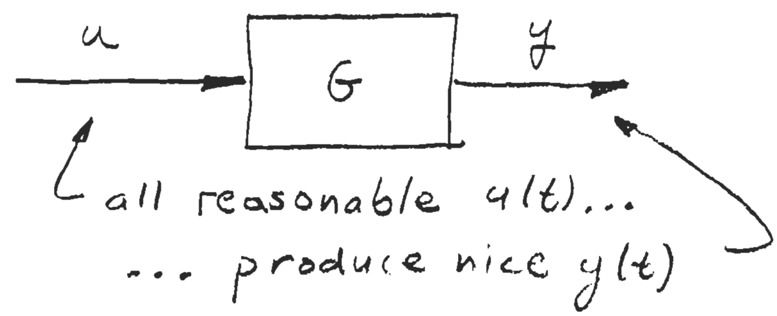
LECTURE S13

Stability

What does it mean for a system to be "stable"?

Be careful — there are many definitions of stability!

One (loose) definition: A system is stable if "reasonable" inputs produce "nice" outputs



Need to define "reasonable" or "nice."

Bounded Signals

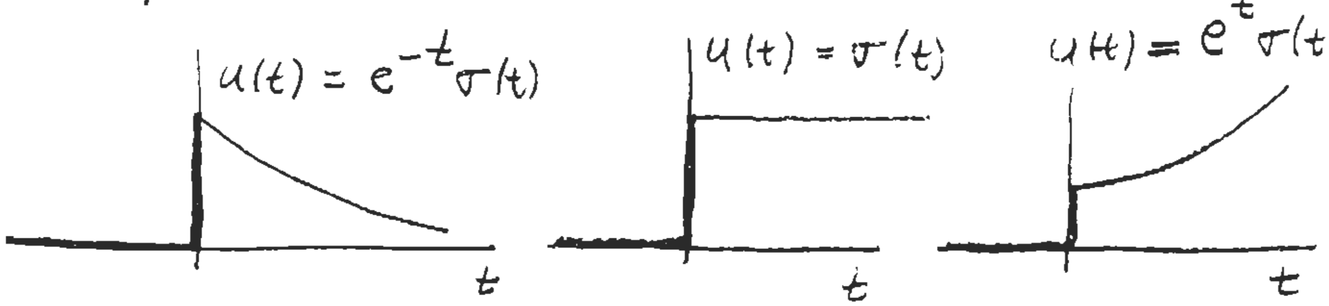
A signal $u(t)$ is **bounded** if

$$|u(t)| \le A < \infty, \quad \text{all } t$$

Bounded signals don't "blow up" as $t \to \infty$.

Bounded signals are (to some people) "reasonable" or "nice."

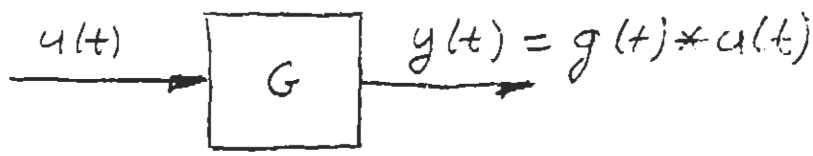
Examples



Bounded Bounded **Unbounded!**

Bounded Input/Bounded Output (BIBO) Stability

The system



is BIBO stable if all bounded inputs $u(t)$ produce bounded outputs $y(t)$.

[Nice inputs produce nice outputs]

Result: The system G is stable if and only if

$$\int_0^\infty |g(t)|\, dt = M < \infty$$

Note: Stability has to depend on $g(t)$ somehow – $g(t)$ completely characterizes the system.

<u>Proof</u> Suppose $\int_0^\infty |g(t)|\, dt = M < \infty$,

and $u(t)$ is bounded, so that

$$|u(t)| \leq A < \infty, \quad \text{all } t$$

Then
$$
\begin{aligned}
|y(t)| &= \left| \int_0^\infty g(\tau)\, u(t-\tau)\, d\tau \right| \\
&\leq \int_0^\infty |g(\tau)\, u(t-\tau)|\, d\tau \\
&= \int_0^\infty |g(\tau)|\,|u(t-\tau)|\, d\tau \\
&\leq \int_0^\infty |g(\tau)|\, d\tau \cdot A \\
&= MA < \infty
\end{aligned}
$$

So bounded input $\Rightarrow$ bounded output.

Suppose $\int_0^\infty |g(t)|\, dt = \infty$. I claim I can find a bounded $u(t)$ that makes $g(t)$ unbounded. Take

$$
u(t) = \begin{cases} 1, & g(-t) > 0 \\ -1, & g(-t) < 0 \\ 0, & g(-t) = 0 \end{cases}
$$

Clearly, $u(t)$ is bounded.

$$\text{Then} \quad y(0) = \int_0^{\infty} g(\tau)\, u(0-\tau)\, d\tau$$

$$= \int_0^{\infty} g(\tau)\, u(-\tau)\, d\tau$$

$$= \int_0^{\infty} |g(\tau)|\, d\tau = \infty$$

So if $\int_0^{\infty} |g(\tau)|\, d\tau = \infty$, can find bounded $u(t)$ that produces unbounded $y(t)$.

So a system is BIBO stable $\Longleftrightarrow$ $\int_0^{\infty} |g(t)|\, dt < \infty$. $\blacksquare$

What does all this have to do with Laplace Transforms?

<u>Fact</u>: The system G is stable if and only if the r.o.c. of the Laplace Transform $G(s)$ converges for $\mathrm{Re}[s] = \sigma = 0$.

<u>Proof</u>: The LT converges for $\mathrm{Re}[s] = 0$

$$\Longleftrightarrow \int_0^{\infty} g(t)\, e^{j\omega t}\, dt \quad \text{converges}$$

$$\Longleftrightarrow \int_0^{\infty} \left| g(t)\, e^{j\omega t} \right| dt < \infty \quad \text{(absolute convergen}$$

$$\Longleftrightarrow \int_0^\infty |g(t)|\,dt < \infty \qquad (|e^{j\omega t}| = 1)$$

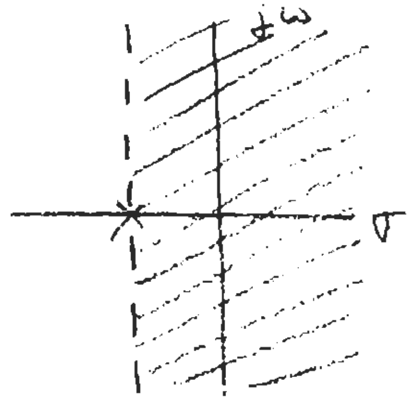
$$\Longleftrightarrow \text{ G BIBO stable.}$$

<u>Examples</u>

$$G(s) = \frac{1}{s+1}, \quad s > -1$$

$$g(t) = e^{-t}\sigma(t)$$

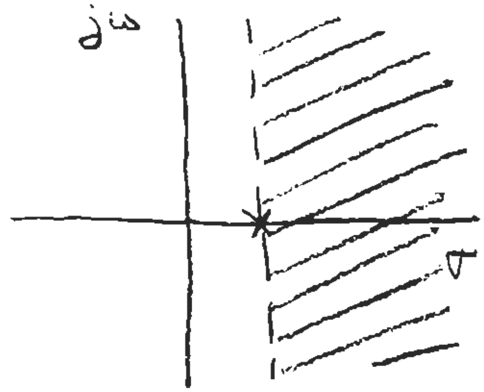
BIBO stable



$$G(s) = \frac{1}{s-1}, \quad s > 1$$

$$g(t) = e^{t}\sigma(t)$$

<u>Not</u> BIBO stable



$$G(s) = 1/\sqrt{s}, \quad s > 0$$

$$g(t) = ?$$

BIBO **Unstable!**