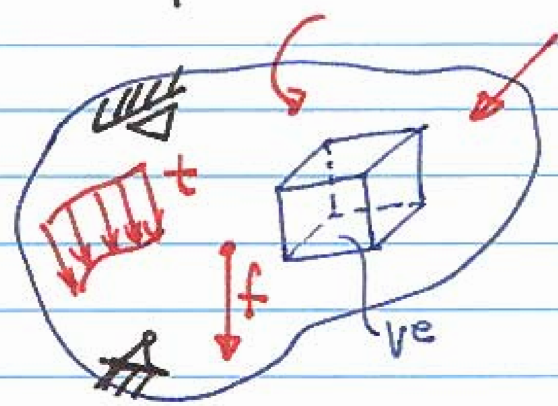
The finite element method (V) for three-dimensional elasticity problems

Potential energy applied to one element

$$\boxed{\Pi^e = \int_{V^e} \frac{1}{2} C_{ijkl}\, \varepsilon_{kl}\, \varepsilon_{ij}\, dV - \int_{V^e} f_i\, u_i\, dV - \int_{S^e} t_i\, u_i\, dS}$$

Here is the picture:



Introduce an approximation for the displacement field "u_i" within the element:

$$\boxed{u_i^e = \phi_k^e\, U_{ik}^e}$$

$i = 1,3 \quad k = 1,n$

n: number of nodes per element

What can be inferred about the approximation for u_1, u_2, u_3?

Approximation for strains:

$$\varepsilon_{ij}^{e} = \frac{1}{2}\left(u_{i,j}^{e} + u_{j,i}^{e}\right) \qquad \text{drop "e's"}$$

$$= \frac{1}{2}\left(\phi_{k,j}\, U_{ik} + \phi_{k,i}\, U_{jk}\right)$$

Procedure is the same as before:

- replace in potential
- minimize with respect to nodal displacements U_{ik}
- $\Rightarrow$ obtain finite element matrices

  stiffness $[K]$

  force vector $[R]$

Expressions look simpler if we write in matrix form

- $u_i \longrightarrow u \equiv \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \in \mathbb{R}^{3\times 1}$

- $\varepsilon_{ij} \longrightarrow \varepsilon \equiv \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{13} \\ \varepsilon_{23} \\ \varepsilon_{12} \end{Bmatrix} \in \mathbb{R}^{6\times 1}$

- $\sigma_{ij} \longrightarrow \sigma \equiv \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{13} \\ \sigma_{23} \\ \sigma_{12} \end{Bmatrix} \in \mathbb{R}^{6\times 1}$

- $\sigma = C\,\varepsilon \,, \qquad C \in \mathbb{R}^{6\times 6}$

- $f_i \longrightarrow f \equiv \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} \in \mathbb{R}^{3\times 1}$

- $t_i \longrightarrow t \equiv \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} \in \mathbb{R}^{3\times 1}$

The potential now reads:

$$\boxed{\Pi^e = \int_{V^e} \frac{1}{2}\, \underbrace{\varepsilon^T C \varepsilon}_{\substack{1\times6\ 6\times6\ 6\times1 \\ 1\times1}}\, dv - \int_{V^e} \underbrace{u^T \cdot f}_{\substack{1\times3\ 3\times1 \\ 1\times1}}\, dv - \int_S \underbrace{u^T \cdot t}_{\substack{1\times3\ 3\times1 \\ 1\times1}}\, ds}$$

Also, approximation in matrix form:

displacements $\boxed{\underset{3\times1}{u^e} = H^e\, U^e}$, U^e: vector of nodal displacements

What is the dimension of U^e? And H^e?

$$(U^e)^T = \{ U_1^1\, U_2^1\, U_3^1 \quad U_1^2\, U_2^2\, U_3^2 \ldots\ldots U_1^n\, U_2^n\, U_3^n \}$$

strains $\boxed{\underset{6\times1}{\varepsilon^e} = \underset{6\times3n}{B^e}\, \underset{3n\times1}{U^e}}$

Obviously H^e is obtained from ϕ_k and B^e from their derivatives.

Replace in potential:

$$\Pi^e = \int_{V^e} \frac{1}{2} \overbrace{U^T B^T}^{\varepsilon^T} C \overbrace{B U}^{\varepsilon} - \int_{V^e} U^T H^T f \, dV - \int_{S^e} U^T H^T t \, dS$$

$$= \frac{1}{2} U^T \underbrace{\int_{V^e} B^T C B \, dV}_{K^e} U - \underbrace{U^T \int_{V^e} H^T f \, dV - U^T \int_{S^e} H^T f \, dS}_{U^T R^e}$$

$$\boxed{\Pi^e = \frac{1}{2} U^T K^e U - U^T R^e}$$

where

$$\boxed{K^e = \int_{V^e} B^T C B \, dV}$$

$$\underbrace{3n \times 6 \quad 6 \times 6 \quad 6 \times 3n}_{3n \times 3n}$$

$$\boxed{R^e = \int_{V^e} H^T f \, dV + \int_{S^e} H^T t \, dS}$$

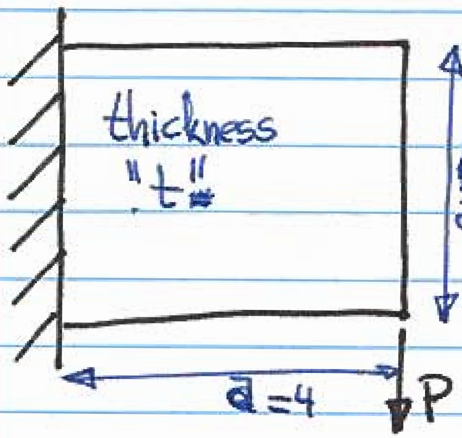
$$\underbrace{3n \times 3 \quad 3 \times 1}_{3n \times 1}$$

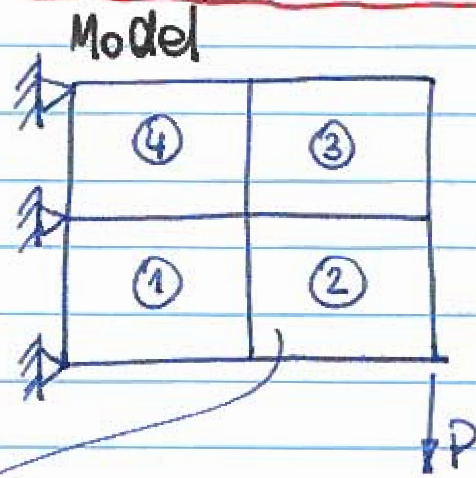
Upon minimization of Π^e with respect to nodal displacements U^e —

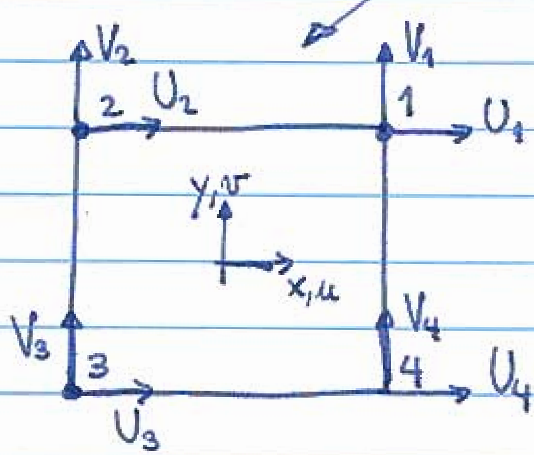
$$\boxed{K^e U^e = R^e}$$

$$3n \times 3n \quad 3n \times 1 \qquad\qquad 3n \times 1$$

- Example: plane stress. linear square element



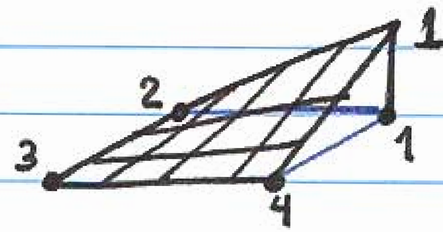




$1 \equiv (1,1)$
$2 \equiv (-1,1)$
$3 \equiv (-1,-1)$
$4 \equiv (1,-1)$

Basis functions:

$$\phi_1 = \frac{1}{4}(1+x)(1+y)$$



$$\phi_2 = \frac{1}{4}(1-x)(1+y)$$

etc.

$$\phi_3 = \frac{1}{4}(1-x)(1-y)$$

$$\phi_4 = \frac{1}{4}(1+x)(1-y)$$

$$u = \begin{Bmatrix} u \\ v \end{Bmatrix} = \underbrace{\begin{bmatrix} \phi_1 & 0 & \phi_2 & 0 & \phi_3 & 0 & \phi_4 & 0 \\ 0 & \phi_1 & 0 & \phi_2 & 0 & \phi_3 & 0 & \phi_4 \end{bmatrix}}_{H} \begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \\ U_4 \\ V_4 \end{Bmatrix}$$

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{Bmatrix} = B \quad U$$

$$B = \begin{bmatrix} \phi_{1,x} & 0 & \phi_{2,x} & 0 & \phi_{3,x} & 0 & \phi_{4,x} & 0 \\ 0 & \phi_{1,y} & 0 & \phi_{2,y} & 0 & \phi_{3,y} & 0 & \phi_{4,y} \\ \phi_{1,y} & \phi_{1,x} & \phi_{2,y} & \phi_{2,x} & \phi_{3,y} & \phi_{3,x} & \phi_{4,y} & \phi_{4,x} \end{bmatrix}$$

$$\phi_{1,x} = \frac{1}{4}(1+y) \qquad \phi_{1,y} = \frac{1}{4}(1+x)$$

$$\phi_{2,x} = -\frac{1}{4}(1+y) \qquad \phi_{2,y} = \frac{1}{4}(1-x)$$

$$\phi_{3,x} = -\frac{1}{4}(1-y) \qquad \phi_{3,y} = -\frac{1}{4}(1-x)$$

$$\phi_{4,x} = \frac{1}{4}(1-y) \qquad \phi_{4,y} = -\frac{1}{4}(1+x)$$

Replace in B, then in K with

$$C = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

See Mathematica file.