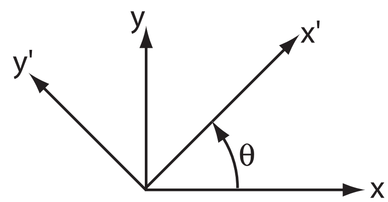
Coordination Transformations for Strain & Stress Rates

To keep the presentation as simple as possible, we will look at purely two-dimensional stress-strain rates. Given an original coordinate system (x, y) and a rotated system ($\hat{x}, \hat{y}$) as shown below:



Recall that the strain rates in the x-y coordinate system are:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}$$

Or, in index notation:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Also, we note that the unit vectors for the rotated axes are:

$$\vec{\hat{i}} = \cos\theta \vec{i} + \sin\theta \vec{j}$$
$$\vec{\hat{j}} = -\sin\theta \vec{i} + \cos\theta \vec{j}$$

Thus, the location of a point in ($\hat{x}, \hat{y}$) is:

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Similarly, the velocity components are related by:

$$\begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

For differential changes, we also have

$$\begin{bmatrix} d\hat{x} \\ d\hat{y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

Thus, defining T as the rotation matrix, we note that:

$$T = \begin{bmatrix} \frac{\partial \hat{x}}{\partial x} & \frac{\partial \hat{x}}{\partial y} \\ \frac{\partial \hat{y}}{\partial x} & \frac{\partial \hat{y}}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Inverting this:

$$T^{-1} = \begin{bmatrix} \frac{\partial x}{\partial \hat{x}} & \frac{\partial x}{\partial \hat{y}} \\ \frac{\partial y}{\partial \hat{x}} & \frac{\partial y}{\partial \hat{y}} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Thus to find $\frac{\partial \hat{u}}{\partial \hat{x}}$ in terms of u, v and their derivatives:

$$\frac{\partial \hat{u}}{\partial \hat{x}} = \frac{\partial \hat{u}}{\partial x}\frac{\partial x}{\partial \hat{x}} + \frac{\partial \hat{u}}{\partial y}\frac{\partial y}{\partial \hat{x}} = \frac{\partial \hat{u}}{\partial x}\cos\theta + \frac{\partial \hat{u}}{\partial y}\sin\theta$$

Then, substituting $\hat{u} = u\cos\theta + v\sin\theta$:

$$\frac{\partial \hat{u}}{\partial \hat{x}} = \left[\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right](u\cos\theta + v\sin\theta)$$

$$= \cos^2\theta\frac{\partial u}{\partial x} + \cos\theta\sin\theta\frac{\partial v}{\partial x} + \cos\theta\sin\theta\frac{\partial u}{\partial y} + \sin^2\theta\frac{\partial v}{\partial y}$$

$$\Rightarrow \boxed{\varepsilon_{\hat{x}\hat{x}} = \cos^2\theta\varepsilon_{xx} + 2\cos\theta\sin\theta\varepsilon_{xy} + \sin^2\theta\varepsilon_{yy}}$$

$$\frac{\partial \hat{v}}{\partial \hat{y}} = \left[-\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}\right](-u\sin\theta + v\cos\theta)$$

$$= \sin^2\theta\frac{\partial u}{\partial x} - \sin\theta\cos\theta\frac{\partial v}{\partial x} - \sin\theta\cos\theta\frac{\partial u}{\partial y} + \cos^2\theta\frac{\partial v}{\partial y}$$

$$\Rightarrow \boxed{\varepsilon_{\hat{y}\hat{y}} = \sin^2\theta\varepsilon_{xx} + 2\sin\theta\cos\theta\varepsilon_{xy} + \cos^2\theta\varepsilon_{yy}}$$

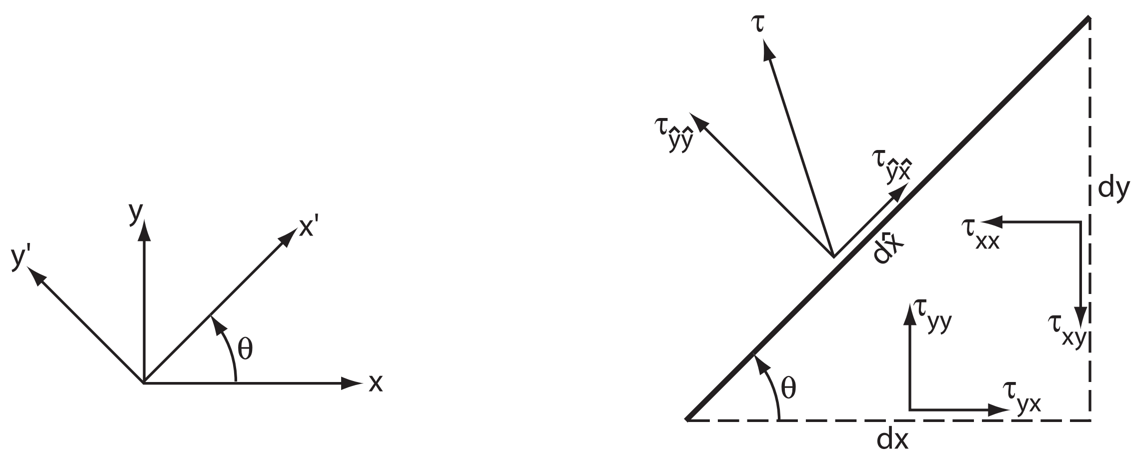
$$\frac{1}{2}\left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}}\right) = \frac{1}{2}\left\{\left[-\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}\right](u\cos\theta + v\sin\theta) + \left[\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right](-\sin\theta + v\cos\theta)\right\}$$

$$\Rightarrow \boxed{\varepsilon_{\hat{x}\hat{y}} = \sin\theta\cos\theta(\varepsilon_{yy} - \varepsilon_{xx}) + (\cos^2\theta - \sin^2\theta)\varepsilon_{xy}}$$

If $x-y$ are the principal strain directions, then $\varepsilon_{xy}=0$ and $\varepsilon_{\hat{x}\hat{x}}, \varepsilon_{\hat{x}\hat{y}}$ & $\varepsilon_{\hat{y}\hat{y}}$ are

$$\boxed{\begin{aligned}\varepsilon_{\hat{x}\hat{x}} &= \cos^2\theta\varepsilon_{xx}+\sin^2\theta\varepsilon_{yy}\\ \varepsilon_{\hat{y}\hat{y}} &= \sin^2\theta\varepsilon_{xx}+\cos^2\theta\varepsilon_{yy}\\ \varepsilon_{\hat{x}\hat{y}} &= \sin\theta\cos\theta(\varepsilon_{yy}-\varepsilon_{xx})\end{aligned}} \quad \text{if } \varepsilon_{xy}=0$$

The next step is to relate the stresses in (x,y) to $(\hat{x},\hat{y})$. Consider a differential surface with $\hat{y}$ normal:



The resultant stress is given as the vector $\vec{\tau}$ and the force on the surface is $\vec{\tau}ds$. Decomposing the stress vector into the coordinate axes gives:

$$\begin{aligned}\vec{\tau}\,d\hat{x} &= \left(\tau_{\hat{y}\hat{x}}\vec{i}+\tau_{\hat{y}\hat{y}}\vec{j}\right)d\hat{x}\\ &= \left(-\tau_{xx}dy+\tau_{yx}dx\right)\vec{i}+\left(\tau_{yy}dx-\tau_{xy}dy\right)\vec{j}\end{aligned}$$

Note that:

$$\begin{aligned}dx &= \cos\theta d\hat{x}\\ dy &= \sin\theta d\hat{x}\\ \vec{i} &= \cos\theta\vec{\hat{i}}-\sin\theta\vec{\hat{j}}\\ \vec{j} &= \sin\theta\vec{\hat{i}}+\cos\theta\vec{\hat{j}}\end{aligned}$$

Thus, the second line becomes:

$$
\begin{aligned}
&(-\tau_{xx}\sin\theta + \tau_{yx}\cos\theta)(\cos\theta\vec{\hat{i}} - \sin\theta\vec{\hat{j}})d\hat{x} \\
&+(\tau_{yy}\cos\theta - \tau_{xy}\sin\theta)(\sin\theta\vec{\hat{i}} + \cos\theta\vec{\hat{j}})d\hat{x} = (\tau_{\hat{y}\hat{x}}\vec{\hat{i}} + \tau_{\hat{y}\hat{y}}\vec{\hat{j}})d\hat{x}
\end{aligned}
$$

So collecting all the $\hat{i}$ & $\hat{j}$ terms (and enforcing $\tau_{xy} = \tau_{yx}$) gives:

$$
\begin{aligned}
\tau_{\hat{y}\hat{x}} &= (\tau_{yy} - \tau_{xx})\sin\theta\cos\theta + \tau_{yx}(\cos^2\theta - \sin^2\theta) \\
\tau_{\hat{y}\hat{y}} &= \tau_{xx}\sin^2\theta + \tau_{yy}\cos^2\theta - 2\tau_{yx}\sin\theta\cos\theta
\end{aligned}
$$

For the principal strain axes,

$$
\begin{aligned}
\tau_{xx} &= 2\mu\varepsilon_{xx} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy}\right) \\
\tau_{yy} &= 2\mu\varepsilon_{yy} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy}\right) \\
\tau_{xy} &= 0
\end{aligned}
$$

Plugging this into $\tau_{\hat{y}\hat{x}}$ and $\tau_{\hat{y}\hat{y}}$ gives

$$
\begin{aligned}
\tau_{\hat{y}\hat{x}} &= 2\mu\underbrace{\left(\varepsilon_{yy} - \varepsilon_{xx}\right)\sin\theta\cos\theta}_{\varepsilon_{\hat{x}\hat{y}}} \\
\tau_{yy} &= 2\mu\underbrace{\left(\varepsilon_{xx}\sin^2\theta + \varepsilon_{yy}\cos^2\theta\right)}_{\varepsilon_{\hat{y}\hat{y}}} + \lambda\underbrace{\left(\varepsilon_{xx} + \varepsilon_{yy}\right)}_{\varepsilon_{\hat{x}\hat{x}} + \varepsilon_{\hat{y}\hat{y}}}
\end{aligned}
$$

Thus, we arrive at the known result:

$$
\begin{aligned}
\tau_{\hat{y}\hat{x}} &= 2\mu\,\varepsilon_{\hat{x}\hat{y}} \\
\tau_{\hat{y}\hat{y}} &= 2\mu\,\varepsilon_{\hat{y}\hat{y}} + \lambda\left(\varepsilon_{\hat{x}\hat{x}} + \varepsilon_{\hat{y}\hat{y}}\right)
\end{aligned}
$$

A similar derivation would give:

$$
\tau_{\hat{x}\hat{x}} = 2\mu\varepsilon_{\hat{x}\hat{x}} + \lambda\left(\varepsilon_{\hat{x}\hat{x}} + \varepsilon_{\hat{y}\hat{y}}\right)
$$