Solutions to Home Assignment #3

Warm-Up Exercises

* Note : In problems 1 ~ 3 we are dealing with plane stress.
We know that ε_{33} is a principal strain, but it does not enter in the rotation of in-plane strains.

1. We want to show geometrically that the transformation via Mohr's circle is the same as the tensor transformation:

$$\hat{\varepsilon}_{\alpha\beta} = l_{\alpha\sigma} l_{\beta\gamma} \varepsilon_{\sigma\gamma} \qquad \text{—— } ①$$

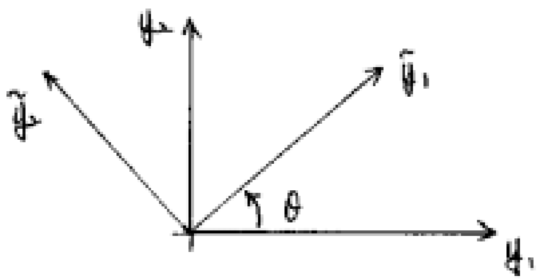
This can be written out as,

$$\hat{\varepsilon}_{11} = l_{\tilde{1}1} l_{\tilde{1}1} \varepsilon_{11} + l_{\tilde{1}2} l_{\tilde{1}2} \varepsilon_{22} + 2 l_{\tilde{1}1} l_{\tilde{1}2} \varepsilon_{12} \qquad \text{—— } ②$$

$$\hat{\varepsilon}_{22} = l_{\tilde{2}1} l_{\tilde{2}1} \varepsilon_{11} + l_{\tilde{2}2} l_{\tilde{2}2} \varepsilon_{22} + 2 l_{\tilde{2}1} l_{\tilde{2}2} \varepsilon_{12} \qquad \text{—— } ③$$

$$\hat{\varepsilon}_{12} = l_{\tilde{1}1} l_{\tilde{2}1} \varepsilon_{11} + l_{\tilde{1}2} l_{\tilde{2}2} \varepsilon_{22} + (l_{\tilde{1}1} l_{\tilde{2}2} + l_{\tilde{1}2} l_{\tilde{2}1}) \varepsilon_{12} \qquad \text{—— } ④$$

Note that the axes y_1-y_2 and $\tilde{y}_1$-$\tilde{y}_2$ are related by θ as:



y_1-y_2 = original axes
$\tilde{y}_1$-$\tilde{y}_2$ = rotated axes.

Using the axes system shown above, the direction cosines, l_{mn} can be determined

* Note : l_{mn} = cosine of angle from $\bar{y}_m$ to y_n

$$l_{\bar{1}1} = \cos\theta \quad \text{--- (5)}$$

$$l_{\bar{1}2} = \cos(90-\theta) = \sin\theta \quad \text{--- (6)}$$

$$l_{\bar{2}1} = \cos(90+\theta) = -\sin\theta \quad \text{--- (7)}$$

$$l_{\bar{2}2} = \cos\theta \quad \text{--- (8)}$$

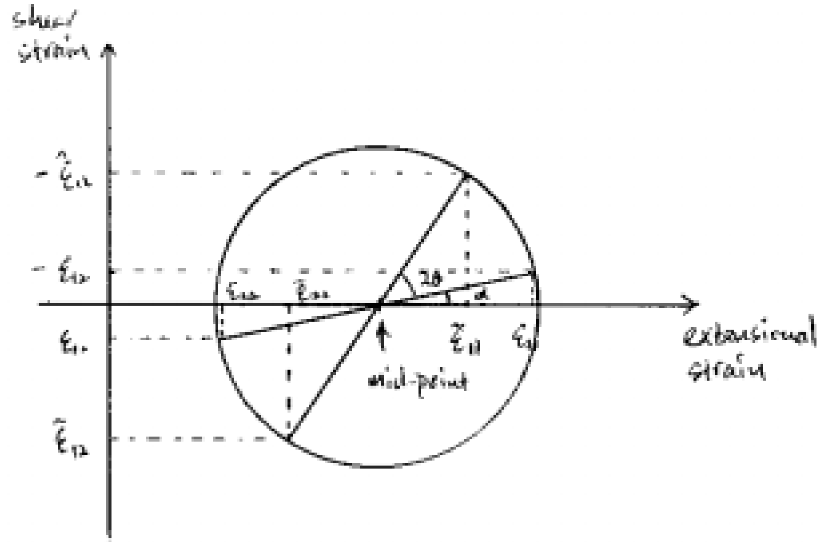
Substituting equations (5) through (8) into equations (2) through (4), we get.

$$\hat{\varepsilon}_{11} = \cos^2\theta\,\varepsilon_{11} + \sin^2\theta\,\varepsilon_{22} + 2\sin\theta\cos\theta\,\varepsilon_{12} \quad \text{--- (9)}$$

$$\hat{\varepsilon}_{22} = \sin^2\theta\,\varepsilon_{11} + \cos^2\theta\,\varepsilon_{22} - 2\sin\theta\cos\theta\,\varepsilon_{12} \quad \text{--- (10)}$$

$$\tilde{\varepsilon}_{12} = -\cos\theta\sin\theta\,\varepsilon_{11} + \sin\theta\cos\theta\,\varepsilon_{22} + (\cos^2\theta - \sin^2\theta)\varepsilon_{12} \quad \text{--- (11)}$$

Now, let's draw the Mohr's circle. We have been given the strain state ε_{11}, ε_{22} and ε_{12}, and we wish to find the strain state in the rotated system at angle θ, $\tilde{\varepsilon}_{11}$, $\tilde{\varepsilon}_{22}$ and $\tilde{\varepsilon}_{12}$.

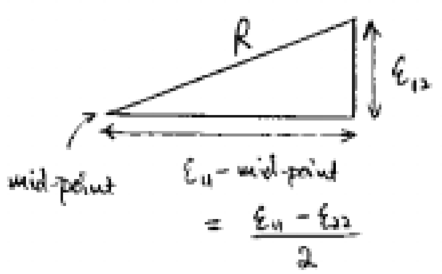


We will denote the angle that the original line makes with the horizontal as α.

The mid-point of the circle can be obtained from geometrical consideration. It is,

$$\text{mid-point} = \frac{\varepsilon_{11} + \varepsilon_{22}}{2} \qquad \text{------} \quad (11)$$

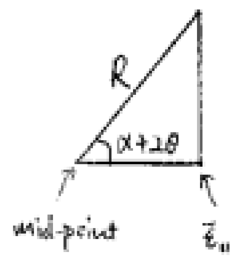
The radius of the circle can also be obtained from the geometry. It is,



$$R = \sqrt{\varepsilon_{12}^2 + \left(\frac{\varepsilon_{11} - \varepsilon_{22}}{2}\right)^2} \qquad \text{---} \quad (12)$$

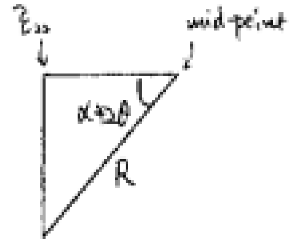
Now consider each of the strain components.

$\tilde{\varepsilon}_{11}$:



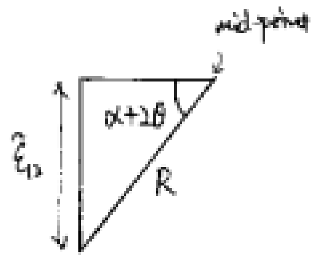
$$\tilde{\varepsilon}_{11} = \frac{\varepsilon_{11} + \varepsilon_{22}}{2} + R\cos(\alpha + 2\theta) \quad \text{—— (13)}$$

$\tilde{\varepsilon}_{22}$:



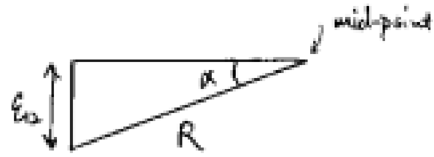
$$\tilde{\varepsilon}_{22} = \frac{\varepsilon_{11} + \varepsilon_{22}}{2} - R\cos(\alpha + 2\theta) \quad \text{—— (14)}$$

$\tilde{\varepsilon}_{12}$:

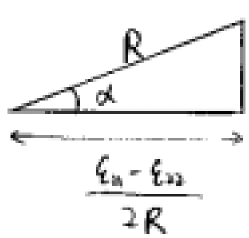


$$\tilde{\varepsilon}_{12} = -R\sin(\alpha + 2\theta) \quad \text{—— (15)}$$

Note that from the Mohr's circle,



$$\sin\alpha = -\frac{\varepsilon_{12}}{R} \quad \text{—— (16)}$$



$$\cos\alpha = \frac{\varepsilon_{11} - \varepsilon_{22}}{2R} \quad \text{—— (17)}$$

In order to expand equations ⑬ through ⑮ out, we use the following trigonometric identities.

$$\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_1 + \cos\theta_1 \sin\theta_1 \quad \text{---} \ ⑯$$

$$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \quad \text{---} \ ⑰$$

This gives us:

$$⑬ : \quad \tilde{\varepsilon}_{11} = \frac{\varepsilon_{11} + \varepsilon_{22}}{2} + R\cos\alpha\cos 2\theta - R\sin\alpha\sin 2\theta \quad \text{---} \ ⑳$$

$$⑭ : \quad \tilde{\varepsilon}_{22} = \frac{\varepsilon_{11} + \varepsilon_{22}}{2} - R\cos\alpha\cos 2\theta - R\sin\alpha\sin 2\theta \quad \text{---} \ ㉑$$

$$⑮ : \quad \tilde{\varepsilon}_{12} = -R\sin\alpha\cos 2\theta - R\cos\alpha\sin 2\theta \quad \text{---} \ ㉒$$

Next, plugging equations ⑯ and ⑰ into equations ⑳ through ㉒, we get:

$$\tilde{\varepsilon}_{11} = \frac{\varepsilon_{11} + \varepsilon_{22}}{2} + R\left(\frac{\varepsilon_{11} - \varepsilon_{22}}{2R}\right)\cos 2\theta - R\left(-\frac{\varepsilon_{12}}{R}\right)\sin 2\theta$$

$$\Rightarrow \tilde{\varepsilon}_{11} = \frac{\varepsilon_{11} + \varepsilon_{22}}{2} + \frac{\varepsilon_{11} - \varepsilon_{22}}{2}\cos 2\theta + \varepsilon_{12}\sin 2\theta \quad \text{---} \ ㉓$$

$$\tilde{\varepsilon}_{22} = \frac{\varepsilon_{11} + \varepsilon_{22}}{2} - R\left(\frac{\varepsilon_{11} - \varepsilon_{22}}{2R}\right)\cos 2\theta + R\left(-\frac{\varepsilon_{12}}{R}\right)\sin 2\theta$$

$$\Rightarrow \tilde{\varepsilon}_{22} = \frac{\varepsilon_{11} + \varepsilon_{22}}{R} - \frac{\varepsilon_{11} - \varepsilon_{22}}{2}\cos 2\theta - \varepsilon_{12}\sin 2\theta \quad \text{---} \ ㉔$$

$$\hat{\varepsilon}_{12} = -R\left(-\frac{\varepsilon_{12}}{R}\right)\cos 2\theta - R\left(\frac{\varepsilon_{11}-\varepsilon_{22}}{2R}\right)\sin 2\theta$$

$$\Rightarrow \hat{\varepsilon}_{12} = \varepsilon_{12}\cos 2\theta - \frac{\varepsilon_{11}-\varepsilon_{22}}{2}\sin 2\theta \quad \text{——— } 25$$

Now we make use of the "half-angle" formulae:

$$\sin 2\theta = 2\sin\theta\cos\theta \quad \text{——— } 26$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \quad \text{——— } 27$$

Plugging into equations 23 to 24, we get:

23 :
$$\bar{\varepsilon}_{11} = \frac{\varepsilon_{11}+\varepsilon_{22}}{2} + \frac{\varepsilon_{11}-\varepsilon_{22}}{2}(\cos^2\theta - \sin^2\theta) + 2\varepsilon_{12}\sin\theta\cos\theta$$

$$\Rightarrow \bar{\varepsilon}_{11} = \frac{\varepsilon_{11}}{2}(1+\cos^2\theta - \sin^2\theta) + \frac{\varepsilon_{22}}{2}(1-\cos^2\theta + \sin^2\theta) + 2\varepsilon_{12}\sin\theta\cos\theta$$

* $1-\sin^2\theta = \cos^2\theta$
$1-\cos^2\theta = \sin^2\theta$

$$\Rightarrow \hat{\varepsilon}_{11} = \frac{\varepsilon_{11}}{2}(\cos^2\theta + \cos^2\theta) + \frac{\varepsilon_{22}}{2}(\sin^2\theta + \sin^2\theta) + 2\varepsilon_{12}\sin\theta\cos\theta$$

$$\therefore \boxed{\hat{\varepsilon}_{11} = \cos^2\theta\,\varepsilon_{11} + \sin^2\theta\,\varepsilon_{22} + 2\sin\theta\cos\theta\,\varepsilon_{12}}$$

This is the same as equation 9.

24 :
$$\hat{\varepsilon}_{22} = \frac{\varepsilon_{11}+\varepsilon_{22}}{2} - \frac{\varepsilon_{11}-\varepsilon_{22}}{2}(\cos^2\theta - \sin^2\theta) - 2\varepsilon_{12}\sin\theta\cos\theta$$

$$\Rightarrow \hat{\varepsilon}_{11} = \frac{\varepsilon_{11}}{2}(\underbrace{1-\cos^2\theta}_{\sin^2\theta} + \sin^2\theta) + \frac{\varepsilon_{22}}{2}(\underbrace{1-\sin^2\theta}_{\cos^2\theta} + \cos^2\theta) - 2\varepsilon_{12}\sin\theta\cos\theta$$

$$\therefore \boxed{\hat{\varepsilon}_{22} = \sin^2\theta\,\varepsilon_{11} + \cos^2\theta\,\varepsilon_{22} - 2\sin\theta\cos\theta\,\varepsilon_{12}}$$

This is the same as equation 10

(15) : $\hat{\varepsilon}_{12} = -\frac{\varepsilon_{11}-\varepsilon_{22}}{2}\, 2\sin\theta\cos\theta + \varepsilon_{12}(\cos^2\theta - \sin^2\theta)$

$$\Rightarrow \boxed{\hat{\varepsilon}_{12} = -\sin\theta\cos\theta\,\varepsilon_{11} + \sin\theta\cos\theta\,\varepsilon_{22} + (\cos^2\theta - \sin^2\theta)\,\varepsilon_{12}}$$

This is the same as equation (11)

2. In problem 1, we found the radius to be (equation (12))

$$R = \sqrt{\varepsilon_{12}^2 + \left(\frac{\varepsilon_{11}-\varepsilon_{22}}{2}\right)^2} \qquad \text{—— (12)}$$

The diameter is,

$$d = 2R = \sqrt{4\varepsilon_{12}^2 + (\varepsilon_{11}-\varepsilon_{22})^2}$$

This diameter does not change, so $4\varepsilon_{12}^2 + (\varepsilon_{11}-\varepsilon_{22})^2$ is invariant.

* Note : the angle, α, does not enter in because d^2 is invariant.
$\rightarrow$ not a function of α.

3. Circle diameter doesn't have any physical significance. Another invariant is the mid-point,

$$\frac{\varepsilon_{11}+\varepsilon_{22}}{2}$$

This also does not have any physical significance, but we do note a useful fact:

The sum of ε_{11} and ε_{22} in any coordinate system does NOT change. This gives us a quick way to check that we've done a transformation properly.

IMPORTANT