\sigma_k^2} = 0$$

$$\hat{x} = \frac{\sum_{k=1}^N \frac{z_k}{\sigma_k^2}}{\sum_{k=1}^N \frac{1}{\sigma_k^2}}$$

We call this the weighted least squares estimate.

So the estimator is a linear combination of all the observations – the constants of the combination being inversely proportional (note proportional) to the variances of the measurement noises. This says that every piece of data should be used and will have a nonzero effect if its variance is less than infinite.

The statistics of the estimate are:

$$\hat{x} = \frac{\sum_{k=1}^N \frac{\bar{z}_k}{\sigma_k^2}}{\sum_{k=1}^N \frac{1}{\sigma_k^2}} = \frac{x \sum_{k=1}^N \frac{1}{\sigma_k^2}}{\sum_{k=1}^N \frac{1}{\sigma_k^2}} = x, \text{ an unbiased estimate}$$

error = 0

$$\text{Var}(\hat{x}) = \frac{\sum_{k=1}^N \text{Var}\left(\frac{z_k}{\sigma_k^2}\right)}{\left(\sum_{k=1}^N \frac{1}{\sigma_k^2}\right)^2}$$

since the central statistics of the z_k are those of the n_k which are independent. But note that the standard deviation of z_k is σ_k , so the standard deviation of $\frac{z_k}{\sigma_k}$ is 1 and that of $\frac{z_k}{\sigma_k^2}$ is $\frac{1}{\sigma_k}$. The variance of $\frac{z_k}{\sigma_k^2}$ is then $\frac{1}{\sigma_k^2}$.

$$\text{Var}(\hat{x}) = \frac{\sum_{k=1}^N \frac{1}{\sigma_k^2}}{\left(\sum_{k=1}^N \frac{1}{\sigma_k^2}\right)^2} = \frac{1}{\sum_{k=1}^N \frac{1}{\sigma_k^2}}$$

An easier way to remember this result is

$$\frac{1}{\sigma_{\hat{x}}^2} = \sum_{k=1}^N \frac{1}{\sigma_k^2}$$

The addition of any new piece of data, no matter how large its variance, thus reduces the variance of \hat{x} .

In the special case of equal quality data, $\sigma_k = \sigma_n$

$$\hat{x} = \frac{\frac{1}{\sigma_n^2} \sum_{k=1}^N z_k}{\frac{1}{\sigma_n^2} \sum_{k=1}^N 1} = \frac{1}{N} \sum_{k=1}^N z_k$$

the ordinary average, or arithmetic mean, of the data.

$$\frac{1}{\sigma_{\hat{x}}^2} = \frac{1}{\sigma_n^2} \sum_{k=1}^N 1 = \frac{N}{\sigma_n^2}$$

$$\sigma_{\hat{x}}^2 = \frac{1}{N} \sigma_n^2$$

$$\sigma_{\hat{x}} = \frac{1}{\sqrt{N}} \sigma$$

The standard deviation of the estimate (the average) goes down with the square root of the number of observations.

This estimator for x can be shown to be the optimum linear estimate of x in the mean squared error sense for arbitrary distributions of the n_k if x is treated as an arbitrary constant. That is, any other linear combination of the z_k will yield a larger mean squared difference between \hat{x} and x if x is an arbitrary constant. For normal noise the min variance linear estimate is the min variance estimate.

An important factor which should bear on the estimation of x and which has not yet been mentioned is the possibility of some a priori information about x . Clearly if we already had a reasonably accurate notion of the value of x and then took some additional data points – say of poor quality – we certainly would not want to derive an estimate based simply on the new data and ignore the a priori information.

Example: Supplemental measurements

Take N_1 measurements starting with no prior information

$$\hat{x}_1 = \frac{\sum_{k=1}^{N_1} \frac{z_k}{\sigma_k^2}}{\sum_{k=1}^{N_1} \frac{1}{\sigma_k^2}}$$

Later, we take more measurements, total of N

$$\hat{x} = \frac{\sum_{k=1}^N \frac{z_k}{\sigma_k^2}}{\sum_{k=1}^N \frac{1}{\sigma_k^2}} = \frac{\sum_{k=1}^{N_1} \frac{z_k}{\sigma_k^2} + \sum_{k=N_1+1}^N \frac{z_k}{\sigma_k^2}}{\sum_{k=1}^N \frac{1}{\sigma_k^2} + \sum_{k=N_1+1}^N \frac{1}{\sigma_k^2}}$$

$$\sum_{k=1}^{N_1} \frac{z_k}{\sigma_k^2} = \hat{x}_1 \sum_{k=1}^{N_1} \frac{1}{\sigma_k^2}$$

$$\left. \begin{aligned} \frac{1}{\sigma_{\hat{x}_1}^2} &= \sum_{k=1}^{N_1} \frac{1}{\sigma_k^2} \\ \sum_{k=1}^{N_1} \frac{z_k}{\sigma_k^2} &= \frac{\hat{x}_1}{\sigma_{\hat{x}_1}^2} \\ \hat{x} &= \frac{\frac{\hat{x}_1}{\sigma_{\hat{x}}^2} + \sum_{k=N_1+1}^N \frac{z_k}{\sigma_k^2}}{\frac{1}{\sigma_{\hat{x}}^2} + \sum_{k=N_1+1}^N \frac{1}{\sigma_k^2}} \end{aligned} \right|$$