Non-linear, Unsteady Transonic Flows

1 SOURCES

Ashley and Landahl: Aerodynamics of wings and bodies

Bisplinghoff and Ashley: Principles of aeroelasticity

Dowell, et al.: A modern course in aeroelasticity

Landahl: Unsteady transonic flow

2 ASSUMPTIONS

- 2-Dimensional
- Inviscid
- Small disturbances (MCL → body surface)
- Shock waves are straight
- Mach number near unity
- Continuous pressure across the wake
- No jump in normal velocity across the wake
- Kutta condition (ΔP vanishes at LE)

- Far-field conditions
- Small shock excurion amplitude

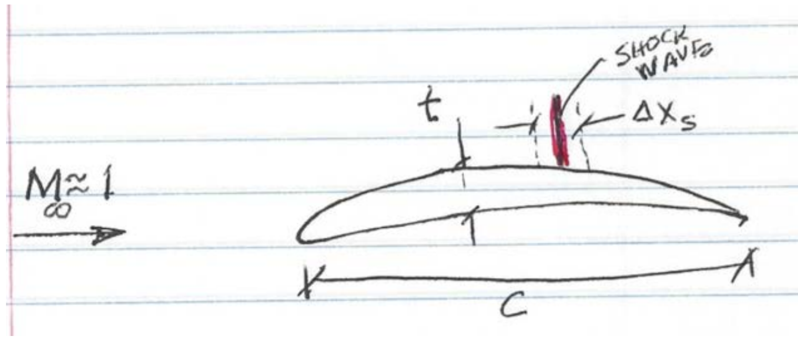

Figure 2.1: Assumptions diagram

3 Unsteady Perturbations

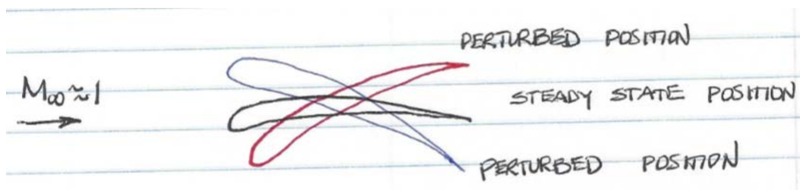

Figure 3.1: Perturbation positions

Symbols

- Q = Magnitude of velocity vector
- $\bar{Q}$ = Velocity vector
- a = speed of sound
- T = Time
- U_∞ = Free-stream velocity
- ρ = Density
- Ψ = Full velocity potential; $\Psi(x, z, T)$

- Φ = Perturbation velocity potential; $\Phi(x, z, T)$
- x, z = Spatial coordinates
- τ = Airfoil thickness ratio; t/c
- α_m = Mean angle of attack
- δ = Amplitude of unsteady motion
- ω = Frequency of unsteady motion
- $\kappa = \frac{\omega c}{U_\infty}$
- B = Instantaneous airfoil position; $B(x, z, t)$

4 Shock excursion amplitude

$$\frac{\Delta x_s}{c} \sim \tau^h \alpha_m^\kappa \left(\frac{\delta}{c}\right)^i \Big/ \kappa^d$$

4.1 Governing equations

Bernoulli's equation

$$\frac{1}{\rho}\frac{D_p}{DT} = -\frac{1}{a^2}\left[\frac{\partial^2 \Psi}{\partial T^2} + \frac{\partial Q^2}{\partial T} + \bar{Q}\bigtriangledown\left(\frac{Q^2}{z}\right)\right] \tag{4.1}$$

Speed of sound

$$a^2 = a_\infty^2 - (\gamma - 1)\left[\Psi_T + \frac{1}{2}\left(\Psi_X^2 + \Psi_Z - U_\infty^2\right)\right] \tag{4.2}$$

Conservation of mass

$$\frac{1}{\rho}\frac{D_p}{DT} = -\bigtriangledown\bar{Q} = -\bigtriangledown^2\Psi \tag{4.3}$$

Combine eqns. (4.1), (4.2), and (4.3) to obtain:

$$(a^2 - \Psi_X^2)\Psi_{XX} + (a^2 - \Psi_Z^2)\Psi_{ZZ} - \Psi_{TT} - 2(\Psi_Z\Psi_X\Psi_{XZ} + \Psi_X\Psi_{XT} + \Psi_Z\Psi_{ZT}) = 0 \tag{4.4}$$

Assume velocity field may be expressed as the sum of a uniform stream and perturbation upon A

Uniform stream

$$\Psi(X, Z, T) = U_\infty[x + \Psi'(X, Z, T) + \ldots] \tag{4.5}$$

Combine equations (4.4) and (4.5) to obtain:

$$\left(1-M_\infty^2\right)\Phi'_{XX}+\Phi'_{ZZ}-2\frac{M_\infty^2}{U_\infty}\Phi'_{XT}-\frac{1}{a_\infty^2}\Phi'_{TT}=M_\infty^2\left[(\gamma+1)\Phi'_x+\Phi'^2_X\right]\Phi'^2_{XX}$$
$$+\frac{\gamma-1}{2}\left(\frac{2}{U_\infty}\Phi'_T+\Phi'^2_X+\Phi'^2_Z\right)\left(\Phi'_{XX}+\Phi'_{ZZ}\right)$$
$$+\left[(\gamma-1)\Phi'_X+\Phi'^2_Z\right]\Phi'_{ZZ}$$
$$+2\left[\left(1+\Phi'_X\right)\Phi'_{XZ}\Phi'_Z+\frac{1}{U_\infty}\left(\Phi'_X\Phi'_{XT}+\Phi'_Z\Phi'_{ZT}\right)\right] \tag{4.6}$$

Neglecting products of small terms and retaining "Transonic Terms", we obtain:

$$\left[\left(1-M_\infty^2\right)-\frac{M_\infty^2}{U_\infty}(\gamma-1)\Phi'_T-M_\infty^2(\gamma+1)\Phi'_X\right]\Phi'_{XX}+\Phi'_{ZZ}-2\frac{M_\infty^2}{U_\infty}\Phi'_{XT}-\frac{1}{a_\infty^2}\Phi_{TT}=0 \tag{4.7}$$

Now introduce nondimensional variables:

$$\mathrm{x}=\frac{X}{c} \qquad \mathrm{z}=\frac{Z}{c}$$
$$\mathrm{t}=T\frac{U_\infty}{c} \qquad \Phi=\frac{\Phi'}{c} \tag{4.8}$$

Equation (4.7) in nondimensional form becomes:

$$\left[\left(1-M_\infty^2\right)-M_\infty^2(\gamma-1)\Phi_t-M_\infty^2(\gamma+1)\Phi_\mathrm{x}\right]\Phi_\mathrm{xx}+\Phi_\mathrm{zz}-2M_\infty^2\Phi_{\mathrm{x}t}-M_\infty^2\Phi_{tt}=0 \tag{4.9}$$

5 Boundary conditions

$B(\mathrm{x},\mathrm{z},t)=0 \rightarrow$ Instantaneous airfoil position

$\frac{(1+\Phi_x)B_x+\Phi_z B_Z}{\sqrt{B_x^2+B_z^2}} \rightarrow$ Fluid velocity normal to the airfoil

$-\frac{B_t}{\sqrt{B_x^2+B_z^2}} \rightarrow$ Velocity of airfoil normal to itself

The airfoil tangency condition may be expressed as:

$$\frac{DB}{Dt}=B_t+(1+\Phi_x)B_x+\Phi_Z B_Z=0 \tag{5.1}$$

For a thin airfoil, $\Phi_x << 1$; therefore, we may write:

$$B_t+B_x+\Phi_z B_z=0 \tag{5.2}$$

Insert the following restrictions:

$$
\begin{aligned}
\frac{\delta}{c} &<< \tau \\
\frac{\delta}{c} &<< \alpha_m
\end{aligned}
\tag{5.3}
$$

This restriction allows us to express the perturbation velocity potential as:

$$
\Phi(\mathrm{x},\mathrm{z},t) = \Phi(\mathrm{x},\mathrm{z}) + \hat{\Phi}(\mathrm{x},\mathrm{z},t) \tag{5.4}
$$

Where $\Phi(\mathrm{x},\mathrm{z})$ and its derivatives are much greater than $\hat{\Phi}(\mathrm{x},\mathrm{z},t)$ and its derivatives. The above formulation is valid for small unsteady perturbations.

Within the above restrictions, equation (4.7) becomes:

$$
\left[\left(1-M^2\right) - M^2(\gamma+1)\Phi_{\mathrm{x}}\right]\Phi_{\mathrm{xx}} + \Phi_{\mathrm{zz}} = 0 \tag{5.5}
$$

$$
\begin{aligned}
&\left[\left(1-M^2\right) - M^2(\gamma+1)\Phi_{\mathrm{x}}\right]\hat{\Phi}_{\mathrm{xx}} - M^2(\gamma+1)\Phi_{\mathrm{xx}}\hat{\Phi}_{\mathrm{x}} \\
&+ \hat{\Phi}_{\mathrm{zz}} - M^2(\gamma-1)\Phi_{\mathrm{xx}}\hat{\Phi}_{\mathrm{t}} - 2M^2\hat{\Phi}_{\mathrm{xt}} - M^2\hat{\Phi}_{\mathrm{tt}} = 0
\end{aligned}
\tag{5.6}
$$

Where we set $M_\infty = M$, why?

Note that equation (5.5) is non-linear and steady. It is used to simulate thickness, camber, and mean angle of attack.

Note that equation (5.6) is linear and unsteady. It is strongly coupled to equation (5.5).

6 Methods of Solution

(a) Numerical simulation

(b) Hodograph plane

2-D, steady, shock-less → Tricomi equation

(c) Parametric differentiation

(d) Variational methods

(e) Weighted residues

(f) Local linearization

(g) Ray tracing

(h) Kernel Function (including Green's function)

(i) Integral methods

(j) Matched asymptotic expansions

Similarity rules