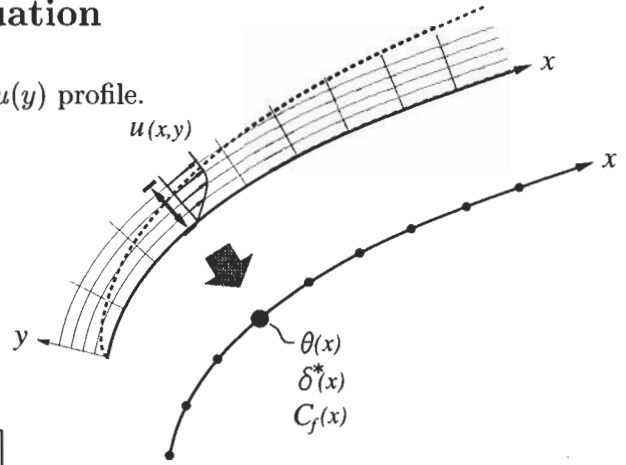


# Von-Karman Integral Momentum Equation

**Idea:** Integrate BL flow in  $y$ , to “wash out” details in  $u(y)$  profile.  
Converts PDEs in  $x, y$  into ODE in  $x$ .



**Combine** Continuity,  $x$ -momentum equations:

$$(u - u_e) \left[ \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \right]$$

$$+ \left[ \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \rho_e u_e \frac{du_e}{dx} + \frac{\partial \tau}{\partial y} \right]$$

$$\Rightarrow \rho u \frac{\partial u}{\partial x} + (u - u_e) \frac{\partial \rho u}{\partial x} + \rho v \frac{\partial u}{\partial y} + (u - u_e) \frac{\partial \rho v}{\partial y} = \rho_e u_e \frac{du_e}{dx} + \frac{\partial \tau}{\partial y}$$

$$\text{or } \frac{\partial}{\partial x} [(u_e - u)\rho u] + \frac{\partial}{\partial y} [(u_e - u)\rho v] = -(\rho_e u_e - \rho u) \frac{du_e}{dx} - \frac{\partial \tau}{\partial y} \quad (*)$$

**Integrate**  $\int_0^{y_e} (*) dy$  term by term:

$$\int_0^{y_e} \frac{\partial}{\partial x} [(u_e - u)\rho u] dy + \int_0^{y_e} \frac{\partial}{\partial y} [(u_e - u)\rho v] dy = - \int_0^{y_e} (\rho_e u_e - \rho u) \frac{du_e}{dx} dy - \int_0^{y_e} \frac{\partial \tau}{\partial y} dy$$

$$\frac{d}{dx} \int_0^{y_e} [(u_e - u)\rho u] dy + 0 = - \frac{du_e}{dx} \int_0^{y_e} (\rho_e u_e - \rho u) dy + \tau_w$$

$\frac{d}{dx} (\rho_e u_e^2 \theta) + \rho_e u_e \delta^* \frac{du_e}{dx} = \tau_w$	Dimensional form
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$\frac{d\theta}{dx} + (H + 2 - M_e^2) \frac{\theta}{u_e} \frac{du_e}{dx} = \frac{C_f}{2}$	Dimensionless form
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## Definitions

$\theta = \int \left(1 - \frac{u}{u_e}\right) \frac{\rho u}{\rho_e u_e} dy$	momentum thickness
$\delta^* = \int \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy$	displacement thickness
$H = \frac{\delta^*}{\theta}$	shape parameter
$C_f = \frac{\tau_w}{\frac{1}{2} \rho_e u_e^2}$	skin friction coefficient