Unit 6
Plane Stress and Plane Strain

<u>Readings</u>:

T & G 8, 9, 10, 11, 12, 14, 15, 16

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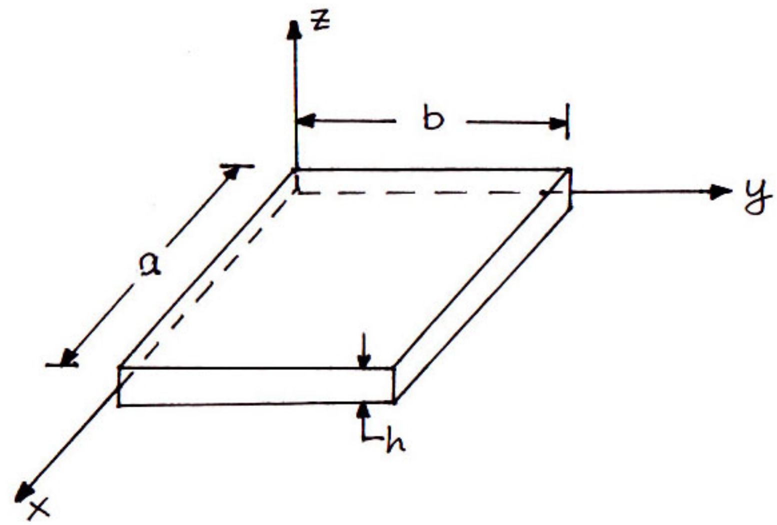
There are many structural configurations where we do not have to deal with the full 3-D case.

- First let's consider the models
- Let's then see under what conditions we can apply them

A. Plane Stress

This deals with stretching and shearing of thin slabs.

Figure 6.1 **Representation of Generic Thin Slab**



The body has dimensions such that

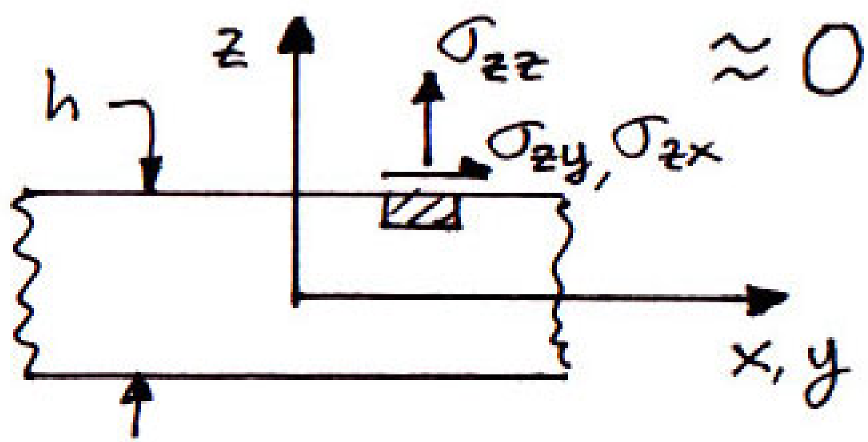
$h << a, b$

(**<u>Key</u>:** where are limits to “<<“??? We’ll consider later)

Thus, the plate is thin enough such that there is no variation of displacement (and temperature) with respect to y_3 (z).

Furthermore, stresses in the z-direction are zero (small order of magnitude).

Figure 6.2 **Representation of Cross-Section of Thin Slab**



Thus, we assume:

$$\sigma_{zz} = 0$$

$$\sigma_{yz} = 0$$

$$\sigma_{xz} = 0$$

$$\frac{\partial}{\partial z} = 0$$

So the equations of elasticity reduce to:

Equilibrium

$$\frac{\partial \sigma_{11}}{\partial y_1} + \frac{\partial \sigma_{21}}{\partial y_2} + f_1 = 0 \qquad (1)$$

$$\frac{\partial \sigma_{12}}{\partial y_1} + \frac{\partial \sigma_{22}}{\partial y_2} + f_2 = 0 \qquad (2)$$

(3rd equation is an identity) $0 = 0$ ($f_3 = 0$)

In general: $$\frac{\partial \sigma_{\beta\alpha}}{\partial y_\beta} + f_\alpha = 0$$

Stress-Strain (fully anisotropic)

Primary (in-plane) strains

$$\varepsilon_1 = \frac{1}{E_1}\left[\sigma_1 - \nu_{12}\sigma_2 - \eta_{16}\sigma_6\right] \qquad (3)$$

$$\varepsilon_2 = \frac{1}{E_2}\left[-\nu_{21}\sigma_1 + \sigma_2 - \eta_{26}\sigma_6\right] \qquad (4)$$

$$\varepsilon_6 = \frac{1}{G_6}\left[-\eta_{61}\sigma_1 - \eta_{62}\sigma_2 + \sigma_6\right] \qquad (5)$$

Invert to get:

$$\sigma_{\alpha\beta} = E^*_{\alpha\beta\sigma\gamma}\,\varepsilon_{\sigma\gamma}$$

Secondary (out-of-plane) strains

⇒ (they exist, but they are not a primary part of the problem)

$$\varepsilon_3 = \frac{1}{E_3}\left[-\nu_{31}\sigma_1 - \nu_{32}\sigma_2 - \eta_{36}\sigma_6\right]$$

$$\varepsilon_4 = \frac{1}{G_4}\left[-\eta_{41}\sigma_1 - \eta_{42}\sigma_2 - \eta_{46}\sigma_6\right]$$

$$\varepsilon_5 = \frac{1}{G_5}\left[-\eta_{51}\sigma_1 - \eta_{52}\sigma_2 - \eta_{56}\sigma_6\right]$$

Note: can reduce these for orthotropic, isotropic (etc.) as before.

Strain - Displacement

Primary

$$\varepsilon_{11} = \frac{\partial u_1}{\partial y_1} \qquad (6)$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial y_2} \qquad (7)$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial y_2} + \frac{\partial u_2}{\partial y_1}\right) \qquad (8)$$

Secondary

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial y_3} + \frac{\partial u_3}{\partial y_1}\right)$$

$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial y_3} + \frac{\partial u_3}{\partial y_2}\right)$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial y_3}$$

Note: that for an orthotropic material

$$\underset{(\varepsilon_{23})}{\varepsilon_4} = \underset{(\varepsilon_{13})}{\varepsilon_5} = 0$$ (due to stress-strain relations)

This further implies from above

(since $\frac{\partial}{\partial y_3} = 0$)

No in-plane variation

$$\frac{\partial u_3}{\partial y_\alpha} = 0$$

but this is not exactly true

⇒ **INCONSISTENCY**

Why? This is an idealized model and thus an approximation. There are, in actuality, triaxial (σ_{zz}, etc.) stresses that we ignore here as being small relative to the in-plane stresses!

(we will return to try to define "small")

Final note: for an orthotropic material, write the tensorial stress-strain equation as:

2-D plane stress

$$\sigma_{\alpha\beta} = E^*_{\alpha\beta\sigma\gamma} \, \varepsilon_{\sigma\gamma} \qquad (\alpha, \beta, \sigma, \gamma = 1, 2)$$

There is not a 1-to-1 correspondence between the 3-D E_{mnpq} and the 2-D $E^*_{\alpha\beta\sigma\gamma}$. The effect of ε_{33} must be incorporated since ε_{33} does not appear in these equations by using the ($\sigma_{33} = 0$) equation.

This gives:

$$\varepsilon_{33} = f(\varepsilon_{\alpha\beta})$$

Also, particularly in composites, another "notation" will be used in the case of plane stress in place of engineering notation:

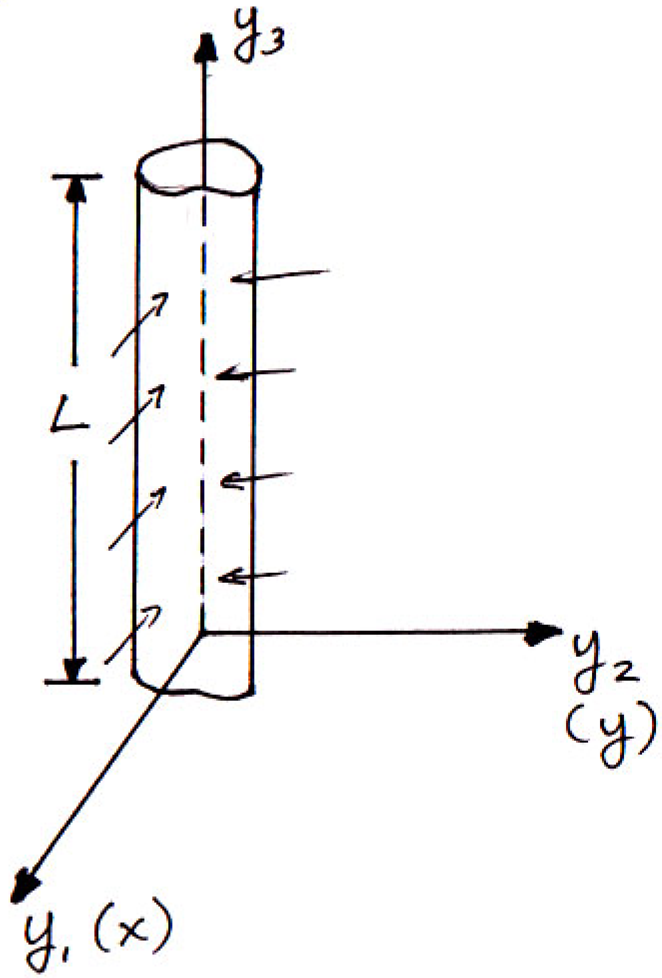
subscript change { x = 1 = L (longitudinal)…along major axis
y = 2 = T (transverse)…along minor axis

The other important "extreme" model is…

B. Plane Strain

This deals with long prismatic bodies:

Figure 6.3 Representation of Long Prismatic Body



Dimension in z - direction is much, much larger than in the x and y directions

L >> x, y

(**Key** again: where are limits to “>>”??? … we’ll consider later)

Since the body is basically “infinite” along z, the important loads are in the x - y plane (none in z) and do not change with z:

$$\frac{\partial}{\partial y_3} = \frac{\partial}{\partial z} = 0$$

This implies there is no gradient in displacement along z, so (excluding rigid body movement):

$$u_3 = w = 0$$

Equations of elasticity become:

Equilibrium:

Primary

$$\frac{\partial \sigma_{11}}{\partial y_1} + \frac{\partial \sigma_{21}}{\partial y_2} + f_1 = 0 \qquad (1)$$

$$\frac{\partial \sigma_{12}}{\partial y_1} + \frac{\partial \sigma_{22}}{\partial y_2} + f_2 = 0 \qquad (2)$$

Secondary

$$\frac{\partial \sigma_{13}}{\partial y_1} + \frac{\partial \sigma_{23}}{\partial y_2} + f_3 = 0$$

σ_{13} and σ_{23} exist but do not enter into <u>primary</u> consideration

<u>Strain - Displacement</u>

$$\varepsilon_{11} = \frac{\partial u_1}{\partial y_1} \quad (3)$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial y_2} \quad (4)$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial y_2} + \frac{\partial u_2}{\partial y_1}\right) \quad (5)$$

Assumptions $\left(\frac{\partial}{\partial y_3} = 0, \; w = 0\right)$ give:

$$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$$

(<u>Plane</u> <u>strain</u>)

Stress - Strain

(Do a similar procedure as in plane stress)

3 Primary

σ_{11} = ... (6)

σ_{22} = ... (7)

σ_{12} = ... (8)

Secondary

$\sigma_{13} = 0$ } orthotropic (≠ 0 for anisotropic)

$\sigma_{23} = 0$

$\sigma_{33} \neq 0$

INCONSISTENCY: No load along z, yet σ_{33} (σ_{zz}) is non zero.

Why? Once again, this is an idealization. Triaxial strains (ε_{33}) actually arise.

You eliminate σ_{33} from the equation set by expressing it in terms of $\sigma_{\alpha\beta}$ via (σ_{33}) stress-strain equation.

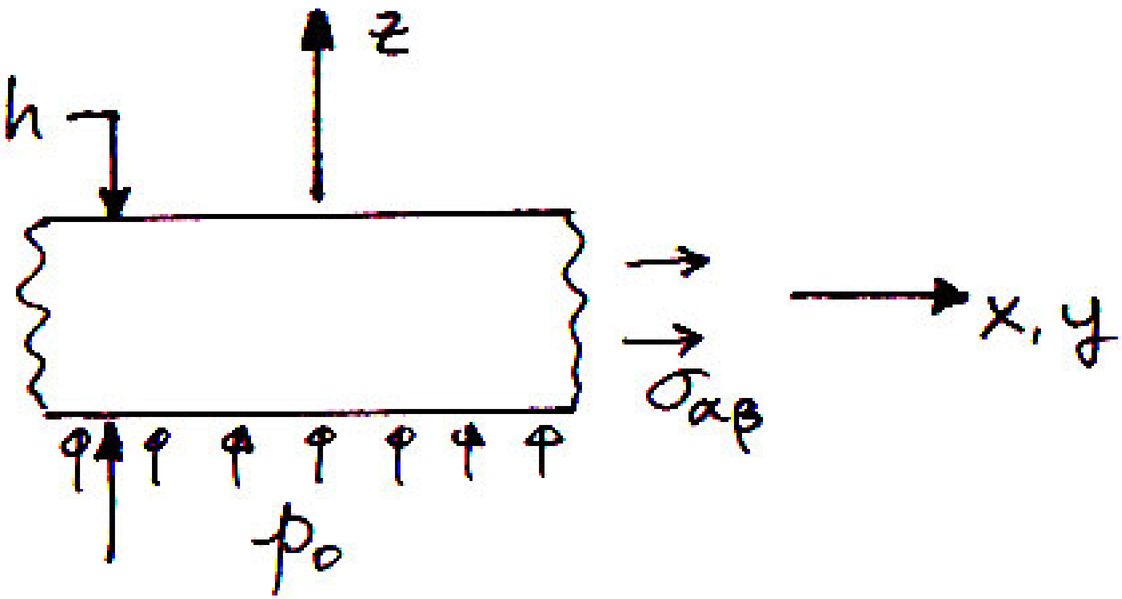
SUMMARY

	Plane Stress	Plane Strain
Geometry:	thickness (y_3) << in-plane dimensions (y_1, y_2)	length (y_3) >> in-plane dimensions (y_1, y_2)
Loading:	$\sigma_{33} << \sigma_{\alpha\beta}$	$\sigma_{\alpha\beta}$ only $\partial / \partial y_3 = 0$
Resulting Assumptions:	$\sigma_{i3} = 0$	$\varepsilon_{i3} = 0$
Primary Variables:	$\varepsilon_{\alpha\beta}$, $\sigma_{\alpha\beta}$, u_α	$\varepsilon_{\alpha\beta}$, $\sigma_{\alpha\beta}$, u_α
Secondary Variable(s):	ε_{33}, u_3	σ_{33}
Note:	Eliminate ε_{33} from eq. set by using $\sigma_{33} = 0$ σ - ε eq. and expressing ε_{33} in terms of $\varepsilon_{\alpha\beta}$	Eliminate σ_{33} from eq. Set by using σ_{33} σ - ε eq. and expressing σ_{33} in terms of $\varepsilon_{\alpha\beta}$

Examples

Plane Stress:

Figure 6.4 **Pressure vessel (fuselage, space habitat) Skin**



in order of 70 MPa (10 ksi)

$p_o \approx$ 70 kPa (~ 10 psi for living environment)

$\Rightarrow \sigma_{zz} << \sigma_{xx}, \sigma_{yy}, \sigma_{xy}$

Plane Strain:

Figure 6.5 **Dams**

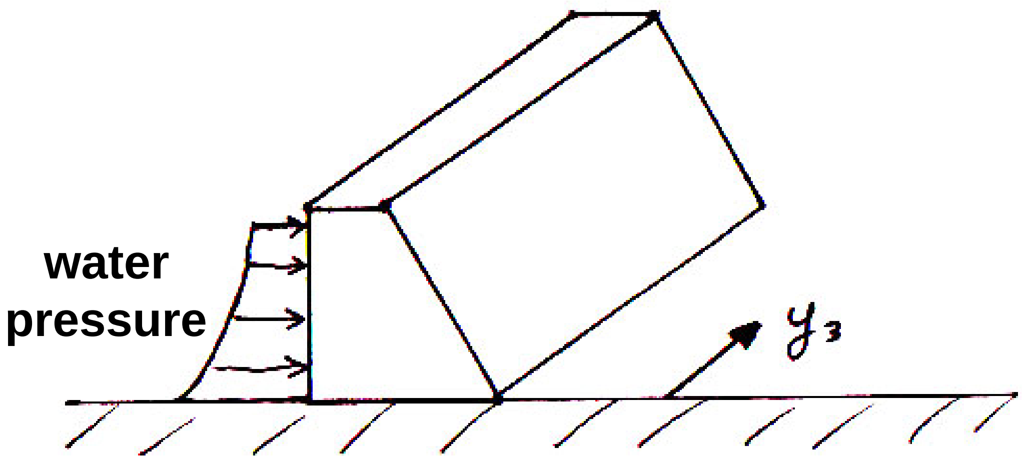
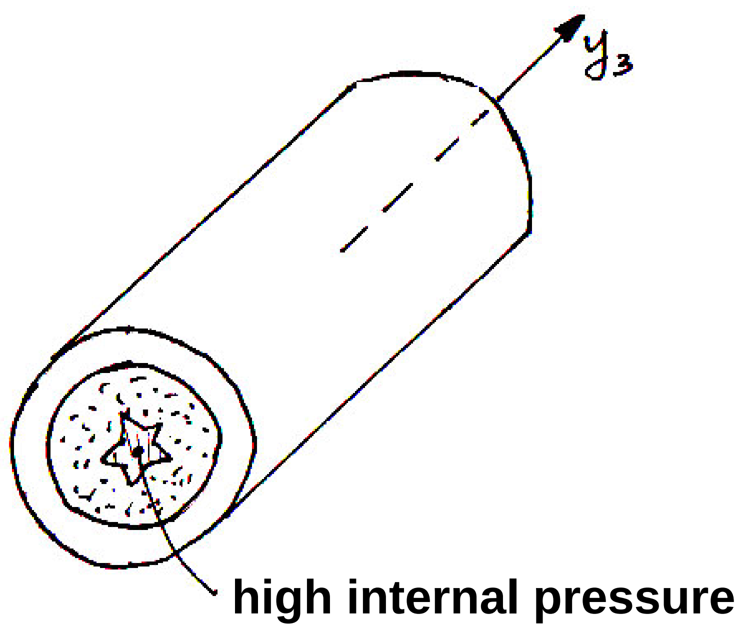


Figure 6.6 **Solid Propellant Rockets**



but…when do these apply???

Depends on…

- loading
- geometry
- material and its response
- issues of scale
- how "good" do I need the answer
- what are we looking for (deflection, failure, etc.)

We've talked about the first two, let's look a little at each of the last three:

--> Material and its response

- Elastic response and coupling changes importance / magnitude of "primary" / "secondary" factors

(Key: are "primary" dominating the response?)

--> Issues of scale

- What am I using the answer for? at what level?
- Example: standing on table

--overall deflection or reactions in legs are not dependent on way I stand (tip toe or flat foot)

⇒ **model of top of table as plate in bending is sufficient**

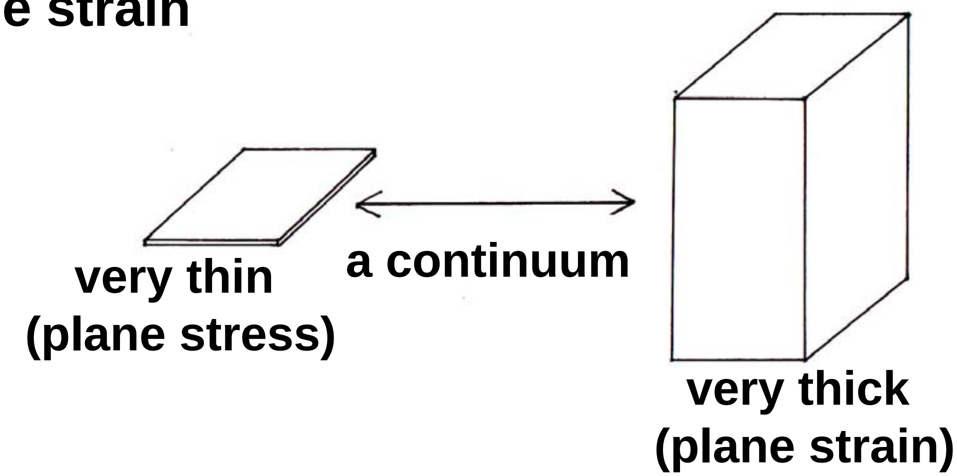
--stresses under my foot very sensitive to specifics

(if table top is foam, the way I stand will determine whether or not I crush the foam)

--> How "good" do I need the answer?

- In preliminary design, need "ballpark" estimate; in final design, need "exact" numbers
- Example: as thickness increases when is a plate no longer in plane stress

Figure 6.7 **Representation of the "continuum" from plane stress to plane strain**



No line(s) of demarkation. Numbers approach idealizations but never get to it.

Must use engineering judgment

AND

Clearly identify key assumptions in model and resulting limitations