Solution Techniques

4.1>

A> Classification

B> Boundary conditions

C> Well-posedness.

Reading: White : 77 - 78
Anderson
Tannehill → Comp. Fluid Mech. and Ht. Transfer } 19 - 31
Pletcher

A> Classification

Consider a linear, 2nd order, 2-D PDE operator

$$L(u) = g(x,y)$$

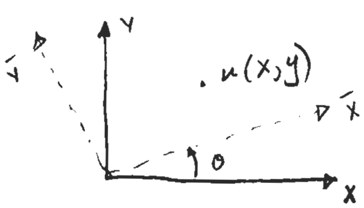
$$L(u) \equiv a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu$$

By coordinate rotation

$$x = \bar{x}\cos\theta - \bar{y}\sin\theta$$
$$y = x\sin\theta + \bar{y}\cos\theta$$

or

$$\bar{x} = x\cos\theta + y\sin\theta$$
$$\bar{y} = -x\sin\theta + y\cos\theta$$



The PDE can be written as

$$\bar{L}(u) \equiv \bar{a}\frac{\partial^2 u}{\partial \bar{x}^2} + \bar{c}\frac{\partial^2 u}{\partial \bar{y}^2} + \bar{d}\frac{\partial u}{\partial \bar{x}} + e\frac{\partial u}{\partial \bar{y}} + fu \,(= g)$$

$$\bar{a} = a\cos^2\theta + b\sin\theta\cos\theta + c\sin^2\theta$$
$$\bar{b} = 0$$
$$\bar{c} = a\sin^2\theta - b\sin\theta\cos\theta + c\cos^2\theta$$

$$\bar{d} = d\cos\theta + e\sin\theta$$

$$\bar{e} = -d\sin\theta + e\cos\theta$$

we can classify the PDE from the characteristic polynomial ← check

If $\bar{a}\bar{c} > 0$ or $b^2 - 4ac < 0$,

PDE is elliptic. Ex. Laplace's Eqn $\nabla^2 u = 0$

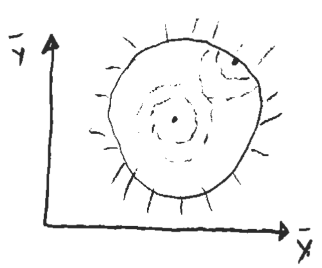
If $\bar{a}\bar{c} < 0$ ($\bar{a}$ or $\bar{c} < 0$),

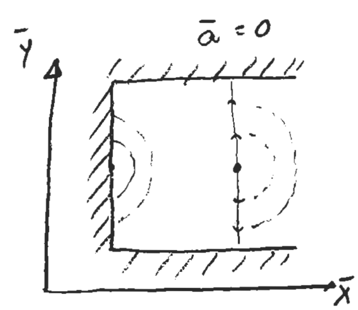
PDE is hyperbolic. Ex wave eqn. $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$

If $\bar{a} = 0$ or $\bar{c} = 0$ (not both),

PDE is parabolic. Ex. diffusion eqn. $\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} = 0$.

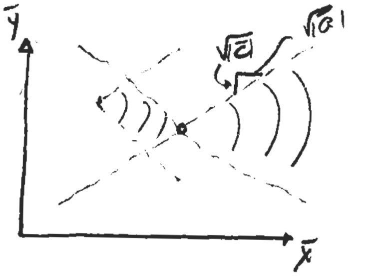
Distinguishing feature are different "domains of dependance" and "domains of influence" of PDE's. (Important for imposing correct boundary conditions & pick appropriate numerical scheme + numerical B.Cs)





Elliptic: each point influences and is influenced by all other points

Parabolic: each point influences points laterally and downstream from it.

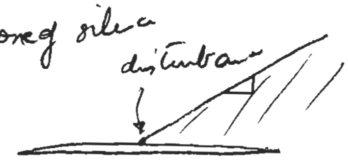


Hyperbolic: each point influences points that lie with its characteristic cone, and is influenced by lying within the characteristic cone of another point

Small disturbance equ.

$\bar{a}\,\varphi_{xx} - \bar{c}\,\varphi_{yy} = 0$

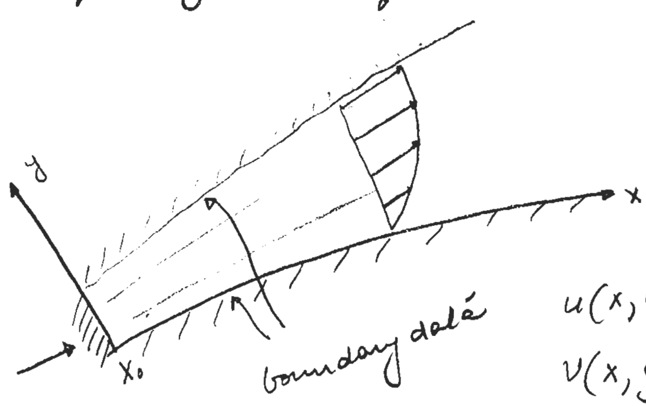
$\tan\mu = \dfrac{1}{\sqrt{M^2-1}} = \sqrt{\dfrac{-\bar{c}}{\bar{a}}}$ $\qquad (1-M^2)\,\varphi_{xx} + \varphi_{yy} = 0$



$\mu = \sin^{-1}(1/M)$ $\qquad$ (for $M_\infty > 1$)

Note for $M_\infty = 1 \rightarrow$ parabolic, $M_\infty < 1 \rightarrow$ elliptic $\qquad \left[\bar{x} = x/\sqrt{1-M_\infty^2} \rightarrow \phi_{\bar{x}\bar{x}} - \phi_{yy} = 0\right.$

Examine the TSL mom. equation $\rightarrow$ parabolic with some hyperbolic character depending on Re_θ



$u(x,0)$, $v(x,0)$

$U(x, y_\infty)$

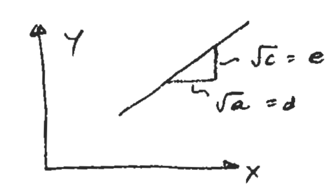
initial data

$u(x_0, y)$

$v(x_0, y)$

$d u_x + e u_y + \cdots$

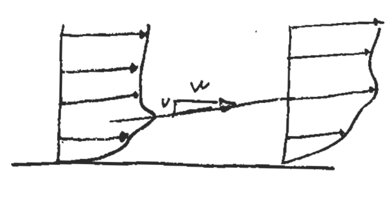
$\Rightarrow \underbrace{d^2}_{a} u_{xx} - \underbrace{e^2}_{c} u_{yy} + \cdots$



$\underline{u}\dfrac{\partial u}{\partial x} + \underline{v}\dfrac{\partial u}{\partial y} = RHS$ $\qquad$ 1st order equ. for $u(x,y)$

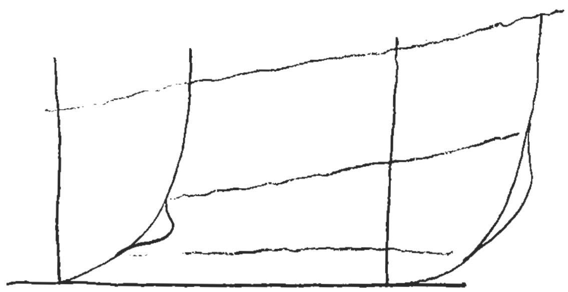
characteristic is streamline

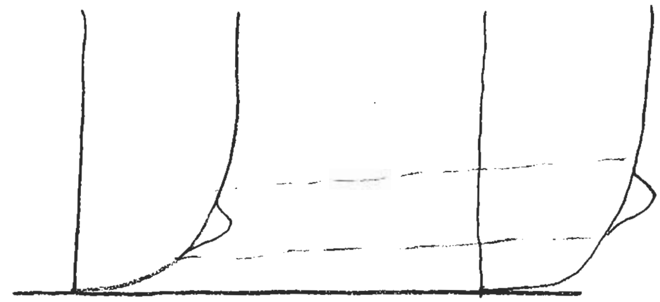
u is convected along a streamline



Example $\qquad \dfrac{D(s)}{Dt} = \vec{u} \cdot \nabla(s) = 0$

TSL Equation contains diffusion term also





Low Re_θ or Re_δ
Rel. fast decay

Large $Re_{\theta,\delta}$, u disturbance convects along streamline with little decay

Rotate axis along streamline,

$$u\frac{\partial u}{\partial x} \simeq \nu \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} \simeq \frac{\nu}{u}\frac{\partial^2 u}{\partial y^2} \simeq O\left(\frac{\nu}{u_e}\cdot\frac{u_e}{\delta^2}\right)$$

$$\Rightarrow \quad \frac{du/u_e}{d(x/\delta)} \simeq O\left(1/Re_\delta\right)$$ measure of decay rate

Re_δ or Re_θ determines relative importance of convection vs. diffusion

B)

B> Boundary Conditions

- Needed to close or full determine a PDE problem
- The problem must be well-posed - implying B.C respect / related to domains of dependance & influence (depend continuously on B.C and I.C data, be unique, and must exist

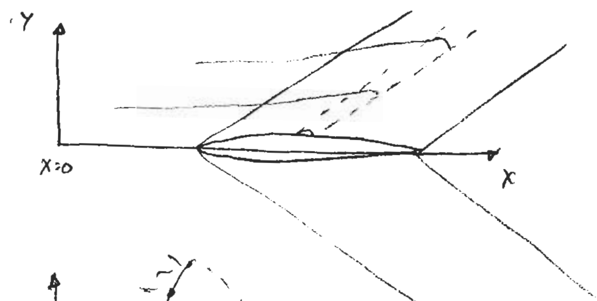
Example

Linearized S.S flow

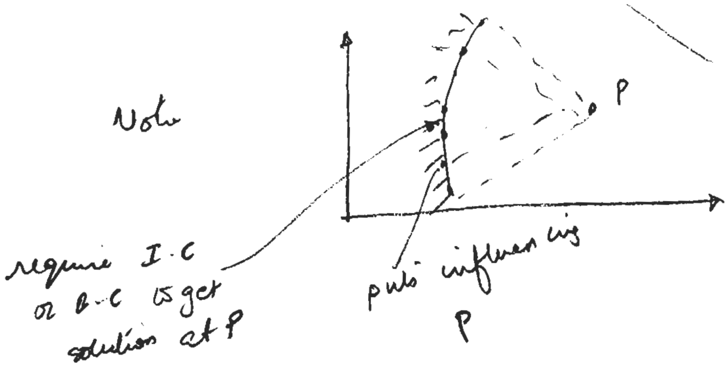
$$(1-M_\infty^2)\,\phi_{xx} + \phi_{yy} = 0$$

$$\phi_y = U_\infty \frac{dy}{dx}\Big|_{wall} \leftarrow \text{airfoil slope}$$

$$\phi(0,y) = 0$$



Note

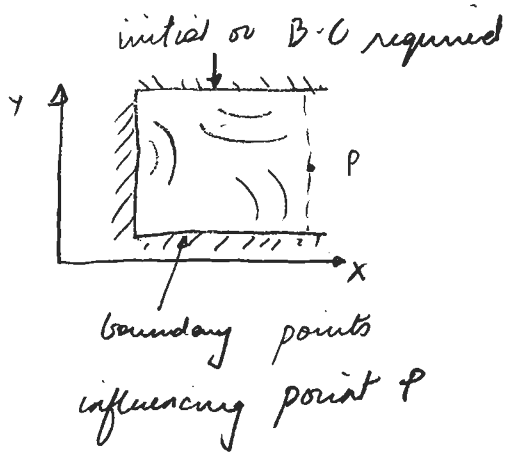


- Dictates "sweep direction" of solution & minimum BC data required

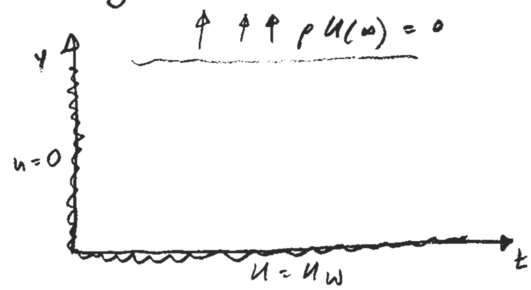
Parabolic Eqns

Example $\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} = 0$ (x is called "time-like" variable

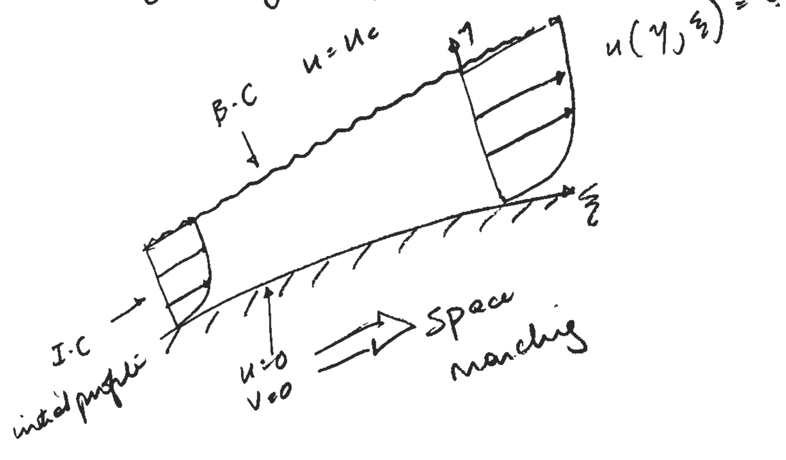
similarly for hyperbolic eqns)
$\phi_{xx} - \phi_{yy} = 0$



In the Rayleigh problem.



For a boundary layer flow



New solution become I.C for next profile.