The finite element method

III

In FEM II we derived finite element equations from PVD:

$$\boxed{\delta W_E^e = \delta W_I^e}$$

and obtained:

$$\boxed{K_{ij}^e \, U_j^e = R_i^e} \qquad i,j = 1,\dots,n$$

where:

n: number of element nodal points

U_j^e: element nodal displacements

$$\boxed{K_{ij}^e = \int_{x_1^e}^{x_n^e} E\,A \,\frac{d\phi_i^e}{dx}\,\frac{d\phi_j^e}{dx}\,dx}$$: element stiffness matrix

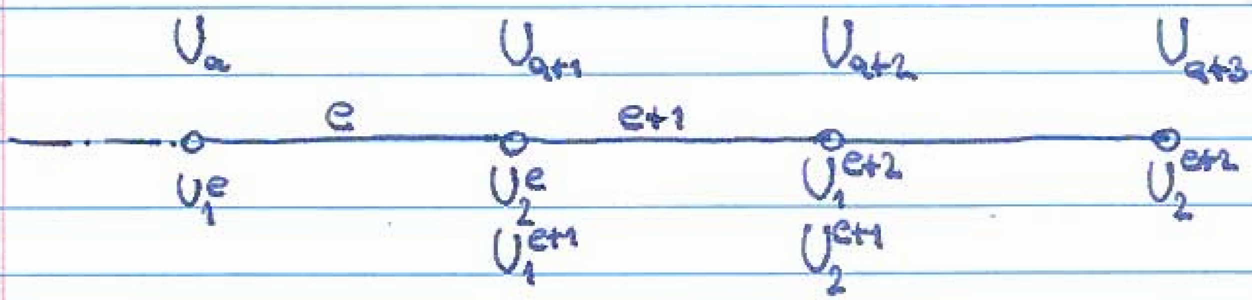
$$\boxed{R_i^e = \int_{x_1^e}^{x_n^e} q(x)\,\phi_i^e\,dx + P_j^e}$$: element nodal forces vector

- Exercise: derive K_{ij}^e, R_i^e for linear and quadratic interpolation for the case of uniform cross section (A), Young's modulus (E) and distributed load ($q(x) = q_0$).

Finite element assembly

- Each finite element is connected to its neighbors at its end nodes.

- As before define a global numbering (identification) for all the nodes



For linear elements:

$$U_1^1 = U_1 \ , \ U_2^1 = U_2 = U_1^2 \ , \ U_2^2 = U_3 = U_1^3 \ \ldots$$

$$\boxed{U_2^e = U_{e+1} = U_1^{e+1}}$$

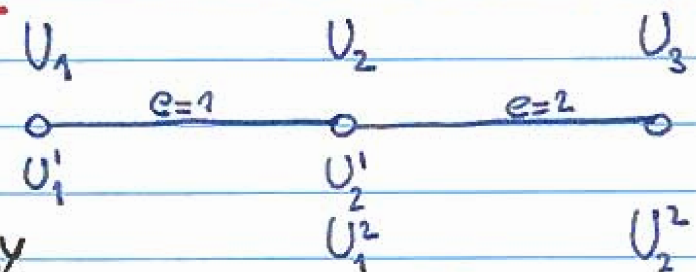
- Concept question: Expression for the local-global nodal mapping for ~~the~~ quadratic element mesh.

The assembly of the element equations is based on the satisfaction of the variational principle for the whole system:

$$\boxed{\delta\Pi = \sum_{i=1}^{n+1} \frac{\partial \Pi}{\partial U_i}\,\delta U_I = 0}$$

where Π is the sum of the element values Π^e over the mesh elements.

Example. Mesh with two linear elements



strain energy

$$\boxed{U^e = \frac{1}{2}\{U^e\}^T[K^e]\{U^e\}}$$ why?

1x1 1xn nxn nx1

$$\varepsilon^e = \frac{U_2^e - U_1^e}{L^e}, \quad \sigma^e = E\varepsilon^e, \quad U_0^e = \frac{1}{2}E(\varepsilon^e)^2$$

$$U^e = A^e L^e U_0^e = \frac{A^e \cancel{L^e}}{2} E \frac{(U_2^e - U_1^e)^2}{L^{e\cancel{2}}} = \frac{A^e E^e}{2L^e}(U_2^e - U_1^e)^2$$

We had obtained K^e as:

$$K^e = \frac{A^e E^e}{L^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\frac{1}{2}\{U^e\}^T[K]\{U^e\} = \frac{A^e E^e}{2L^e}\{U_1^e \; U_2^e\}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}U_1^e\\ U_2^e\end{Bmatrix}$$

$$= \frac{A^e E^e}{2L^e}\{U_1^e - U_2^e \;\; U_2^e - U_1^e\}\begin{Bmatrix}U_1^e\\ U_2^e\end{Bmatrix}$$

$$= \frac{A^e E^e}{2L^e}(U_1^e - U_2^e)^2 = U^e$$

q.e.d.

Replacing local with global numbering:

strain energy $e=1$

$$U^1 = \frac{1}{2}\{U_1^1 \; U_2^1\}\begin{bmatrix}K_{11}^1 & K_{12}^1\\ K_{21}^1 & K_{22}^1\end{bmatrix}\begin{Bmatrix}U_1^1\\ U_2^1\end{Bmatrix}$$

$$= \frac{1}{2}\{U_1 \; U_2 \; U_3\}\begin{bmatrix}K_{11}^1 & K_{12}^1 & 0\\ K_{21}^1 & K_{22}^1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\end{Bmatrix}$$

strain energy $e=2$

$$U^2 = \frac{1}{2}\{U_1^2 \; U_2^2\}\begin{bmatrix}K_{11}^2 & K_{12}^2\\ K_{21}^2 & K_{22}^2\end{bmatrix}\begin{Bmatrix}U_1^2\\ U_2^2\end{Bmatrix}$$

$$= \frac{1}{2}\{U_1 \; U_2 \; U_3\}\begin{bmatrix}0 & 0 & 0\\ 0 & K_{11}^2 & K_{12}^2\\ 0 & K_{21}^2 & K_{22}^2\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\end{Bmatrix}$$

$$U^{TOTAL} = U^1 + U^2$$

$$= \frac{1}{2}\{U_1\ U_2\ U_3\}\begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 \\ sym & K_{22}^1 + K_{11}^2 & K_{12}^2 \\ & & K_{22}^2 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}$$

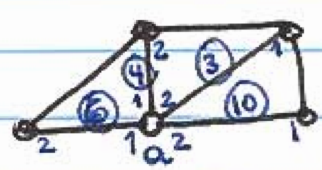
$$\Longrightarrow \boxed{K^{TOTAL} = \sum_{e=1}^{E} K^e}$$

but here the "sum" sign has a special meaning: "assembly" which involves proper placement of the element stiffness matrix coefficients in the global matrix.

Remarks on assembly of global matrix

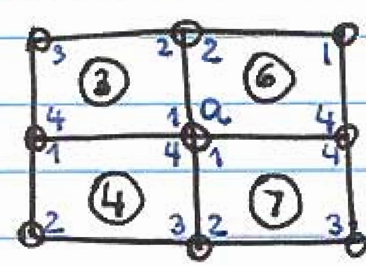
- Diagonal element corresponding to global node "i" gets contributions from all the elements writing to it.

Trusses, beams:



$$K_{aa} = K_{22}^{③} + K_{11}^{④} + K_{11}^{⑥} + K_{22}^{⑩}$$

Plane elasticity



$$\underset{(x\text{-dir})}{K_{aa}} = K_{11}^{⑦} + K_{33}^{⑥} + K_{11}^{③} + K_{44}^{④}$$

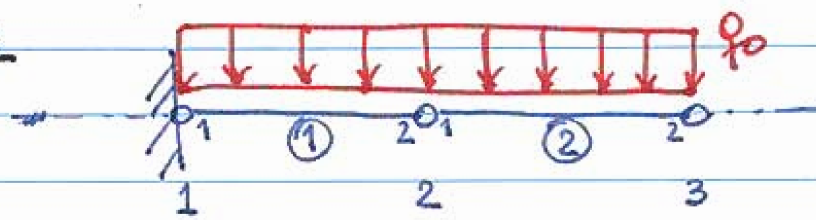
- $K_{ij} = 0$ if node "i" and "j" are in different elements
- K_{ij} preserves the symmetry of the element K_{ij}^e

Assembly of global force vector R_i

Given R_i^e, $e = 1, \ldots, E$

Find: $\boxed{R_i = \sum_{e=1}^{E} R_i^e}$

Example:



$$R^e = \frac{L^e q_0}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$R = R^1 + R^2 = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \frac{L^1 q_0}{2} + \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} \frac{L^2 q_0}{2}$$

$$\boxed{R = \frac{q_0}{2} \begin{Bmatrix} L^1 \\ L^1 + L^2 \\ L^2 \end{Bmatrix}}$$