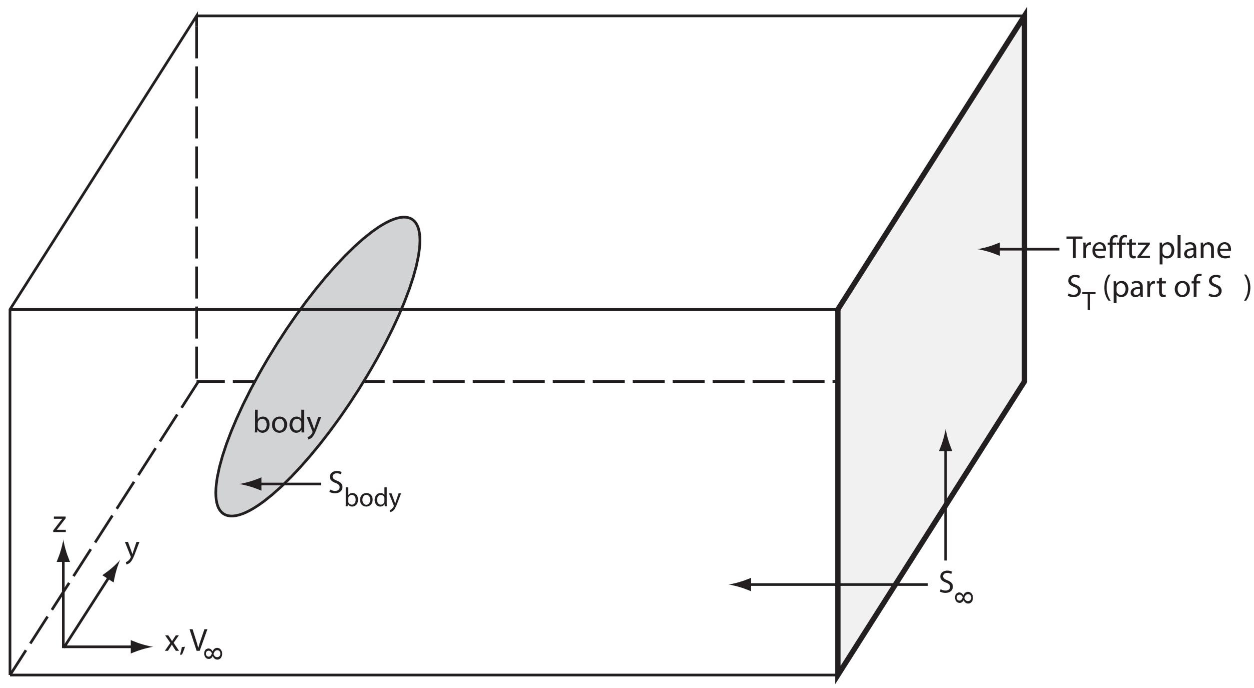
Trefftz Plane Analysis of Induced Drag

Consider an inviscid, incompressible potential flow around a body (say a wing). We define a control volume surrounding the body as follows



Upstream flow is V_∞ and is in $x-$direction. Thus, drag is the force in $x-$direction. Apply integral momentum in x to find induced drag.

$$\iint_{S_{body}+S_\infty} \rho \vec{u}\vec{u}\cdot\vec{n}\,dS = -\iint_{S_{body}+S_\infty} p\vec{n}\,dS$$

First, on the body $\vec{u}\cdot\vec{n}=0$, so:

$$\iint_{S_\infty} \rho \vec{u}\vec{u}\cdot\vec{n}\,dS = -\iint_{S_{body}+S_\infty} p\vec{n}\,dS$$

Next, also on the body,

$$-\iint_{S_{body}} p\vec{n}\,dS = \text{ force of body acting on fluid}$$

We are interested in the exact opposite, i.e. the force acting on the body. In x, this is the drag, in z this is the lift, and in y this is a yaw or side force:

$$\Rightarrow \qquad -\iint_{S_{body}} p\vec{n}dS = -D\vec{i} - Y\vec{j} - L\vec{k}$$

$$\Rightarrow \qquad D\vec{i} + Y\vec{j} + L\vec{k} = -\iint_{S_\infty} p\vec{n}dS - \iint_{S_\infty} \rho\vec{u}\,\vec{u}\cdot\vec{n}dS$$

Now, let's pull out the drag:

$$D = -\iint_{S_\infty} p\vec{n}\cdot\vec{i}\,dS - \iint_{S_\infty} \rho\vec{u}\,\vec{u}\cdot\vec{n}dS$$

The next piece is to apply Bernoulli to eliminate the pressure:

$$p = p_\infty + \frac{1}{2}\rho V_\infty^2 - \frac{1}{2}\rho(u^2 + v^2 + w^2)$$

$$\Rightarrow \qquad D = -\iint_{S_\infty}\left[p_\infty + \frac{1}{2}\rho V_\infty^2 - \frac{1}{2}\rho(u^2 + v^2 + w^2)\right]\vec{n}\cdot\vec{i}\,dS - \iint_{S_\infty}\rho u\,\vec{u}\cdot\vec{n}dS$$

But, $\displaystyle\iint_{S_\infty}(p_\infty + \frac{1}{2}\rho V_\infty^2)\vec{n}\cdot\vec{i}dS = (p_\infty + \frac{1}{2}\rho V_\infty^2)\underbrace{\iint_{S_\infty}\vec{n}\cdot\vec{i}dS}_{=\,0\text{ for a closed surface}}$

$$D = \iint_{S_\infty}\frac{1}{2}\rho(u^2 + v^2 + w^2)\vec{n}\cdot\vec{i}\,dS - \iint_{S_\infty}\rho u\,\vec{u}\cdot\vec{n}dS$$

Next, we divide the velocity into a freestream and a perturbation:

$$\begin{aligned} u &= V_\infty + \hat{u} \\ v &= \hat{v} \\ w &= \hat{w} \end{aligned}$$

where $\hat{u}, \hat{v}, \hat{w}$ are perturbation velocities (not necessarily small).

Substitution gives:

$$D = \frac{1}{2}\rho\iint_{S_\infty}(V_\infty^2 + 2V_\infty\hat{u} + \hat{u}^2 + \hat{v}^2 + \hat{w}^2)\vec{n}\cdot\vec{i}dS - \rho\iint_{S_\infty}(V_\infty + \hat{u})\vec{u}\cdot\vec{n}dS$$

But, we note that

$\displaystyle\rho\iint_{S_\infty}V_\infty\vec{u}\cdot\vec{n}dS = \rho V_\infty\iint_{S_\infty}\vec{u}\cdot\vec{n}dS = 0$ from conservation of mass

$$\Rightarrow \quad \boxed{D = \rho V_\infty \iint_{S_\infty} \hat{u}\,\vec{n}\cdot\vec{i}\,dS + \frac{1}{2}\rho \iint_{S_\infty} (\hat{u}^2 + \hat{v}^2 + \hat{w}^2)\vec{n}\cdot\vec{i}\,dS - \rho \iint_{S_\infty} \hat{u}\,\vec{u}\cdot\vec{n}dS}$$

If we take the control volume boundary far away from the wing, then the velocity perturbations go to zero except downstream. Downstream the presence of trailing vortices will create non-zero perturbations (more on this in a bit).

So, $\hat{u}, \hat{v}, \hat{w} \rightarrow 0$ except on S_T.

$$\Rightarrow \quad D = \rho V_\infty \cancel{\iint_{S_T} \hat{u} dS} + \frac{1}{2}\rho \iint_{S_T} (\hat{u}^2 + \hat{v}^2 + \hat{w}_2) dS - \rho \iint_{S_T} \hat{u}\,(\cancel{V_\infty} + \hat{u}) dS$$

$$\Rightarrow \quad \boxed{D = \frac{1}{2}\rho \iint_{S_T} (\hat{v}^2 + \hat{w}^2 - \hat{u}^2) dS}$$

The final step is to note that far downstream the $x-$velocity perturbation must die away (in inviscid flow). The reason is that the trailing vortices, which far downstream must be in the $x-$direction, cannot induce an $x-$component of velocity.

So, this brings us to the final answer

$$\boxed{D = \frac{1}{2}\rho \iint_{S_T} (\hat{v}^2 + \hat{w}^2) dS}$$

In other words, the induced drag is the kinetic energy which is transferred into the crossflow (i.e. the trailing vortices)!