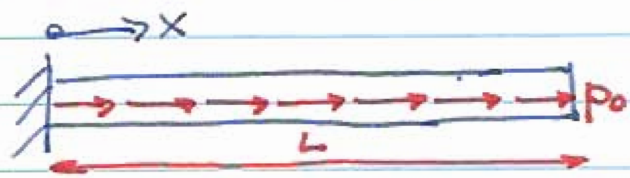
The finite element method IV

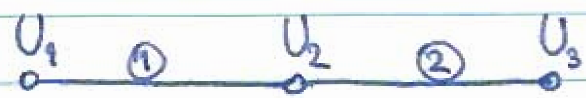
Imposition of boundary conditions (displacement)

We had obtained the assembled finite element equation

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 \\ & K_{22}^1 + K_{11}^2 & K_{12}^2 \\ \text{sym} & & K_{22}^2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix} p_0 \frac{L}{4}$$

for the following problem and model:





The global stiffness matrix is singular, which is an indication that we have not imposed the displacement boundary conditions (the variational approach does not enforce them automatically).

The appropriate boundary condition in this case is:

$$\boxed{U_1 = 0}$$

Replacing in our system of equations:

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 \\ & K_{22}^1 + K_{11}^2 & K_{12}^2 \\ & & K_{22}^2 \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 2 \\ 1 \end{Bmatrix} \frac{p_0 L}{4}$$

Eliminating the first row and column:

$$\begin{bmatrix} K_{22}^1 + K_{11}^2 & K_{12}^2 \\ K_{21}^2 & K_{22}^2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} \frac{p_0 L}{4}$$

where $K_{22}^1 = K_{11}^2 = K_{22}^2 = \frac{2AE}{L}$

$K_{12}^2 = K_{21}^2 = -\frac{2AE}{L}$

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \frac{p_0 L}{4} \frac{L}{2AE} \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$$

$$\rightarrow \boxed{\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \frac{p_0 L^2}{8EA} \begin{Bmatrix} 3 \\ 4 \end{Bmatrix}}$$

Post processing

Reaction at support: replacing solution in first eqn.

$$-\frac{2AE}{L} U_2 = R_1$$

$$-\frac{2\cancel{AE}}{\cancel{L}} \frac{p_o L^{\cancel{2}}}{8\cancel{EA}} 3 = R_1 \quad \text{or} \quad \boxed{R_1 = -\frac{6}{8} p_o L}$$

<u>Note:</u> this is an approximate value. Why? CQ1

The approximate displacements are:

$$u(x) = \begin{cases} \phi_1 U_1^1 + \phi_2 U_2^1 = 0 + \left(\frac{x\,2}{\cancel{L}}\right) \frac{3}{8} \frac{p_o L^{\cancel{2}}}{EA} & e=1 \\ \phi_1 U_1^2 + \phi_2 U_2^2 = \left[\left(1 - \frac{\cancel{x}\,2}{L}\right) 3 + \frac{2\cancel{x}}{L} 4\right] \frac{p_o L^2}{8EA} & e=2 \end{cases}$$

$\left(x - \frac{L}{2}\right)$ $\left(x - L/2\right)$

$$\varepsilon = \frac{du}{dx} = \begin{cases} \frac{3}{4} \frac{p_o L}{EA} & e=1 \quad (x < L/2) \\ \underbrace{\left(\frac{-3\,2}{L} + \frac{8}{L}\right) \frac{p_o L^2}{8EA}}_{\frac{1}{4} \frac{p_o L}{8EA}} & e=2 \quad \frac{L}{2} < x < L \end{cases}$$

$$\sigma = E\varepsilon = \begin{cases} \dfrac{3}{4}\dfrac{p_o L}{A} & e=1 \\ \dfrac{1}{4}\dfrac{p_o L}{A} & e=2 \end{cases}$$

The exact solution for this problem is given by

$$\begin{cases} u(x) = \dfrac{p_o L}{EA}\, x \left(1 - \dfrac{x}{2L}\right) \\ \varepsilon = \dfrac{p_o}{EA}(L-x) \\ \sigma = \dfrac{p_o}{A}(L-x) \end{cases}$$

The exact solution at nodes 2 and 3 is:

$$\boxed{u\left(\frac{L}{2}\right) = \frac{3}{8}\frac{p_o L^2}{EA}}, \quad \boxed{u(L) = \frac{1}{2}\frac{p_o L^2}{EA}}$$

which coincides with the finite element solution !!

Is this true for all values of "x"? CQ2