Lecture 5

2.3) A) N-S Equations (units, scales)

B) Physical parameters (Non-dimensional forms)

C) Dynamic Similarity — Inferences from N-S Equs.

Reading: Bat. 164 – 173 White: 81 – 94.

Sch. 15 – 23. Kuethe & Chow: 461 – 462

A) N-S Equations

Continuity: $D\rho/Dt + \rho \nabla\cdot\vec{u} = 0$

Momentum $\rho \dfrac{D\vec{u}}{Dt} = \rho\vec{f} - \nabla p + \dfrac{\partial}{\partial x_j}\left(\mu\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\,\lambda\,\nabla\cdot\vec{u}\right)$

* Energy.

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \nabla\cdot(k\,\nabla T) + \Phi \qquad \text{(White 69–72)}$$

temperature (T), dissipation function (Φ)

enthalpy → $h = e + p/\rho$

A1) Units and scales

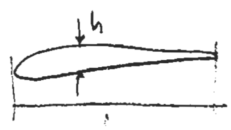
A <u>unit</u> is a known reference quantity

Standard unit : m, kg, sec. (fixed scale)

Natural Unit : L, $\rho_\infty L^3$, L/V_∞ (adjustable scale)

<u>Example</u> : steady, incomp, inviscid flow

V_∞, ρ_∞



$\left(h/L = (t/c) = AR \right)$

B) Non-Dimensionalization: changing for standard to natural units

For steady, incompressible, inviscid flow

	std		Nat
Length	m	→	L
Mass	kg	→	$\rho_\infty L^3$
Time	sec	→	L/V_∞

If we analyse steady, compressible, viscous flow, then a_∞ (speed of sound) and dynamic viscosity μ_∞ are added so that

$$\text{Length} \quad L \quad \text{or} \quad \frac{\mu_\infty}{\rho_\infty U_\infty} \;\; {}^{/\,LV} \qquad \text{and} \qquad \text{Time} \quad \frac{L}{U_\infty} \quad \text{or} \quad L/a_\infty$$

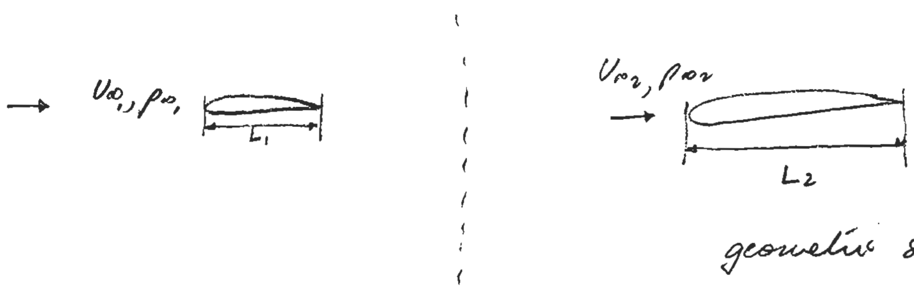
(alternate scales)

can be used. There are redundant scales for length & time. The ratio of such alternate scales gives non-dimensional parameters

$$(\text{Length}) \quad \frac{L}{\dfrac{\mu_\infty}{\rho_\infty U_\infty}} = \underbrace{Re_L}_{\text{Reynolds \#}} \qquad \text{and } (\text{Time}) \quad \frac{L/a_\infty}{L/U_\infty} = \underbrace{M_\infty}_{\text{Mach \#}}$$

c) <u>Dynamic Similarity</u>

Suppose we have two airfoils A & B



geometric similarity

A & B are <u>dynamically</u> similar if they are identical in natural units. ⇒ significant non-dimensional parameters are the same (Re_L, M_∞, etc.)

Non-Dimensionalization of N-S equations

Reference Quantities: L_{ref}, U_{ref}, ρ_{ref}, μ_{ref}

$$x_i^* = \frac{x_i}{L_{ref}}, \quad t^* = \frac{t}{L_{ref}/U_{ref}}, \quad u_i^* = \frac{u_i}{U_{ref}}, \quad \rho^* = \rho/\rho_{ref}$$

$$\mu^* = \mu/\mu_{ref}$$

Convective derivative $\quad \frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{u}\cdot\vec{\nabla})$

$$\Rightarrow \quad \frac{D}{Dt} = \frac{U_{ref}}{L_{ref}}\frac{\partial}{\partial t^*} + \frac{U_{ref}}{L_{ref}} \cdot u_i^* \frac{\partial}{\partial x_i^*}$$

$$= \frac{U_{ref}}{L_{ref}}\left(\frac{\partial}{\partial t^*} + u_i^* \frac{\partial}{\partial x_i^*}\right) = \frac{U_{ref}}{L_{ref}}\frac{D}{Dt^*}$$

1) Conservation of Mass.

$$\frac{D\rho}{Dt} + \rho\frac{\partial u_i}{\partial x_i} = 0$$

$$\Rightarrow \quad \rho_{ref} \cdot \frac{U_{ref}}{L_{ref}} \cdot \frac{D\rho^*}{Dt^*} + \rho_{ref}\frac{U_{ref}}{L_{ref}} \cdot \rho^* \frac{\partial u_i^*}{\partial x_i^*} = 0$$

$$\Rightarrow \quad \frac{D\rho^*}{Dt^*} + \rho^*\frac{\partial u_i^*}{\partial x_i^*} = 0$$

For 2D steady incompressible flows

$$\frac{\partial u_1^*}{\partial x_1^*} + \frac{\partial u_2^*}{\partial x_2^*} = 0 \quad \left(= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)$$

2) Conservation of Momentum

$$\rho \frac{D\vec{u}}{Dt} = \rho\vec{f} + \vec{\nabla}\cdot\overline{\overline{\sigma}} = \rho\vec{f} - \nabla p + \nabla\cdot\overline{\overline{\tau}}$$

Assume $\vec{f} = 0$, incompressible flow $\nabla\cdot\overline{\overline{\tau}} = \mu \nabla^2 \vec{u}$
μ = constant

$$\rho \frac{Du_i}{Dt} = \mu \frac{\partial^2 u_i}{\partial x_i \partial x_j} - \frac{\partial p}{\partial x_i}$$

$$P_{ref} = \rho_{ref} V_{ref}^2$$

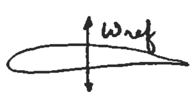
$$p^* = p / P_{ref}$$

$$\Rightarrow \quad \rho_{ref} \frac{V_{ref}^2}{L_{ref}} \cdot \rho^* \frac{Du_i^*}{Dt^*} = \mu_{ref} \frac{U_{ref}}{L_{ref}^2} \mu^* \frac{\partial^2 u_i^*}{\partial x_i^* \partial x_j^*} - \frac{\rho_{ref} U_{ref}^2}{L_{ref}} \frac{\partial p^*}{\partial x_i^*}$$

$$\rho^* \frac{Du_i^*}{Dt^*} = \left(\frac{1}{\underline{\underline{Re_{ref}}}}\right) \mu^* \frac{\partial^2 u_i^*}{\partial x_i^* \partial x_j^*} - \frac{\partial p^*}{\partial x_i^*}$$

$$\frac{Du^*}{Dt^*} = \frac{1}{Re} \frac{\partial^2 u_i^*}{\partial x_i^* \partial x_j^*} - \frac{\partial p^*}{\partial x_i^*}$$

- unsteady, incompressible viscous flow



L_{ref}, V_{ref}, ρ_{ref}, μ_{ref}, ω_{ref}

$$t_{ref} = \frac{L_{ref}}{V_{ref}} \quad \text{or} \quad \frac{1}{\omega_{ref}}$$

$$\Rightarrow \quad \frac{\omega_{ref} V_{ref}}{L_{ref}} = St_{ref} \quad \text{or} \quad k_{ref}$$

Strouhal — reduced frequency.

- steady compressible viscous flow

$L_{ref}, \; ----, \; a_{ref}, \; K_{ref}$

speed of sound — thermal conductivity

$$M_{ref} = \frac{U_{ref}}{a_{ref}} \, , \quad P_{ref} = \frac{C_p \mu_{ref}}{K_{ref}}$$

temperature variation / ht transfer

For ~~steady~~, incompressible, viscous flow

$$\nabla \cdot \vec{u} = 0$$

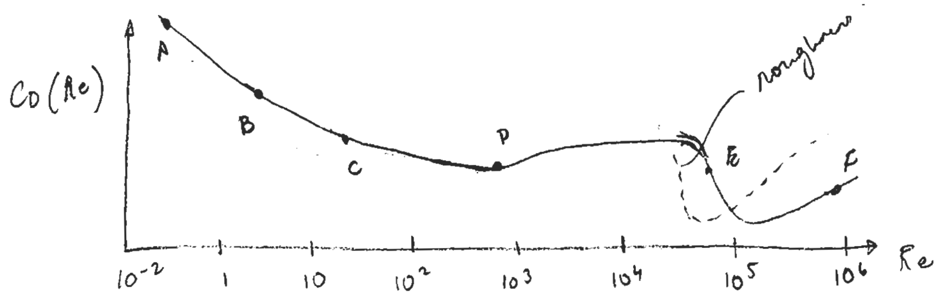
$$\underset{1}{\frac{\partial \vec{u}}{\partial t}} \quad \underset{2}{(\vec{u} \cdot \nabla)\vec{u}} = \underset{3}{-\frac{1}{\rho}\nabla p} + \underset{4}{\frac{1}{Re}\,\nu\,\nabla^2 \vec{u}}$$

* dropped

B.C depend on geometry, ie h/L ~ AR ...

Flow Regimes as a function of Reynolds #

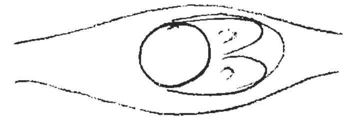
Consider a spherical body



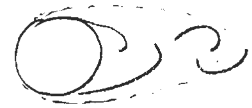
A) $Re \ll 1$ Stokes Flow (3, 4)

B) $Re < 1$ Oseen Flow (3, 4, 2-linearized)

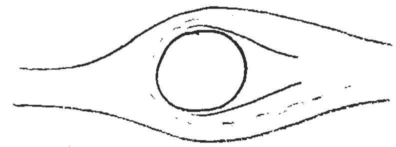
C) $Re \simeq 1 - 100$ – steady, separated flow (2, 3, 4)



D) $Re \simeq 100 - 1000$ – unsteady sep. flow (1, 2, 3, 4) (NS)
↑ vortex shedding



E) $Re \approx 200 - 400 \times 10^3$ transition moves onto surface



F) $Re \approx 500 \times 10^3$ turbulent flow (~~2, 3, simplified 4~~)
↑ TSL