Topic #18

16.31 Feedback Control Systems

Deterministic LQR

- Optimal control and the Riccati equation
- Weight Selection

Linear Quadratic Regulator (LQR)

- Have seen the solutions to the LQR problem, which results in linear full-state feedback control.
  - Would like to get some more insight on where this came from.

- Deterministic Linear Quadratic Regulator

  Plant:

$$\dot{\mathbf{x}} = A\mathbf{x} + B_u\mathbf{u}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$
$$\mathbf{z} = C_z\mathbf{x}$$

  Cost:

$$J_{LQR} = \frac{1}{2}\int_0^{t_f} \left[\mathbf{z}^T R_{\text{zz}}\mathbf{z} + \mathbf{u}^T R_{\text{uu}}\mathbf{u}\right] dt + \frac{1}{2}\mathbf{x}^T(t_f)P(t_f)\mathbf{x}(t_f)$$

  - Where $R_{\text{zz}} > 0$ and $R_{\text{uu}} > 0$
  - Define $R_{\text{xx}} = C_z^T R_{\text{zz}} C_z \geq 0$

- **Problem Statement:** Find input $\mathbf{u}$ $\forall t \in [t_0, t_f]$ to min J_{LQR}
  - This is not necessarily specified to be a feedback controller.

- Control design problem is a constrained optimization, with the constraints being the dynamics of the system.

Constrained Optimization

- The standard way of handling the constraints in an optimization is to add them to the cost using a **Lagrange multiplier**
  - Results in an unconstrained optimization.

- **Example:** $\min f(x,y) = x^2 + y^2$ subject to the constraint that $c(x,y) = x + y + 2 = 0$

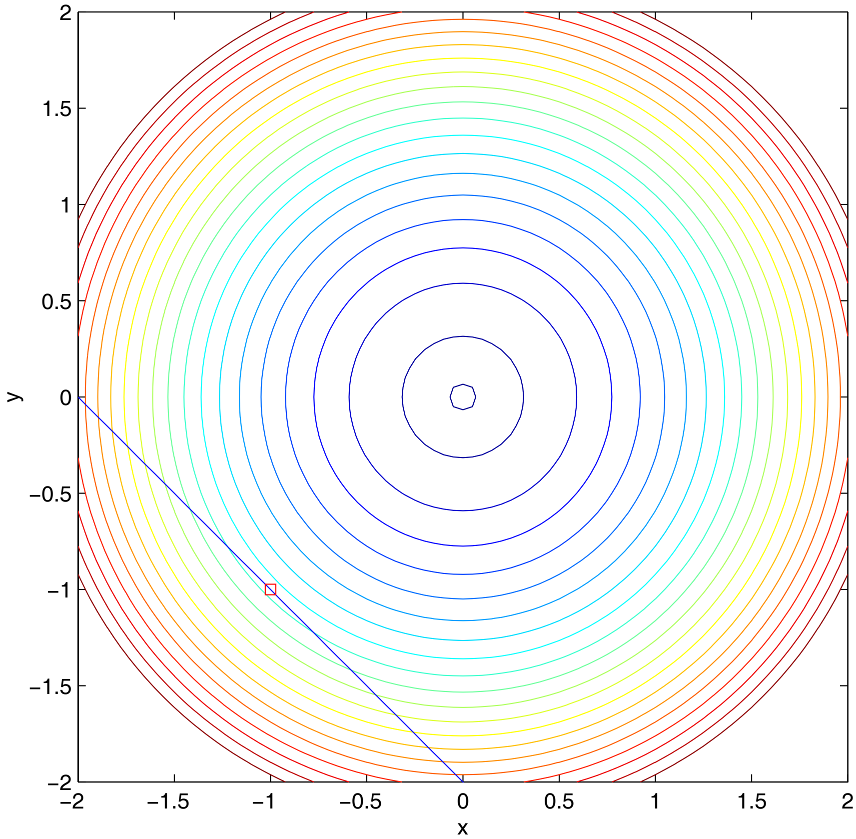


Fig. 1: Optimization results

- Clearly the unconstrained minimum is at $x = y = 0$

- To find the constrained minimum, form augmented cost function

$$L \triangleq f(x, y) + \lambda c(x, y) = x^2 + y^2 + \lambda(x + y + 2)$$

  - Where λ is the Lagrange multiplier
  - Note that if the constraint is satisfied, then $L \equiv f$

- The solution approach without constraints is to find the stationary point of $f(x, y)$ ($\partial f/\partial x = \partial f/\partial y = 0$)
  - With constraints we find the stationary points of L

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \lambda} = 0$$

  which gives

$$\begin{aligned} \frac{\partial L}{\partial x} &= 2x + \lambda = 0 \\ \frac{\partial L}{\partial y} &= 2y + \lambda = 0 \\ \frac{\partial L}{\partial \lambda} &= x + y + 2 = 0 \end{aligned}$$

- This gives 3 equations in 3 unknowns, solve to find

$$x^\star = y^\star = -1$$

- Key point here is that due to the constraint, the selection of x and y during the minimization are not independent
  - Lagrange multiplier captures this dependency.

LQR Optimization

- LQR optimization follows the same path, but it is complicated by the fact that the cost involves an integration over time
  - See 16.323 OCW notes for details

- To optimize the cost, follow the same procedure of augmenting the constraints in the problem (the system dynamics) to the cost (integrand, then integrate by parts) to form the **Hamiltonian**:

$$H = \frac{1}{2}\left(\mathbf{x}^T R_{\text{xx}}\mathbf{x} + \mathbf{u}^T R_{\text{uu}}\mathbf{u}\right) + \mathbf{p}^T\left(A\mathbf{x} + B_u\mathbf{u}\right)$$

  - $\mathbf{p} \in \mathbb{R}^{n\times 1}$ is called the **Adjoint variable** or **Costate**
  - It is the **Lagrange multiplier** in the problem.

- The necessary conditions for optimality are then that:
  1. $\dot{\mathbf{x}} = \dfrac{\partial H}{\partial \mathbf{p}}^T = A\mathbf{x} + B_u\mathbf{u}$ with $\mathbf{x}(t_0) = \mathbf{x}_0$
  2. $\dot{\mathbf{p}} = -\dfrac{\partial H}{\partial \mathbf{x}}^T = -R_{\text{xx}}\mathbf{x} - A^T\mathbf{p}$ with $\mathbf{p}(t_f) = P_{t_f}\mathbf{x}(t_f)$
  3. $\dfrac{\partial H}{\partial \mathbf{u}} = 0 \Rightarrow R_{\text{uu}}\mathbf{u} + B_u^T\mathbf{p} = 0$, so $\mathbf{u}^\star = -R_{\text{uu}}^{-1}B_u^T\mathbf{p}$

- Can check for a minimum by looking at $\dfrac{\partial^2 H}{\partial \mathbf{u}^2} \geq 0$ (need to check that $R_{\text{uu}} \geq 0$)

- Key point is that we now have that

$$\dot{\mathbf{x}} \;=\; A\mathbf{x} + B_u\mathbf{u}^\star = A\mathbf{x} - B_u R_{\rm uu}^{-1} B_u^T \mathbf{p}$$

which can be combined with equation for the adjoint variable

$$\dot{\mathbf{p}} = -R_{\rm xx}\mathbf{x} - A^T\mathbf{p} = -C_z^T R_{\rm zz} C_z \mathbf{x} - A^T\mathbf{p}$$

$$\Rightarrow \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} A & -B_u R_{\rm uu}^{-1} B_u^T \\ -C_z^T R_{\rm zz} C_z & -A^T \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix}$$

which is called the **Hamiltonian Matrix**.

  - Matrix describes closed loop dynamics for both $\mathbf{x}$ and $\mathbf{p}$.
  - Dynamics of $\mathbf{x}$ and $\mathbf{p}$ are coupled, but $\mathbf{x}$ known initially and $\mathbf{p}$ known at terminal time, since $\mathbf{p}(t_f) = P_{t_f}\mathbf{x}(t_f)$
  - Two point boundary value problem $\Rightarrow$ typically hard to solve.

- However, in this case, we can introduce a new matrix variable P and it is relatively easy to show that:

  1. $\mathbf{p} = P\mathbf{x}$
  2. It is relatively easy to find P.

- In fact, P must satisfy

$$0 = A^T P + PA + C_z^T R_{\rm zz} C_z - P B_u R_{\rm uu}^{-1} B_u^T P$$

  - Which, is the matrix **algebraic Riccati Equation**.

- The control gains are then

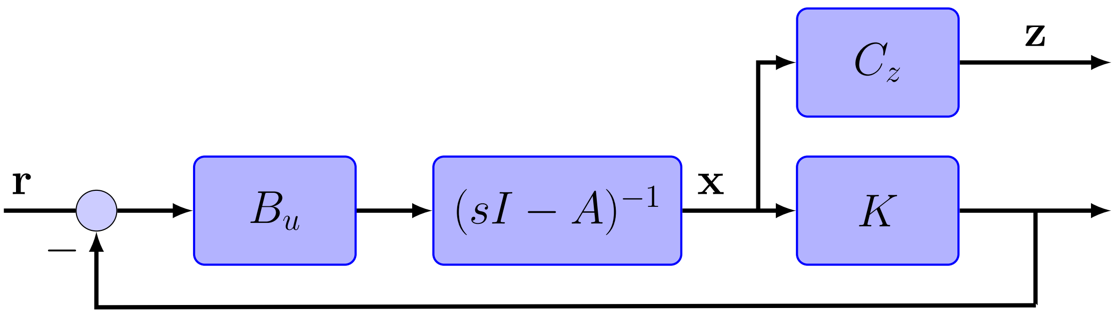
$$\mathbf{u}_{\rm opt} \;=\; -R_{\rm uu}^{-1} B_u^T \mathbf{p} = -R_{\rm uu}^{-1} B_u^T P\mathbf{x} = -K\mathbf{x}$$

- **So the optimal control inputs can in fact be computed using linear feedback on the full system state**

LQR Stability Margins

- LQR approach selects closed-loop poles that **balance** between system errors and the control effort.
  - Easy design iteration using $R_{\rm uu}$
  - Sometimes difficult to relate the desired transient response to the LQR cost function.
- Particularly nice thing about the LQR approach is that the designer is focused on system performance issues
- Turns out that the news is even better than that, because LQR exhibits very good stability margins
- Consider the LQR stability robustness.

$$
\begin{aligned}
J &= \frac{1}{2}\int_0^\infty \mathbf{z}^T\mathbf{z} + \rho\mathbf{u}^T\mathbf{u}\; dt \\
\dot{\mathbf{x}} &= A\mathbf{x} + B_u\mathbf{u}, \qquad \mathbf{z} = C_z\mathbf{x}, \qquad R_{\rm xx} = C_z^TC_z
\end{aligned}
$$



- Study robustness in the frequency domain.
  - Loop transfer function $L(s) = K(sI-A)^{-1}B_u$
  - Cost transfer function $C(s) = C_z(sI-A)^{-1}B_u$

- Can develop a relationship between the open-loop cost $C(s)$ and the closed-loop return difference $I+L(s)$ called the **Kalman Frequency Domain Equality**

$$[I + L(-s)]^T [I + L(s)] = 1 + \frac{1}{\rho} C^T(-s)C(s)$$

- Written for MIMO case, but look at the SISO case to develop further insights ($s = \mathbf{j}\omega$)

$$\begin{aligned}[I + L(-\mathbf{j}\omega)] [I + L(\mathbf{j}\omega)] &= (I + L_r(\omega) - \mathbf{j}L_i(\omega))(I + L_r(\omega) + \mathbf{j}L_i(\omega)) \\ &\equiv |1 + L(\mathbf{j}\omega)|^2\end{aligned}$$

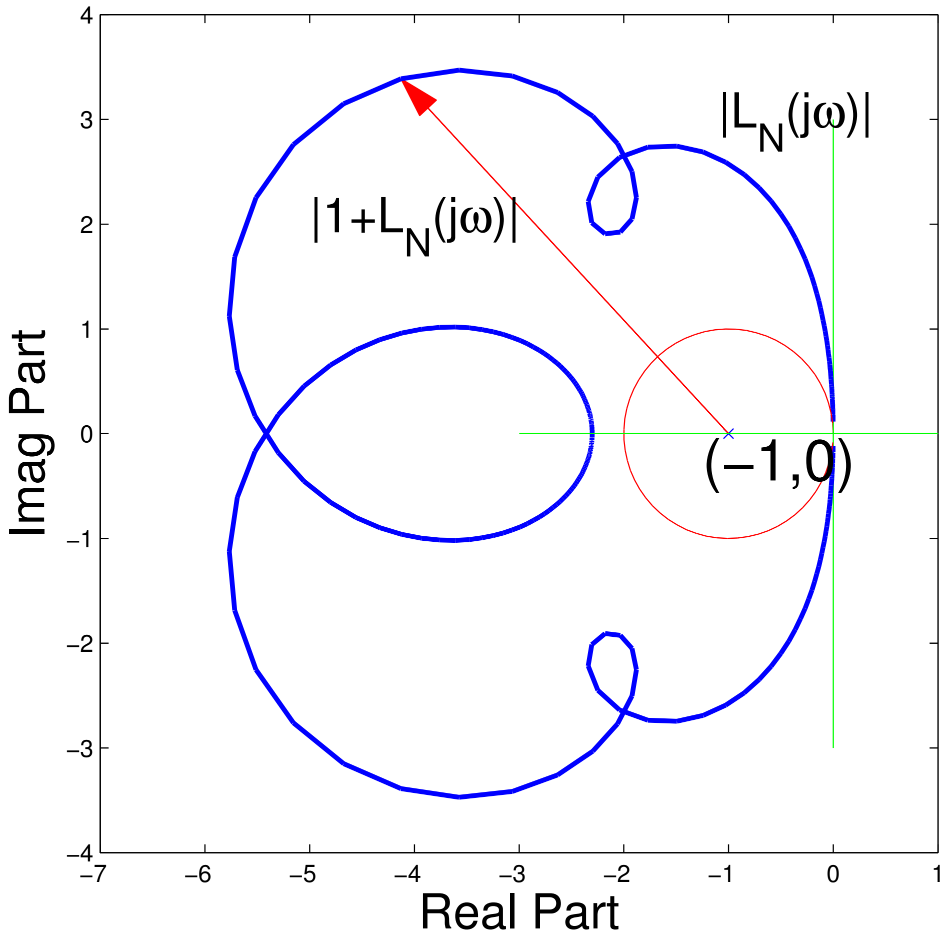
and

$$C^T(-\mathbf{j}\omega)C(\mathbf{j}\omega) = C_r^2 + C_i^2 = |C(\mathbf{j}\omega)|^2 \geq 0$$

- Thus the KFE becomes

$$|1 + L(\mathbf{j}\omega)|^2 = 1 + \frac{1}{\rho}|C(\mathbf{j}\omega)|^2 \geq 1$$

- **Implications:** The Nyquist plot of $L(\mathbf{j}\omega)$ will always be outside the unit circle centered at $(-1,0)$.



- Great, but why is this so significant? Recall the SISO form of the **Nyquist Stability Theorem:**

  If the loop transfer function $L(s)$ has P poles in the RHP s-plane (and $\lim_{s\to\infty} L(s)$ is a constant), then for closed-loop stability, the locus of $L(\mathbf{j}\omega)$ for $\omega : (-\infty, \infty)$ must encircle the critical point $(-1,0)$ P times in the **counterclockwise** direction (Ogata528)

- So we can directly prove stability from the Nyquist plot of $L(s)$. But what if the model is wrong and it turns out that the actual loop transfer function $L_A(s)$ is given by:

$$L_A(s) = L_N(s)[1 + \Delta(s)], \quad |\Delta(\mathbf{j}\omega)| \leq 1, \quad \forall\omega$$

- We need to determine whether these perturbations to the loop TF will change the decision about closed-loop stability

  $\Rightarrow$ can do this directly by determining if it is possible to **change the number of encirclements of the critical point**

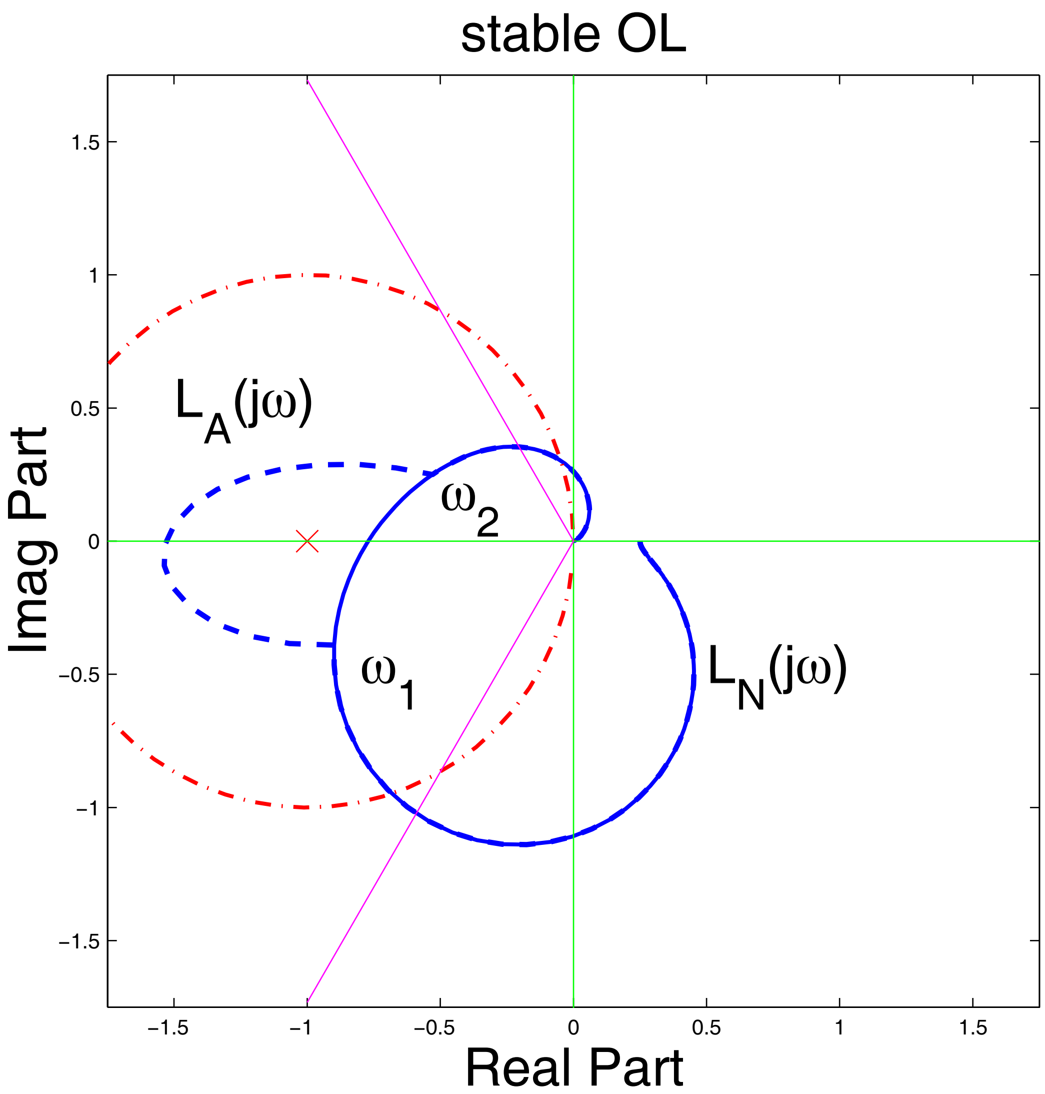
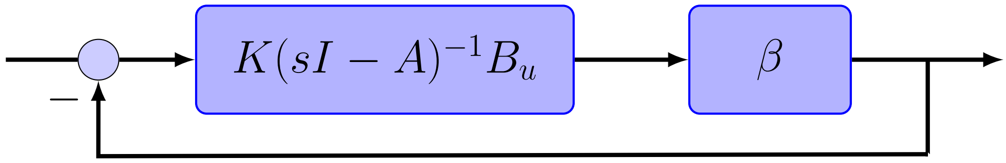


Fig. 2: Perturbation to the LTF causing a change in the number of encirclements

- Claim is that "since the LTF $L(\mathbf{j}\omega)$ is guaranteed to be far from the critical point for all frequencies, then LQR is VERY robust."
  - Can study this by introducing a modification to the system, where nominally $\beta = 1$, but we would like to consider:
    - The gain $\beta \in \mathbb{R}$
    - The phase $\beta \in e^{j\phi}$



- In fact, can be shown that:
  - If open-loop system is stable, then any $\beta \in (0, \infty)$ yields a stable closed-loop system. For an unstable system, any $\beta \in (1/2, \infty)$ yields a stable closed-loop system
    $\Rightarrow$ gain margins are $(1/2, \infty)$
  - Phase margins of at least $\pm 60°$
- Both of these robustness margins are very large on the scale of what is normally possible for classical control systems.

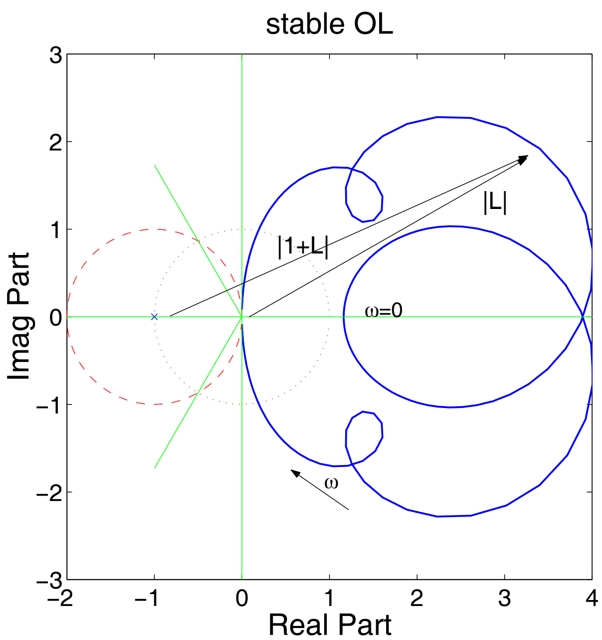


Fig. 3: Example of LTF for an open-loop stable system

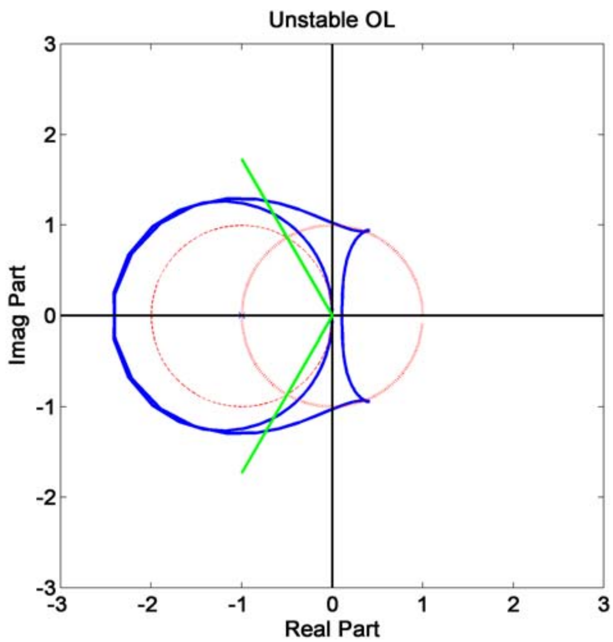


Fig. 4: Example loop transfer functions for open-loop unstable system.

16.30 / 16.31 Feedback Control Systems
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