16.06 Principles of Automatic Control
Recitation 10

Design a compensator strategy for a system

$$G(s) = 5\frac{\left(1 - \frac{s}{300}\right)}{\left(1 + \frac{s}{20}\right)\left(1 + \frac{s}{4}\right)}$$

to have a closed loop system that will have PM = 45°, and rise time as fast as possible.

Let's first identify the "big picture" strategy. In typical problems, there is no limit on ω_c (t_r as fast as possible would mean ω_c as high as possible, since $t_r \approx \frac{1}{\omega_c}$). But because of the RHP zero, we would need to set ω_c at $\frac{300}{2}$ or $\frac{300}{3}$, which would provide 20% undershoot, which is too much (not accessible), so let's aim to set ω_c at 100 rad/sec.

So the big picture strategy consists of two parts:

1. Design lead compensator in the vicinity of ω_c.
2. Then design a lag compensator to get the desired gain at low frequency, at $\omega \sim \frac{\omega_c}{10}$.

Therefore, since we will add a lag compensator, with a zero a decade before the crossover frequency, we will need to account for $\sim 6°$ extra phase when designing the lead compensator (that we will lose due to the phase lag contribution of the lag compensator).

Step 1. Lead compensator at $\omega_c = 100\,\text{rad/sec}$:

$$K\frac{1 + \frac{s}{a}}{1 + \frac{s}{b}}$$

$$\begin{aligned}\text{phase of } G(s)\Big|_{\omega=100} &= -\tan^{-1}\left(\frac{100}{20}\right) - \tan^{-1}\left(\frac{100}{4}\right) - \tan^{-1}\left(\frac{100}{300}\right)\\ &= -78.69° - 87.71° - 18.43°\\ &= -184.83°\end{aligned}$$

and we need a PM = 45°.

$\therefore$ we need the compensator to provide

$45^\circ + 6^\circ + (-180^\circ - (-184.83^\circ)) = 55.83^\circ$

$\therefore\ 55.83^\circ = 2\tan^{-1}\left(\sqrt{\frac{b}{a}}\right) - 90^\circ$

$\Rightarrow \sqrt{\frac{b}{a}} = 3.25$ and, since $\omega_c = 100 = \sqrt{(ab)}$:

$\begin{cases} b = & 325 \\ a = & 30.77 \end{cases}$

To find K, we use the magnitude condition:

$$1 = |K(s)G(s)|_{\omega_c=100} = 5\frac{\sqrt{1+(\frac{100}{300})^2}}{\sqrt{1+5^2}\sqrt{1+25^2}} \cdot \frac{\sqrt{1+(\frac{100}{30.77})^2}}{\sqrt{1+(\frac{100}{325})^2}} \cdot K$$

$$\therefore\ K \sim 7.45$$

$\therefore$ lead compensator:

$$K(s) = 7.45\frac{(1+\frac{s}{30.77})}{(1+\frac{s}{325})}$$

Step 2. Design the lag compensator: <u>zero</u> 1 decade below crossover:

$$\frac{s+a}{s+b} \rightarrow \frac{s+10}{s+b}$$

but low frequency gain is $\sim 7.45 \cdot 5 \sim 37.25\,\text{rad/s}$.

$$K_{p\,\text{requirement}} \rightarrow \text{need } 200 \rightarrow \text{lag ratio} \sim 5.37 = \frac{10}{b} \rightarrow b \sim 1.86$$

$\therefore$ lag compensator:

$$\frac{s+10}{s+1.86}$$

$$\boxed{K_{\text{fin}}(s) = \frac{7.45\left(1+\frac{s}{30.77}\right)}{\left(1+\frac{s}{325}\right)}\frac{(s+10)}{(s+1.86)}}$$

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