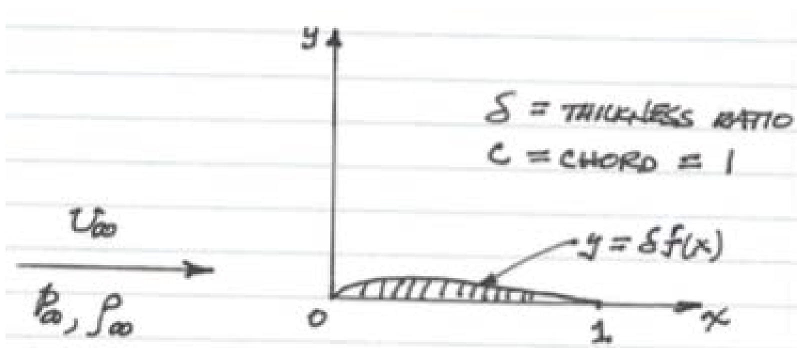
Slender Body Theory

Let's take another look at the Prandtl-Glauert rule. We shall rely more on regular perturbation methods and boundary conditions. Our flow field is:



Assume inviscid, irrotational, compressible flow of an ideal, perfect gas over a slender body, $\delta << 1$. We have:

$$\vec{Q} = \nabla\Phi = \Phi_x \vec{i} - \Phi_y \vec{j}$$

The equations for conservation of mass and linear momentum are:

$$(a^2 - \Phi_x^2)\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + (a^2 - \Phi_y^2)\Phi_{yy} = 0$$

$$\frac{a^2}{\gamma - 1} + \frac{1}{2}(\Phi_x^2 + \Phi_y^2) = \frac{a_\infty^2}{\gamma - 1} + \frac{1}{2}U_\infty^2$$

Where

a = speed of sound γ = specific heat ratio

The subscript ∞ denotes conditions far from the airfoil body. The boundary condition on the body surface may be expressed as:

$$\frac{\Phi_y[x,\delta f(x)]}{\Phi_x[x,\delta f(x)]} = \delta \frac{df}{dx} = \delta f'(x)$$

And far from the body

$$\Phi(x,y) \to U_\infty x, as|x| \to \infty$$

Consider the limiting case $\delta \to 0$, holding M_∞ fixed. Assume an asymptotic expansion for the velocity potential of the following form:

$$\Phi(x,y;\delta,M_\infty) \sim U_\infty[x + \epsilon_0(\delta)\Phi_0(x,y;M_\infty) + \epsilon_1(\delta)\Phi_1(x,y;M_\infty) + ...]$$

What does the first term on the right-hand side represent? Is the term correct? Why? Why not?

Consider the surface boundary condition. First, we expand $\Phi_y[x,\delta f(x)]$ in a Taylor series about (x,0):

$$\Phi_y(x,\delta f(x)) \sim \Phi_y(x,0) + \delta f(x)\Phi_{yy}(x,0) + ...$$

Substituting and taking the derivative with respect to y:

$$\Phi_y(x,\delta f(x)) \sim U_\infty[\epsilon_0\Phi_{0_y}(x,0;M_\infty) + \epsilon_1\Phi_{1_y}(x,0;M_\infty) + ...] + U_\infty\delta f(x)[\epsilon_0\Phi_{0_{yy}}(x,0;M_\infty) + ...]$$

Thus:

$$\Phi_y(x,\delta f(x)) \sim U_\infty[\epsilon_0\Phi_{0_y}(x,0;M_\infty) + \epsilon_1\Phi_{1_y}(x,0;M_\infty) + \delta\epsilon_0 f(x)\Phi_{0_{yy}}(x,0;M_\infty) + ...]$$

We may also calculate $\Phi_x(x,\delta f(x))$:

$$\Phi_x(x,\delta f(x)) \sim U_\infty(1 + \epsilon_0\Phi_{0_x}(x,0;M_\infty) + ...)$$

The surface boundary condition takes the following form:

$$\frac{\Phi_y(x,\delta f(x))}{\Phi_x(x,\delta f(x))} \sim \frac{U_\infty[\epsilon_0\Phi_{0_y}(x,0;M_\infty) + \epsilon_1\Phi_{1_y}(x,0;M_\infty) + \delta\epsilon_0 f(x)\Phi_{0_{yy}}(x,0;M_\infty) + ...]}{U_\infty[1 + \epsilon_0\Phi_{0_x}(x,0;M_\infty) + ...]}$$

$$\sim (\epsilon_0\Phi_{0_y} + \epsilon_1\Phi_{1_y} + \delta\epsilon_0 f\Phi_{0_{yy}})(1 - \epsilon_0\Phi_{0_x}) + ...$$

$$\sim \epsilon_0\Phi_{0_y} + \epsilon_1\Phi_{1_y} + \delta\epsilon_0 f\Phi_{0_{yy}} - \epsilon_0^2\Phi_{0_x}\Phi_{0_y} + ...$$

Or

$$\delta f'(x) = \epsilon_0 \Phi_{0_y} + \epsilon_1 \Phi_{1_y} + \delta \epsilon_0 f \Phi_{0_{yy}} - \epsilon_0^2 \Phi_{0_x} \Phi_{0_y} + \ldots$$

For our assumed asymptotic sequence ϵ^n, we balance the leading term with no contradiction, i.e., the distinguished limit:

$$\epsilon_0(\delta) = \delta$$

Hence

$$\frac{\partial \Phi_0}{\partial y}(x, 0; M_\infty) = f'(x)$$

Balancing the next order of terms:

$$\epsilon_1 = \delta \epsilon_0 = \delta^2$$

And

$$0 \sim \epsilon_1 \Phi_{1_y} + \delta \epsilon_0 + \Phi_{0_{yy}} - \epsilon_0^2 \Phi_{0_x} \Phi_{0_y}$$

$$0 \sim \delta^2 \Phi_{1_y} + \delta^2 f \Phi_{0_{yy}} - \delta^2 \Phi_{0_x} \Phi_{0_y}$$

$$0 \sim \Phi_{1_y}(x, 0; M_\infty) + f \Phi_{0_{yy}}(x, 0; M_\infty) - \Phi_{0_y}(x, 0; M_\infty) \Phi_{0_y}(x, 0; M_\infty)$$

$$\Phi_{1_y}(x, 0; M_\infty) = \frac{\partial \Phi_1}{\partial y}(x, 0; M_\infty) = [\Phi_{0_x} \Phi_{0,y} - f \Phi_{0_{yy}}]_{(x,0,M_\infty)}$$

There is no contradiction since Φ_1 is determined from Φ_0. This means that our perturbation solution yields a linearized boundary condition at the body surface at each order of ϵ^n.

The conservation equations take the following form:

$$a^2 = a_\infty^2 - (\gamma - 1) U_\infty^2 \delta \Phi_{0_x} + O(\delta^2)$$

$$(M_\infty^2 - 1) \Phi_{0_{xx}} - \Phi_{0_{yy}} = 0$$

And for Φ_1:

$$(M_\infty^2 - 1) \Phi_{1_{xx}} - \Phi_{1_{yy}} = M_\infty^2 [(\gamma - 1) M_\infty^2 - 2] \Phi_{0_x} \Phi_{0_{xx}} - 2 M_\infty^2 \Phi_{0_y} \Phi_{0_{xy}}$$

The above zeroth order equation, Φ_0, is the basis of slender body theory. The Prandtl-Glauert rule is shown by re-scaling the x-coordinate:

For $M_\infty < 1$:

$$\xi = \frac{x}{\sqrt{1-M_\infty^2}}$$

$$\Phi_{0_{\xi\xi}} + \Phi_{0_{yy}} = 0$$

$$C_p = \frac{P - P_\infty}{\frac{1}{2}P_\infty U_\infty^2} \sim -2\delta\Phi_{0_x} \sim \frac{2\delta}{\sqrt{1-M_\infty^2}}\Phi_{0_\xi}$$

Note we have assumed:

$$\sqrt{1-M_\infty^2} >> \delta$$

$$M_\infty \delta << 1$$

Where:

$$\sqrt{M_\infty^2 - 1} \sim \delta \rightarrow \text{Transonic flow}$$
$$M_\infty \delta \sim 1 \rightarrow \text{Hypersonic flow}$$