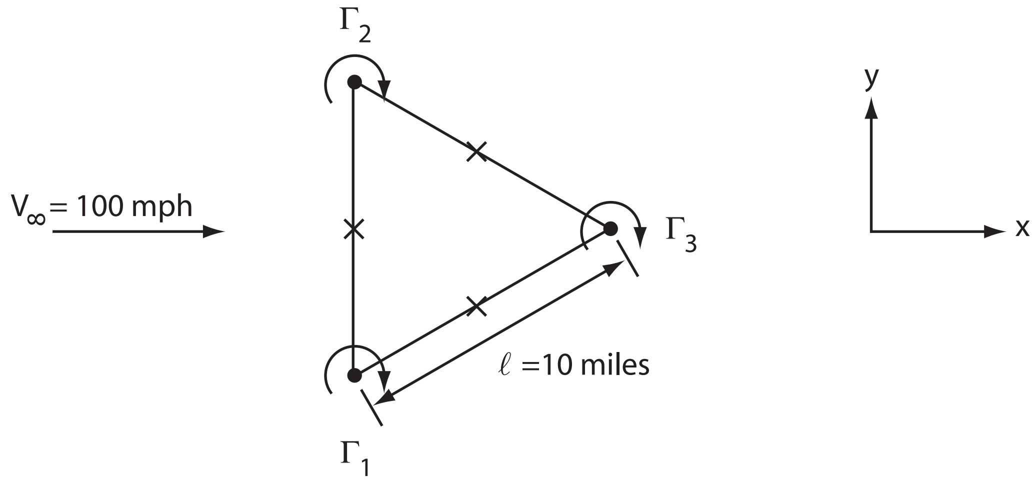
Solution



- $\vec{u} \cdot \vec{n} = 0$ at control pt #1:

  The velocity at control pt #1 is the sum of the freestream + 3 point vortices' velocities at that point:

$$\vec{u}_1 = V_\infty \vec{i} + \frac{\Gamma_1}{2\pi\left(\frac{\ell}{2}\right)}\vec{i} - \frac{\Gamma_2}{2\pi\left(\frac{\ell}{2}\right)}\vec{i} + \frac{\Gamma_3}{2\pi\left(\frac{\ell}{2}\right)}\vec{j}$$

  The normal at control pt #1 is:

$$\vec{n}_1 = -\vec{i}$$

$$\Rightarrow \vec{u}_1 \cdot \vec{n}_1 = -V_\infty - \frac{\Gamma_1}{2\pi\left(\frac{\ell}{2}\right)} + \frac{\Gamma_2}{2\pi\left(\frac{\ell}{2}\right)} = 0$$

  Rearranging:

$$\boxed{\frac{\Gamma_1}{\pi\ell} - \frac{\Gamma_2}{\pi\ell} = -V_\infty} \qquad (1)$$

- $\vec{u} \cdot \vec{n} = 0$ at control pt #2:

  Now, following the same procedure for control pt #2:

$$\vec{n}_2 = \frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}$$

$$\vec{u}_2 \cdot \vec{n}_2 = \frac{V_\infty}{2} - \frac{\Gamma_2}{\pi\ell} + \frac{\Gamma_3}{\pi\ell} = 0$$

$$\boxed{\frac{\Gamma_2}{\pi\ell} - \frac{\Gamma_3}{\pi\ell} = \frac{V_\infty}{2}} \qquad (2)$$

- $\vec{u} \cdot \vec{n} = 0$ at control pt #3:

$$\vec{n}_3 = \frac{1}{2}\vec{i} - \frac{\sqrt{3}}{2}\vec{j}$$

$$\vec{u}_3 \cdot \vec{n}_3 = \frac{V_\infty}{2} + \frac{\Gamma_1}{\pi\ell} - \frac{\Gamma_3}{\pi\ell} = 0$$

$$\Rightarrow \boxed{-\frac{\Gamma_1}{\pi\ell} + \frac{\Gamma_3}{\pi\ell} = \frac{V_\infty}{2}} \qquad (3)$$

Final System of Equations

Combine the numbered equations:

$$\underbrace{\begin{bmatrix} \frac{1}{\pi\ell} & -\frac{1}{\pi\ell} & 0 \\ 0 & \frac{1}{\pi\ell} & -\frac{1}{\pi\ell} \\ -\frac{1}{\pi\ell} & 0 & \frac{1}{\pi\ell} \end{bmatrix}}_{\text{Influence matrix}} \underbrace{\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{bmatrix}}_{\substack{\text{vortex} \\ \text{strengths} \\ \text{(unknowns)}}} = \underbrace{\begin{bmatrix} -V_\infty \\ \frac{V_\infty}{2} \\ \frac{V_\infty}{2} \end{bmatrix}}_{V_\infty \bullet \vec{n}_i}$$

The problem with these equations is that they have infinitely many solutions. One clue is that the determinant of the matrix is zero. In particular we can add a constant strength to any solution because:

$$\begin{bmatrix} & & \\ & \text{influence matrix} & \\ & & \end{bmatrix} \begin{bmatrix} \Gamma_0 \\ \Gamma_0 \\ \Gamma_0 \end{bmatrix} = 0$$

$\Rightarrow$ Given a solution $\begin{bmatrix}\Gamma_0\\ \Gamma_0\\ \Gamma_0\end{bmatrix}$, then

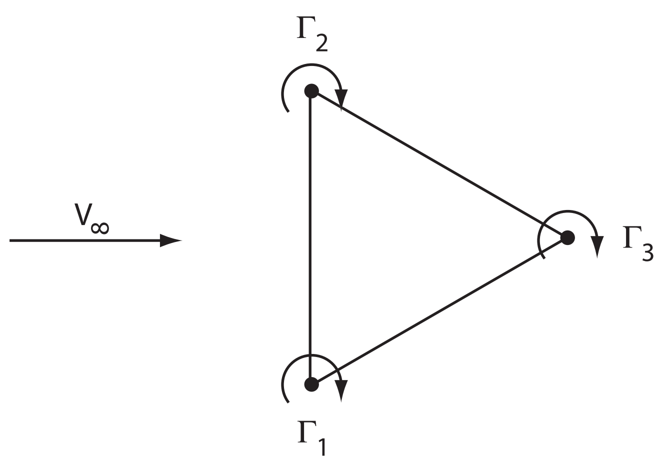
$\begin{bmatrix}\Gamma_1\\ \Gamma_2\\ \Gamma_3\end{bmatrix}+\begin{bmatrix}\Gamma_0\\ \Gamma_0\\ \Gamma_0\end{bmatrix}$ is also a solution where Γ_0 is arbitrary.

So, how do we resolve this?
Answer: the Kutta condition!

$$p_{t.e.}+\frac{1}{2}\rho V_{upper}^2 = p_{t.e.}+\frac{1}{2}\rho V_{lower}^2$$

$$\Rightarrow \quad \boxed{V_{upper}=V_{lower}\neq 0}$$

What's the Kutta condition for the windy city problem:



Kutta: $\Gamma_3 = 0 \Rightarrow$ no flow around node 3!

So, we can now solve our system of equations starting with $\Gamma_3 = 0$

$$\Rightarrow \boxed{\begin{aligned}\Gamma_1 &= -\frac{\pi}{2}V_\infty \ell\\ \Gamma_2 &= \frac{\pi}{2}V_\infty \ell\\ \Gamma_3 &= 0\end{aligned}}$$