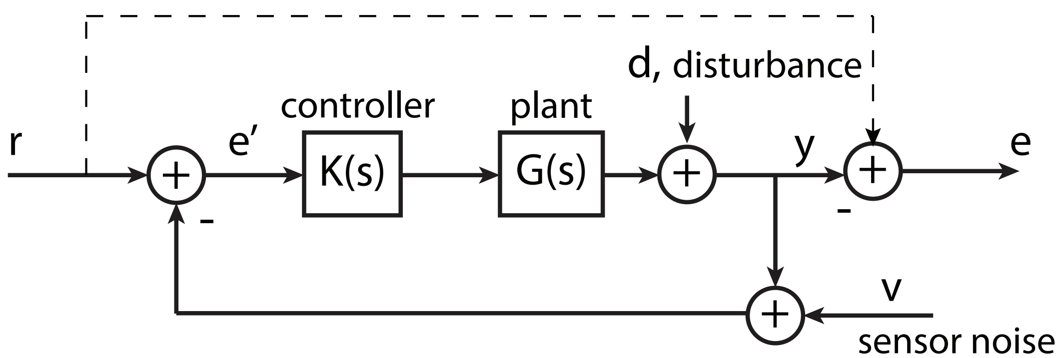
16.06 Principles of Automatic Control
Lecture 9

Unity Feedback Control With Noise

Consider a typical unity feedback control system



e' is the error perceived by the control system; e is the actual error. The important transfer functions are

$$\begin{aligned}
\frac{Y}{R}(s) &= \frac{1}{1+K(s)G(s)} \equiv S(s) \\
\frac{E}{D}(s) &= \frac{-1}{1+K(s)G(s)} \equiv -S(s) \\
\frac{E}{V}(s) &= \frac{-K(s)G(s)}{1+K(s)G(s)} \equiv -T(s)
\end{aligned}$$

$S(s)$ =Sensitivity transfer function

$T(s)$ =Complementary Sensitivity transfer function

For low sensitivity to disturbances, want:

$$|S(s)| \ll 1$$

For good tracking of the reference input, want:

$$|S(s)| \ll 1$$

For low sensitivity to sensor noise or errors, want:

$$|T(s)| \ll 1$$

But these goals are mutually exclusive, since

$$S(s) + T(s) = 1$$

So there is a fundamental trade-off between good tracking performance and low sensitivity to sensor noise.

How is this trade-off addressed?

In most (not all) systems, want good tracking performance at low frequency, low sensitivity to sensor noise at high frequency:

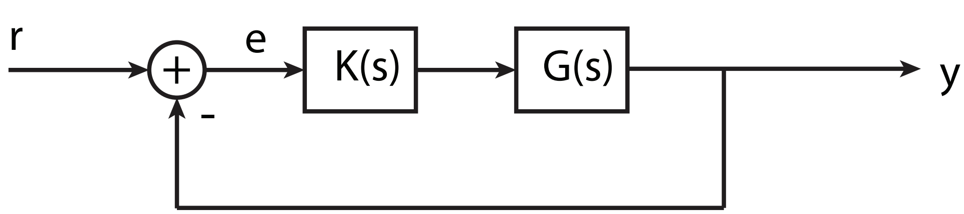
- Reference inputs are low frequency

- Sensor noise is usually high frequency

So let's look at the lowest frequency, $\omega = 0$ ($s = 0$)...

Steady-State Errors

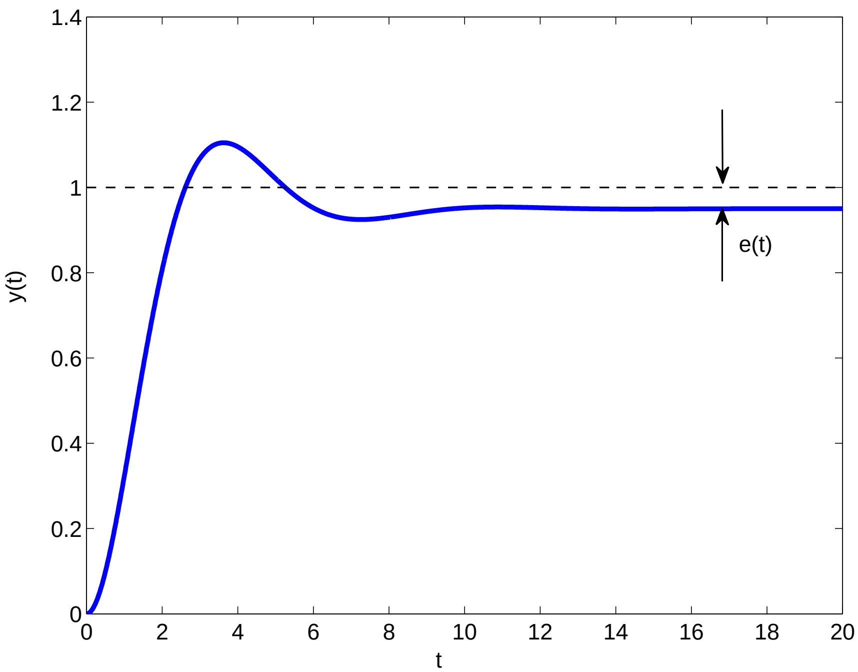
Consider a unity feedback system without sensor noise or disturbance:



For stability, define

$$L(s) = K(s)G(s) = \text{"Loop Gain"}$$

What is the steady-state error to a unit step input?



Use LTs:

$$E(s) = S(s)R(s) = \frac{1}{1+L(s)}R(s) = \frac{1}{1+L(s)}\frac{1}{s}$$

To find the *steady* error, use final value theorem:

$$\lim_{s\to 0} e(t) = \lim_{s\to 0} sE(s) = \frac{1}{1+L(0)}$$

If L(0) is finite, we define

$$K_p \equiv L(0) = \text{"positive error constant"}$$

Furthermore, if L(0) is finite, we say that a system is a “type 0 system”.

So a type 0 system will always have a finite error in response to a steady input r, but the error can be made small by making the position error constant large.

To make the steady error zero, we must have that L(0) is *infinite*. Suppose we can express L(s) as

$$L(s) = \frac{L_0(0)}{s}$$

where $L_0(0) \neq 0$, $L_0(0)$ is finite. Then L is a “type 1 system” (one pole at $s = 0$). We have that

$$\lim_{s\to 0} e(t) = \lim_{s\to 0} s \frac{1}{1+\frac{L_0(s)}{s}} \frac{1}{s} = \lim_{s\to 0} \frac{s}{s+L_0(s)} = 0$$

since $L_0(0) \neq 0$.

What if we want to track a unit ramp instead?

$$\begin{aligned} r(t) =& tr(t) \\ \Rightarrow R(s) =& \frac{1}{s^2} \end{aligned}$$

The steady-state error for a type 0 system will be

$$\begin{aligned} e_{ss} =& \lim_{s\to 0} sS(s)R(s) \\ =& \lim_{s\to 0} s \frac{1}{1+L(s)} \frac{1}{s^2} \\ =& \lim_{s\to 0} \frac{1}{1+L(0)} \frac{1}{s} = \infty \end{aligned}$$

The steady-state error for a type 1 system will be

$$\begin{aligned} e_{ss} =& \lim_{s\to 0} sS(s)R(s) \\ =& \lim_{s\to 0} s \frac{1}{1+\frac{L_0(s)}{s}} \frac{1}{s^2} \\ =& \lim_{s\to 0} \frac{1}{s+L_0(s)} \\ =& \frac{1}{L_0(s)} \end{aligned}$$

which is finite. We define

$$K_v = L_0(s) = \text{"velocity error constant"}$$

More generally, suppose that L(s) has the form

$$L(s) = \frac{L_0(s)}{s^n}$$

L is said to be a type n system, and the error constant is K_p,or K_v or $K_a ... = L_0(0)$.

$$
\begin{aligned}
K_p =& K_0 = \lim_{s \to 0} L(s), \;\; n = 0 \\
K_v =& K_1 = \lim_{s \to 0} sL(s), \; n = 1 \\
K_a =& K_2 = \lim_{s \to 0} s^2 L(s), \; n = 2 \\
&\vdots
\end{aligned}
$$

Input			
Type	$\sigma(\mathbf{t})$	$\underline{\mathbf{t}\sigma(\mathbf{t})}$	$\frac{\mathbf{t^2}}{\mathbf{2}}\sigma(\mathbf{t})$
Type 0	$e_{ss} = \frac{1}{1+K_p}$	$e_{ss} = \infty$	$e_{ss} = \infty$
Type 1	0	$\frac{1}{K_v}$	∞
Type 2	0	0	$\frac{1}{K_a}$

Obviously, this generalizes, but we usually care most about K_p and K_v - higher order inputs are rare.