A) Iteration Stability

B) Interaction Laws

Reading : Handout

A) Iteration Stability

Classical Iteration

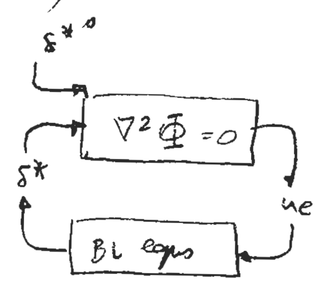
0) Assume some δ^*

1) add δ^* to geometry contour (or impose $\frac{\partial \phi}{\partial n} = V_{wall}$)

2) Solve $\nabla^2 \Phi = 0$ $\longrightarrow$ calculate $u_e = \frac{\partial \Phi}{\partial x}$

3) Solve BL eqns $\longrightarrow$ calc. δ^*

4) Iterate



<u>Problem</u> : Almost never works due to numerical instability

<u>Stability Analysis</u>

- Assume converged soln.
- perturb solution
- See if perturbation grows or decays with iteration $\to \infty$

Apply to BL flow over a wall. Let $\Phi = u_\infty (x + \varphi)$

$$\therefore\ u = \Phi_x = u_\infty (1 + \varphi_x), \quad \frac{du}{dx} = U_\infty \varphi_{xx}$$

$$\nabla^2 \Phi = \nabla^2 \varphi = 0$$

Linearize BL equations

$$\frac{d\delta^*}{dx} = \underbrace{\theta \frac{dH}{dx}}_{K.E} + \underbrace{H \frac{d\theta}{dx}}_{Mom}$$

$$\rightarrow \quad \frac{d\delta^*}{dx} = A + B \frac{du_e}{dx}$$

assume const. (depend on base soln)

Using displacement surface model

$$\Delta = \delta^*$$

and $\alpha = \dfrac{d\delta^*}{dx}$ — slope of displacement surface

Inviscid BC on outer flow is

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \varphi}{\partial y} = \alpha \quad \text{(no flow through displacement surf.)}$$

Introduce perturbation in $\alpha \rightarrow \alpha_0 + \tilde{\alpha}$, where

$$\tilde{\alpha} = \epsilon e^{ikx}$$

The corresponding perturbation potential is

$$\tilde{\varphi} = -\frac{\epsilon}{k} e^{ikx} \cdot e^{-ky}, \quad \nabla^2 \tilde{\varphi} = 0$$

(wavy wall problem)

decays in y ($\tilde{\varphi} = 0$ at $y \rightarrow \infty$)

C

Perturbation in u_e is

$$\tilde{u}_e = \tilde{\varphi}_x u_\infty \quad \text{and} \quad \frac{d\tilde{u}_e}{dx} = U_\infty \tilde{\varphi}_{xx} = U_\infty k \alpha$$

From iteration model we have

$$\frac{d\tilde{\delta}^*}{dx} = B \frac{d\tilde{u}_e}{dx}$$

$$\therefore \quad \tilde{\alpha}^{n+1} = B \frac{d\tilde{u}_e^n}{dx} = B u_\infty k \epsilon e^{ikx}$$

$$= B u_\infty k \alpha^n$$

$$\Rightarrow \quad \boxed{g = B u_\infty k} \quad \text{amplification factor}$$

Must have $|g| < 1$ for decay / convergence (stable iteration)

Express $\quad g = B u_\infty k = \left(\frac{B u_e}{\delta^*}\right)\left(\frac{u_\infty}{u_e}\right) \delta^* k \qquad \frac{B u_e}{\delta^*} = f(H)$

$\left(\frac{B u_e}{\delta^*}\right)$ depends on local BL $\qquad B = \frac{\delta^*}{u_e}\left[\frac{H^*}{dH^*/dH}(H-1) + (H+2)\right]$

Example : for similar flows

$$\frac{B u_e}{\delta^*} \simeq \begin{cases} -5 & \text{—— } \beta_u = 1 \text{ (stagn. pnt)} \\ -50 & \text{—— } \beta_u = 0 \text{ (Blasius)} \\ -\infty & \text{—— } \beta_u = -0.09 \text{ (sep.)} \end{cases}$$

H
2.26
2.59
4.00

Also, $(u_\infty / u_e) \approx 1$ in this case

k $= 2\pi / \lambda$ is the wave # of the perturbation. Worst case is

$$k \simeq \pi / \Delta x = \pi / (\text{streamwise grid spacing})$$

— sawtooth

Hence, we have

$$g \simeq \left| B \frac{u_c}{\delta^*} \right| \left(\frac{\delta^* \pi}{\Delta x} \right) \leq 1$$

$$\therefore \quad \frac{\Delta x}{\delta^* \pi} \geq \left| \pi B \frac{u_c}{\delta^*} \right| = \begin{cases} 5 \\ 50 \\ \infty \end{cases}$$

Consider a flat plate airfoil

$$\Delta x \approx 50 \cdot \pi \cdot \delta^* \approx 150\,\delta^*$$

$$\delta^* \approx c/150 \quad \Rightarrow \quad \Delta x = \text{[illegible]} \cdot c \rightarrow 2 \text{ grid points}$$

Tough to make stable and accurate. Close to separation, hopelessly unstable.

Summary

① Closer profile is to separation, the less stable classical iteration procedure

② $\frac{\Delta x}{\delta^*}$ must be quite large for stability, hence severe accuracy-stability tradeoff

③ Guaranteed to fail if separation is present.

One fix: under relaxation

$$\tilde{\alpha}^{n+1} = \tilde{\alpha}^n + \omega \left(B u_\infty k \epsilon e^{ikx} - \tilde{\alpha}^n \right)$$

$$\frac{\Delta x}{\delta^* \pi} \geq \omega \left| B \frac{u_c}{\delta^*} \right| = \begin{cases} 5\omega \\ 50\omega \\ \infty \end{cases}$$

Slow convergence if stabilized with $0 < \omega < 1$

~~Inverse iteration and separation~~

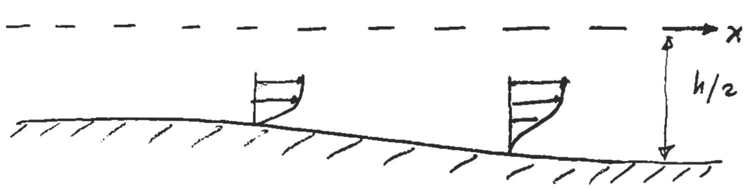
Interaction laws

Replace classical $\Phi \to u_e \to BL$ with interaction laws.

① $\Phi - \delta^*$ or $\Phi - m$ relation for outer flow
$m \equiv \rho u_e \delta^*$

② $u_e - \delta^*$ or $u_e - m$ for internal flow

Example: Consider quasi 1-D flow



$\longrightarrow u_e = \frac{\dot m}{\rho h(x)}$

$$\dot m_0 = \rho u_e h - \dot m = \rho u_e (h - \delta^*) = \text{const.}$$

$$\therefore \quad u_e = \left(\frac{\dot m}{\rho}\right) \frac{1}{h - \delta^*} \qquad \text{replaces} \quad u_e = \dot m / \rho h$$

$\Rightarrow$ 3 ODE's for 3 unknowns – θ, δ^*, u_e

$$\frac{d\theta}{dx} = f_1(\theta, \delta^*, u_e)$$

$$\frac{d\delta^*}{dx} = f_2(\qquad)$$

(see hand/out)

$$\frac{du_e}{dx} = f_3(\qquad)$$

Boundary layer is allowed to change the outer flow through the interaction model/law – crucial diff.

Express in similarity form

β_θ - - -
β_H - - -
β_u - - -

Solve 3x3 system

$$\left[\quad\right]\{\beta\} = \{R\}$$

$$\beta = \begin{Bmatrix} \beta_\theta \\ \beta_H \\ \beta_u \end{Bmatrix} \rightarrow \text{or } \beta\delta^*$$

Forward integration (Euler, Trapezoid a

$$\theta_{i+1} = \theta_i + \frac{d\theta}{dx}\cdot \Delta x$$

$$= \theta_i + \frac{\theta_i}{x_i}\beta_\theta \Delta x$$

$$= \theta_i \left(\frac{x_{i+1}}{x_i}\right)^{\beta_\theta}$$

$\delta^*_{i+1} =$ $\qquad$ $(\beta_H = \beta\delta^* - \beta_\theta)$

$u_{e_{i+1}} =$

Solving require closure relations

$C_f\ (H, Re_\theta)$
C_D
H^*
$\frac{dH^*}{dx}$

given for laminar and turb cases.

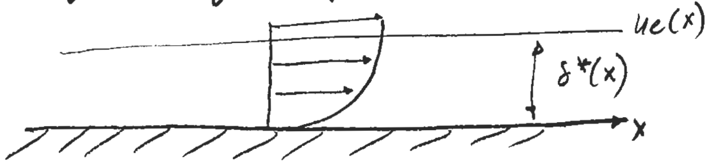
2D IBLT

Local interaction laws not strictly valid/correct in 2D since inv. flow may be elliptic – global influence

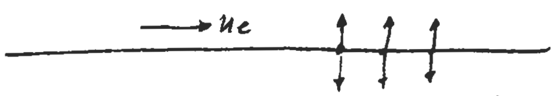
$\Rightarrow$ u_e depends on $\delta^*(x)$ everywhere

One approximate solution: Hilbert integral (thin airfoil theory)

Consider infinite flat plate



$\Downarrow$



$$\sigma \equiv \Delta(\vec{V}\cdot\hat{n}) = 2V_w = 2\frac{d(u_e\delta^*)}{dx} = 2\frac{d}{dx}(m/\rho)$$

$$u_e(x) = u_\infty + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sigma(\xi)}{x-\xi}\,d\xi \qquad \frac{1}{\pi}\int\frac{d(m/\rho)}{d\xi}\cdot\frac{d\xi}{x-\xi}$$

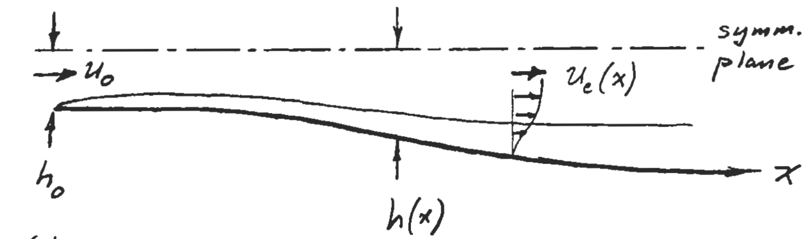
Implemented in incremental sense

$$u_e^{n+1}(x) - u_e^{n}(x) = \frac{1}{\pi}\int\frac{d\left[(m/\rho)^{n+1}-(m/\rho)^{n}\right]}{x-\xi} \quad \text{(int. law)}$$

In 3D

Quasi-1D IBLT Solution Procedure

Assumed geometry $\Rightarrow$



Unknowns: $\theta(x)$ $\delta^*(x)$ $u_e(x)$

Constant mass flow: $\dot{m} = \rho u_0 h_0$

Governing ODE's:

$$\boxed{\frac{d\theta}{dx} + (H+2)\frac{\theta}{u_e}\frac{du_e}{dx} = \frac{C_f}{2}} \quad ①$$

$$\frac{\theta}{H^*}\frac{dH^*}{dx} = \boxed{\frac{\theta}{H^*}\frac{dH}{dx}\frac{dH^*}{dH} = \frac{2C_D}{H^*} - \frac{C_f}{2} + (H-1)\frac{\theta}{u_e}\frac{du_e}{dx}} \quad ②$$

$$u_e = \frac{\dot{m}/\rho}{h-\delta^*} \quad \Rightarrow \quad \boxed{\frac{du_e}{dx} = \frac{u_e}{h-\delta^*}\left[\frac{d\delta^*}{dx} - \frac{dh}{dx}\right]} \quad ③$$

To allow numerical integration, these need to be put in the form:

$$\frac{d\theta}{dx} = f_1(\theta, \delta^*, u_e) \qquad \frac{d\delta^*}{dx} = f_2(\theta, \delta^*, u_e) \qquad \frac{du_e}{dx} = f_3(\theta, \delta^*, u_e)$$

This can be done either anlytically or numerically.
First we write ①, ②, ③ in terse form as follows:

$$\frac{x}{\theta}\frac{d\theta}{dx} + (H+2)\frac{x}{u_e}\frac{du_e}{dx} = \frac{x}{\theta}\frac{C_f}{2} \quad \Rightarrow \quad \beta_\theta + (H+2)\beta_u = \frac{x}{\theta}\frac{C_f}{2}$$

$$\frac{x}{H}\frac{dH}{dx}\left(\frac{H}{H^*}\frac{dH^*}{dH}\right) - (H-1)\frac{x}{u_e}\frac{du_e}{dx} = \frac{x}{\theta}\left(\frac{2C_D}{H^*} - \frac{C_f}{2}\right) \quad \Rightarrow \quad \beta_H\left(\frac{H}{H^*}\frac{dH^*}{dH}\right) - (H-1)\beta_u = \frac{x}{\theta}\left(\frac{2C_D}{H^*} - \frac{C_f}{2}\right)$$

$$\frac{x}{u_e}\frac{du_e}{dx} - \frac{\delta^*}{h-\delta^*}\frac{x}{\delta^*}\frac{d\delta^*}{dx} = \frac{-x}{h-\delta^*}\frac{dh}{dx} \quad \Rightarrow \quad \beta_u - \frac{\delta^*}{h-\delta^*}\beta_{\delta^*} = \frac{-x}{h-\delta^*}\frac{dh}{dx}$$

Where

$$\beta_\theta \equiv \frac{x}{\theta}\frac{d\theta}{dx}, \quad \beta_u \equiv \frac{x}{u_e}\frac{du_e}{dx}, \quad \ldots \text{ etc.}$$

Also, note that
$$\beta_H = \frac{x}{\delta^*/\theta}\frac{d(\delta^*/\theta)}{dx} = \theta\frac{x}{\delta^*}\left[\frac{1}{\theta}\frac{d\delta^*}{dx} - \frac{\delta^*}{\theta^2}\frac{d\theta}{dx}\right] = \frac{x}{\delta^*}\frac{d\delta^*}{dx} - \frac{x}{\theta}\frac{d\theta}{dx}$$

or $$\beta_H = \beta_{\delta^*} - \beta_\theta$$

42-381 50 SHEETS 5 SQUARE
42-382 100 SHEETS 5 SQUARE
42-389 200 SHEETS 5 SQUARE
NATIONAL

The 3 ODE's can now be written as:

$$\begin{bmatrix} 1 & 0 & H+2 \\ -\frac{H}{H^*}\frac{dH^*}{dH} & \frac{H}{H^*}\frac{dH^*}{dH} & 1-H \\ 0 & -\frac{\delta^*}{h-\delta^*} & 1 \end{bmatrix} \begin{bmatrix} \beta_\theta \\ \beta_{\delta^*} \\ \beta_u \end{bmatrix} = \begin{bmatrix} \frac{x}{\theta}\frac{C_f}{2} \\ \frac{x}{\theta}\left(\frac{2C_D}{H^*} - \frac{C_f}{2}\right) \\ -\frac{x}{h-\delta^*}\frac{dh}{dx} \end{bmatrix}$$

At any streamwise station x_i, the coefficient matrix and righthand side can be evaluated. This allows us to solve the 3×3 system for β_θ, β_{δ^*}, β_u. The x-derivatives

$$\frac{d\theta}{dx} = \frac{\theta}{x}\beta_\theta \qquad \frac{d\delta^*}{dx} = \frac{\delta^*}{x}\beta_{\delta^*} \qquad \frac{du_e}{dx} = \frac{u_e}{x}\beta_u$$

can then be used to determine θ, δ^*, u_e at x_{i+1} using Forward-Euler (say), or some higher-order method such as Predictor-Corrector or Runge-Kutta.

Alternative Integration | The integration method will be inaccurate if $\frac{\Delta x}{x}$ is not small (such as near the leading edge). One solution to this problem is to integrate using β_θ, β_{δ^*}, β_u directly, e.g.:

$$\beta_\theta = \frac{x}{\theta}\frac{d\theta}{dx} = \frac{d\theta/\theta}{dx/x} = \frac{d(\ln\theta)}{d(\ln x)} \simeq \frac{\Delta(\ln\theta)}{\Delta(\ln x)} = \frac{\ln\theta_{i+1} - \ln\theta_i}{\ln x_{i+1} - \ln x_i} = \frac{\ln(\theta_{i+1}/\theta_i)}{\ln(x_{i+1}/x_i)}$$

so $\ln(\theta_{i+1}/\theta_i) = \beta_\theta \ln(x_{i+1}/x_i) \Rightarrow \theta_{i+1} = \theta_i \left(\frac{x_{i+1}}{x_i}\right)^{\beta_\theta}$

Likewise $\quad \delta^*_{i+1} = \delta^*_i \left(\frac{x_{i+1}}{x_i}\right)^{\beta_{\delta^*}} \qquad u_{e_{i+1}} = u_{e_i}\left(\frac{x_{i+1}}{x_i}\right)^{\beta_u}$

Note that these are <u>exact</u> for similar flows (i.e. near the leading edge) no matter how large $\Delta x/x = (x_{i+1} - x_i)/x_i$ is. For small $\Delta x/x$ the above power-law integration is equivalent to normal Forward Euler to first order in $\Delta x/x$.

Note: The Classical BL case is obtained by neglecting δ^* in the 3rd line in the 3×3 system above:

→ $\beta_u = -\frac{x}{h}\frac{dh}{dx}$ or simply $u_e = \frac{\dot{m}/\rho}{h}$ is known a priori.

Quasi-1D IBLT Solution Procedure 16.13

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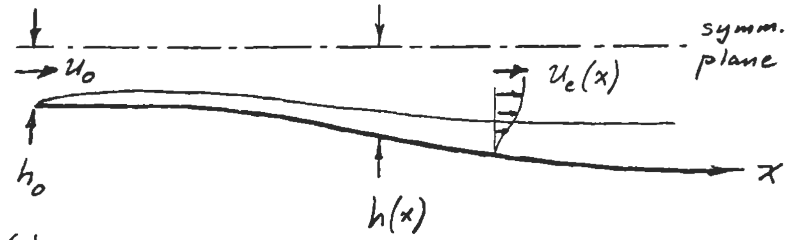

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$$\frac{\theta}{H^*}\frac{dH^*}{dx} = \boxed{\frac{\theta}{H^*}\frac{dH}{dx}\frac{dH^*}{dH} = \frac{2C_D}{H^*} - \frac{C_f}{2} + (H-1)\frac{\theta}{u_e}\frac{du_e}{dx}} \quad (2)$$

$$u_e = \frac{\dot{m}/\rho}{h-\delta^*} \quad \Rightarrow \quad \boxed{\frac{du_e}{dx} = \frac{u_e}{h-\delta^*}\left[\frac{d\delta^*}{dx} - \frac{dh}{dx}\right]} \quad (3)$$

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