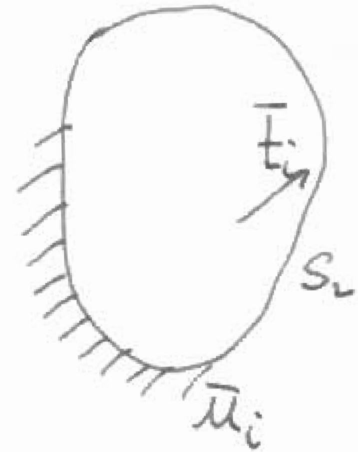
Boundary value problem

$$\begin{cases} \sigma_{ij,j} + f_i = 0 & \text{in } B \\ \sigma_{ij} n_j = \bar{t}_i & \text{on } S_2 \end{cases}$$

$$\begin{cases} u_i = \bar{u}_i & \text{on } S_1 \\ \varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) & \text{in } B \end{cases}$$



$\varepsilon^e, \varepsilon^p$ incompatible

Variational principle (minimum potential energy)

Assume: $\exists\ \mathcal{U}(\varepsilon, y)$ internal energy potential
$y \equiv (\varepsilon^p, q)$

$$\sigma_{ij} = \frac{\partial \mathcal{U}(\varepsilon, y)}{\partial \varepsilon_{ij}}$$

Lubliner (1972):

$$\mathcal{U}(\varepsilon, y) = \mathcal{U}^e(\epsilon - \varepsilon^p) + \mathcal{U}^p(q) \quad \text{(decoupled)}$$

Example: linear elasticity $\quad \mathcal{U}^e = \frac{1}{2} C_{ijkl}\, \varepsilon^e_{ij}\, \varepsilon^e_{kl}$

<u>Equilibrium:</u> minimize potential energy for B with respect to u_i, <u>for fixed y</u>

$$J = \int_B \mathcal{U}(\varepsilon, y)\, dV - \int_B f_i u_i\, dV - \int_{S_2} t_i u_i\, ds$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\dot{y}(x,t) = f\big(\varepsilon(x,t), y(x,t)\big) \quad \text{kinetic equation}$$

<u>Algorithms</u>: Need to integrate constitutive relation in time at all quadrature points.

Time-stepping algorithms for constitutive relations

Given $\varepsilon_n, \sigma_n, q_n, \varepsilon_n^p$ and

ε_{n+1} (strain driven)

Compute $\sigma_{n+1}, q_{n+1}, \varepsilon_{n+1}^p$

General algorithm:

$$\sigma_{n+1} = \hat{\sigma}(\varepsilon_{n+1}; \varepsilon_n, \sigma_n, q_n, \varepsilon_n^p, \Delta t)$$

combine state vector $\Lambda \equiv (\varepsilon, \sigma, q, \varepsilon^p)$

$$\boxed{\sigma_{n+1} = \hat{\sigma}(\varepsilon_{n+1}; \Lambda_n, \Delta t)}$$

strain driven

Integrate into global solution procedure

Equilibrium: Principle of virtual displacements

$$\int_B \sigma_{ij}\, \eta_{i,j}\, dV - \int_B f_i\, \eta_i\, dV - \int_{S_2} t_i\, \eta_i\, dS = 0$$

$\forall \eta$ admissible

finite element discretization:

$$\sum_e \int_{\Omega^e} B^T \sigma \, dV - f_{ext} = 0$$

Enforce at time (load step) $t = t_{n+1}$

$$\sum_e \int_{\Omega^e} B^T \sigma_{n+1} \, dV - f^{ext}_{n+1} = 0$$

Insert update:

$$\sum_e \int_{\Omega^e} B^T \hat{\sigma}(\varepsilon_{n+1}; \Lambda_n, \Delta t) \, dV - f^{ext}_{n+1} = 0$$

Compatibility: $\varepsilon_{n+1} = B u_{n+1}$

$$\boxed{\sum_e \int_{\Omega^e} B^T \hat{\sigma}(B u_{n+1}; \Lambda_n, \Delta t) \, dV - f^{ext}_{n+1} = 0}$$

$\rightarrow$ System of nonlinear algebraic equations for u_{n+1}

Updated "u_{n+1}" satisfies:

$$\begin{cases} \sum_e \int_{\Omega^e} B^T \sigma_{n+1} \, dV - f^{ext}_{n+1} = 0 \\ \sigma_{n+1} = \hat{\sigma}(\varepsilon_{n+1}; \Lambda_n, \Delta t) \\ \varepsilon_{n+1} = B \, u_{n+1} \end{cases}$$

<u>Numerical quadrature:</u>

$$\sum_e \sum_q w_p^e \, B^T(\xi_q) \, \hat{\sigma}(\xi_q) - f^{ext}_{n+1} = 0$$

state variables sampled at quadrature points

<u>Newton-Raphson solution procedure</u>

$$t_n \longrightarrow t_{n+1} \quad , \quad f^{ext}_n \rightarrow f^{ext}_{n+1}$$

(k)-th iteration: $u^{(k)}_{n+1} \longrightarrow u^{(k+1)}_{n+1}$

$$u^{(k+1)}_{n+1} = u^{(k)}_{n+1} + \Delta u$$

$$f^{int}(u^{(k)}_{n+1} + \Delta u) = f^{ext}_{n+1} \quad , \text{ linearize}$$

$$f^{int}(u^{(k)}_{n+1}) + \underbrace{\frac{\partial f^{int}}{\partial u}(u^{(k)}_{n+1})}_{K(u^{(k)}_{n+1})} \Delta u + \text{h.o.t} = f^{ext}_{n+1}$$

$$K(u_{n+1}^{(k)})\,\Delta u = r(u_{n+1}^{(k)}) = f_{n+1}^{ext} - f^{int}(u_{n+1}^{(k)})$$

Consistent tangent stiffness

$$K = \frac{\partial f^{int}}{\partial u} = \frac{\partial}{\partial u}\left[\sum_e \int_{\Omega^e} B^T \hat{\sigma}(Bu; \Lambda_n, \Delta t)\, dV\right]$$

$$= \sum_e \left[\int_{\Omega^e} B^T \underbrace{\frac{\partial \hat{\sigma}}{\partial \varepsilon}(Bu; \Lambda_n, \Delta t)}_{C(Bu)} B\, dV\right]$$

$$\boxed{C = \frac{\partial \hat{\sigma}}{\partial \varepsilon}(\varepsilon; \Lambda_n, \Delta t)}$$ CONSISTENT TANGENT MODULI

Note C is obtained by linearization of the constitutive update algorithm

$$\Longrightarrow \quad \boxed{K = \sum_e K^e = \sum_e \int_{\Omega^e} B^T C B\, dV}$$

Newton-Raphson solution:

$$r(u_{n+1}^{(k)}) = f_{n+1}^{ext} - \sum_e \int_{\Omega^e} B^T \hat{\sigma}(\overbrace{B u_{n+1}^{(k)}}^{\varepsilon_{n+1}^{(k)}}; \Lambda_n, \Delta t)\, dV$$

$$K(u_{n+1}^{(k)}) = \sum_e \int_{\Omega^e} B^T C(\varepsilon_{n+1}^{(k)}; \Lambda_n, \Delta t) B\, dV$$

Note state remains fixed at Δ_n during the Newton Raphson iterations. The state is updated at the end of the load (time) step.

Constitutive update algorithms

Backward Euler, fully implicit

$$\dot{\sigma} = C\,(\dot{\varepsilon} - \dot{\lambda}\, r(\sigma, q))$$

$$\dot{q} = \dot{\lambda}\, h(\sigma, q)$$

$$\dot{\lambda} = \begin{cases} \dfrac{\phi(\sigma, q)}{\eta} & \text{if } \phi \geq 0 \\ 0 & \text{if } \phi < 0 \end{cases}$$

$$\begin{cases} \sigma_{n+1} = \sigma_n + C(\Delta\varepsilon - \Delta\lambda\, r_{n+1}) \\ q_{n+1} = q_n + \Delta\lambda\, h_{n+1} \\ \Delta\lambda = \dfrac{\phi_{n+1}}{\eta} \end{cases}$$

where

$$r_{n+1} = r(\sigma_{n+1}, q_{n+1})\;;\; h_{n+1} = h(\sigma_{n+1}, q_{n+1});\; \phi_{n+1} = \phi(\sigma_{n+1}, q_{n+1})$$

This defines a system of nonlinear algebraic equations in: σ_{n+1}, q_{n+1}, $\Delta\lambda$.

Rate independent limit: $\eta \to 0$ $\boxed{\phi_{n+1}=0}$ yield criterion at $n+1$

Geometrical interpretation

Elastic predictor

$$\left.\begin{aligned} \sigma^*_{n+1} &= \sigma_n + C\,\Delta\varepsilon \\ q^*_{n+1} &= q_n \,, \quad \Delta\lambda^* = 0 \end{aligned}\right\} \text{neglects plasticity}$$

Two possibilities:

- $\phi^*_{n+1} = \phi(\sigma^*_{n+1}, q^*_{n+1}) \leq 0$

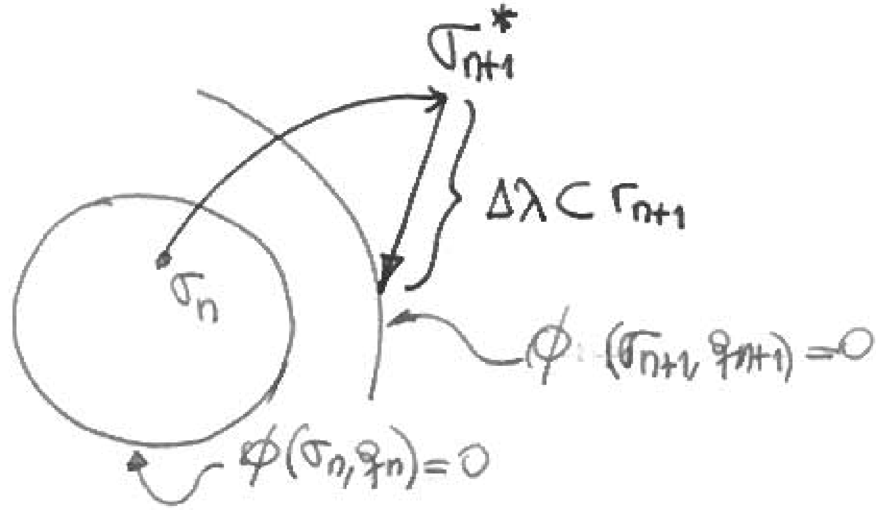
  Then $\sigma_{n+1} = \sigma^*_{n+1}$, $q_{n+1} = q^*_{n+1}$, $\Delta\lambda = \Delta\lambda^*$

  DONE

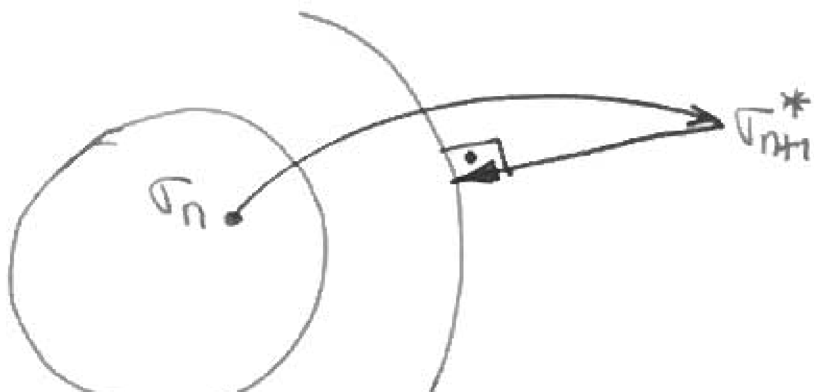
- otherwise $\longrightarrow$ plastic corrector

$$\sigma_{n+1} = \sigma^*_{n+1} - \Delta\lambda\, r_{n+1}$$

$$q_{n+1} = q^*_{n+1} + \Delta\lambda\, h_{n+1}\,, \quad \Delta\lambda = \phi_{n+1}/\eta$$



if $r \propto \dfrac{\partial \phi}{\partial \sigma} \Longrightarrow$



<u>Closest point projection algorithms</u> $\Big\{$ always defined if elastic domain is convex

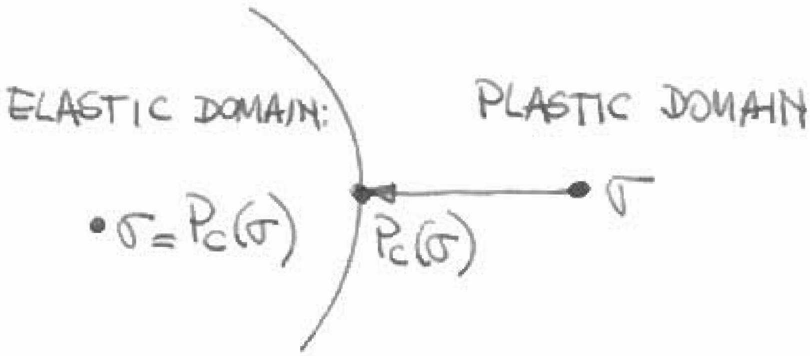
σ_{n+1} is closest to σ_{n+1}^{*}:

$(\sigma_{n+1} - \sigma_{n+1}^{*}) : C^{-1} : (\sigma_{n+1} - \sigma_{n+1}^{*})$ is minimized

(plastic work)

σ_{n+1} closest to σ^*_{n+1} in norm $\|\sigma\| = \sigma : C^{-1} : \sigma$

$\longrightarrow \|\sigma^*_{n+1} - \sigma_{n+1}\|$ minimum

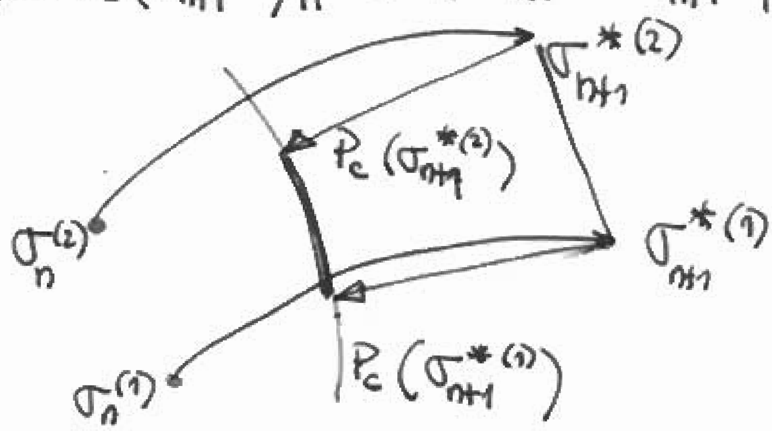


$P_C \equiv$ closest point projection onto boundary of elastic domain (yield surface)
Well defined for any <u>convex</u> elastic domain with or without corners

$$\boxed{\sigma_{n+1} = P_{C_{n+1}}(\sigma^*_{n+1})}$$

P_C is contractive if elastic domain is convex

$$\|P_C(\sigma^{*(1)}_{n+1}) - P_C(\sigma^{*(2)}_{n+1})\| \leq \|\sigma^{*(1)}_{n+1} - \sigma^{*(2)}_{n+1}\|$$



$\Longrightarrow$ fully implicit algorithm is contractive (errors in initial conditions are reduced by algorithm)

STRONG STATEMENT OF STABILITY

Specific model: $\underline{J_2\text{- isotropic hardening - fully implicit}}$

$$\dot{\varepsilon}_{ij}^{p} = \dot{\lambda}\,\frac{3}{2}\frac{s_{ij}}{\bar{\sigma}} \quad ; \quad \bar{\sigma} = \left(\frac{3}{2}\,s_{ij}s_{ij}\right)^{1/2};$$

$$r_{ij} = \frac{3}{2}\,\frac{s_{ij}}{\bar{\sigma}}$$

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}^{e}$$

$$\dot{\lambda} = \dot{\varepsilon}_0 \left[\left(\frac{\bar{\sigma}}{\sigma_0}\right)^{m} - 1\right] = \frac{\phi}{\eta} \quad \text{if} \quad \phi \geq 0$$

$$\eta = \frac{\sigma_y}{\dot{\varepsilon}_0} \quad ; \quad \phi = \sigma_y\left[\left(\frac{\bar{\sigma}}{\sigma_0}\right)^{m} - 1\right]$$

$$\sigma_0(\bar{\varepsilon}) = \sigma_y\left(1 + \frac{\bar{\varepsilon}}{\varepsilon_0}\right)^{1/n}$$

$$\bar{\varepsilon} = \int_0^t \dot{\lambda}\,dt$$

$$\sigma_{n+1} = \sigma_n + C\left(\Delta\varepsilon - \Delta\lambda\, r_{n+1}\right)$$

$$\Delta\lambda = \frac{\Delta t}{\eta}\,\phi\left(\bar{\sigma}_{n+1}, \sigma_{0,n+1}\right)$$

$$\sigma_{0,n+1} = \sigma_0\left(\bar{\varepsilon}_n + \Delta\lambda\right)$$

Isotropic elasticity

$$\boxed{p_{n+1} = p_n + K\,\Delta\varepsilon_{kk}}$$ pressure update is elastic

$$s_{n+1} = s_n + 2\mu\left(\Delta e - \Delta\lambda\,\frac{3}{2}\,\frac{s_{n+1}}{\bar{\sigma}_{n+1}}\right)$$

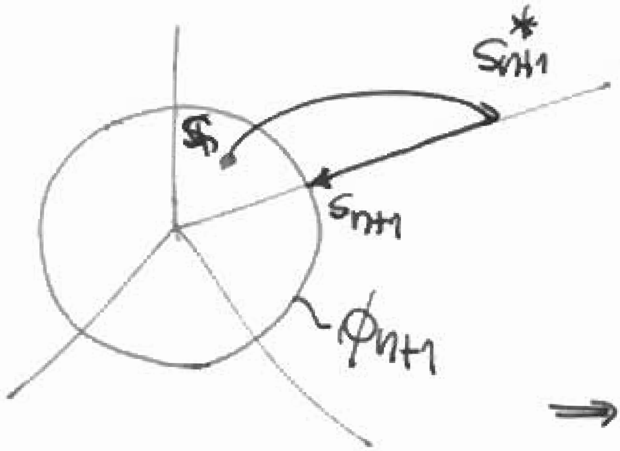
$$\Delta\varepsilon = \frac{\Delta\varepsilon_{kk}}{3}\,I + \Delta e$$

$$s^*_{n+1} = s_n + 2\mu\,\Delta e$$

$$\bar{\sigma}^*_{n+1} = \left(\frac{3}{2}\,s^*_{n+1} : s^*_{n+1}\right)^{1/2}$$

$$s_{n+1} = s^*_{n+1} - \left(\Delta\lambda\,\frac{3}{\cancel{2}}\,\frac{s_{n+1}}{\bar{\sigma}_{n+1}}\right)\cancel{2}\mu$$

$$= s^*_{n+1} - 3\mu\,\Delta\lambda\,\frac{s_{n+1}}{\bar{\sigma}_{n+1}}$$



$$s_{n+1} = C\, s^*_{n+1}$$

$$\bar{\sigma}_{n+1} = C\, \bar{\sigma}^*_{n+1}$$

$$\Longrightarrow \frac{s_{n+1}}{\bar{\sigma}_{n+1}} = \frac{s^*_{n+1}}{\bar{\sigma}^*_{n+1}}$$

$$C\, s^*_{n+1} = s^*_{n+1} - 3\mu\, \Delta\lambda \frac{s^*_{n+1}}{\bar{\sigma}^*_{n+1}}$$

$$\left(C - 1 + 3\mu \frac{\Delta\lambda}{\bar{\sigma}^*_{n+1}}\right) s^*_{n+1} = 0$$

$$\Longrightarrow \boxed{C = 1 - 3\mu \frac{\Delta\lambda}{\bar{\sigma}^*_{n+1}}}$$

$$\begin{cases} \bar{\sigma}_{n+1} = \bar{\sigma}^*_{n+1} - 3\mu\,\Delta\lambda \\ \Delta\lambda = \dfrac{\Delta t}{\eta}\, \phi\left(\bar{\sigma}_{n+1}, \sigma_{0,n+1}\right) \\ \sigma_{0,n+1} = \sigma_0\left(\bar{\varepsilon}_n + \Delta\lambda\right) \end{cases}$$

3 equations, 3 unknowns. Can be transformed into 1 equation with 1 unknown $(\Delta\lambda)$

$$\boxed{\Delta\lambda = \frac{\Delta t}{\eta}\,\phi\left(\bar{\sigma}^{*}_{n+1} - 3\mu\,\Delta\lambda,\ \sigma_0(\varepsilon_n + \Delta\lambda)\right)}$$

One scalar equation with one unknown "$\Delta\lambda$", solve by local Newton Raphson iteration.
For power-law viscosity:

$$f(\Delta\lambda) = \left(\frac{\Delta\lambda}{\dot{\varepsilon}_0\,\Delta t} + 1\right)^{1/m} - \frac{\bar{\sigma}^{*}_{n+1} - 3\mu\,\Delta\lambda}{\sigma_0(\varepsilon_n + \Delta\lambda)} = 0$$