LECTURE S9

Selected Laplace Transforms

1. $g(t) = \delta(t)$

$$G(s) = \mathcal{L}[\delta(t)] = \int_0^\infty \delta(t)\, e^{-st}\, dt$$

$$= \int_0^\infty \delta(t)\, e^{-s\cdot 0}\, dt \qquad \text{(sifting property)}$$

$$= \int_0^\infty \delta(t)\, dt = \underline{1}$$

$$\boxed{\mathcal{L}[\delta(t)] = 1, \quad \text{all } s}$$

Intepretation:

For a system with impulse response $g(t)$ = impulse, $G(s) = 1$, so

- Every exponential in produces the same exponential out.
- In fact, this is true for every signal – $y(t) = g(t) * u(t) = \delta(t) * u(t) = u(t)$.

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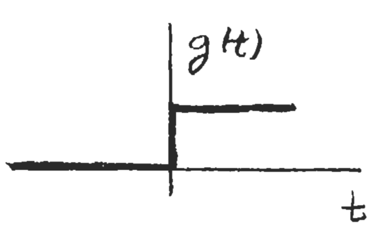
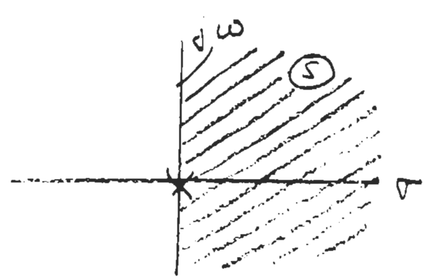
2. Unit step: $g(t) = \sigma(t)$

$$G(s) = \mathcal{L}[\sigma(t)] = \frac{1}{s}, \quad \mathrm{Re}[s] > 0$$





Note that $\sigma(t) = \int^{t} \delta(\tau)\, d\tau$, so G is an "integrator."

$$\int^{t} e^{s\tau}\, d\tau = \underbrace{\frac{1}{s}}_{G(s)} e^{st}$$

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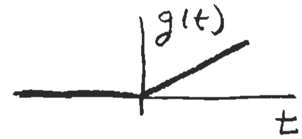
3. Unit ramp: $g(t) = u_{-2}(t)$



$$= \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

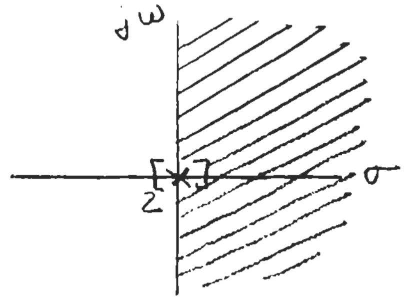
$$G(s) = \mathcal{L}[t] = \int_0^{\infty} t\, e^{-st}\, dt$$

Integrate by parts:

$$G(s) = t\left(\frac{-1}{s}\right) e^{-st} \Big|_0^{\infty} - \int_0^{\infty} 1 \cdot \left(\frac{-1}{s}\right) e^{-st}\, dt$$

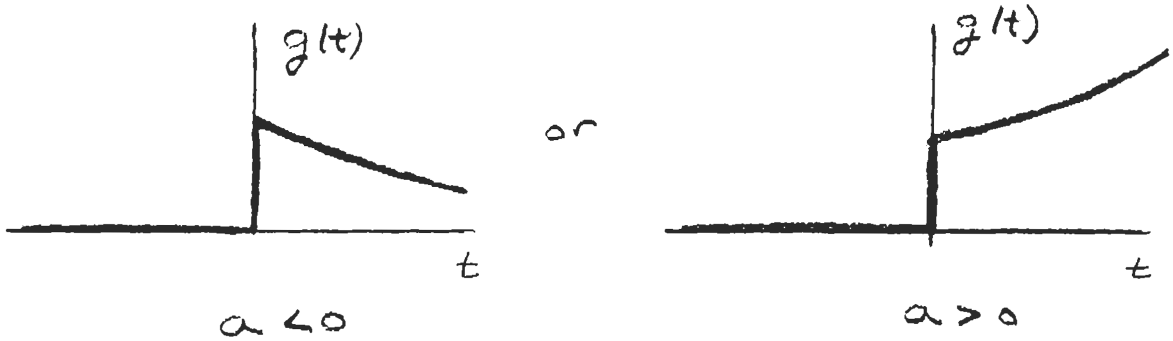
$$= \underbrace{0}_{\text{if } s>0} - \underbrace{0 + \frac{1}{s^2}}_{\text{if } s>0}$$

$$= \frac{1}{s^2}, \quad \mathrm{Re}[s] > 0$$



This is a "double integrator"

4. Exponential: $g(t) = e^{at} \cdot \sigma(t)$



$$G(s) = \int_0^\infty e^{at}\, e^{-st}\, dt$$

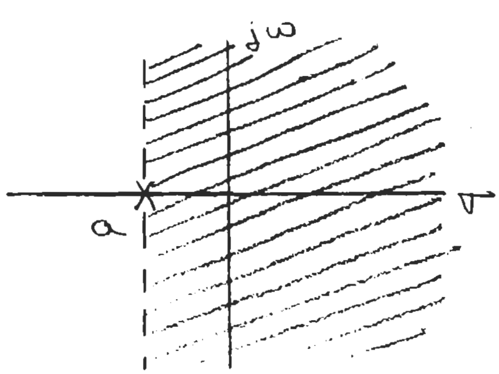
$$= \int_0^\infty e^{(a-s)t}\, dt$$

$$= \frac{1}{a-s} e^{(a-s)t} \Big|_0^\infty$$

$$= 0 - \frac{1}{a-s}$$

if $\mathrm{Re}[a-s] < 0$

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}, \quad \mathrm{Re}[s] > \mathrm{Re}[a]$$



5. Cosine: $g(t) = \cos \omega_o t$

$$G(s) = \int_0^\infty \cos \omega_o t \, e^{-st} \, dt$$

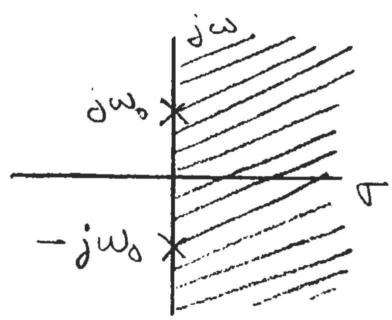
$$= \int_0^\infty \frac{1}{2} \left(e^{j\omega_o t} + e^{-j\omega_o t} \right) e^{st} \, dt$$

$$= \frac{1}{2} \left(\frac{1}{s - j\omega_o} + \frac{1}{s + j\omega_o} \right)$$

$$[\mathrm{Re}(s) > 0] \quad [\mathrm{Re}(s) > 0]$$

Therefore,

$$\mathcal{L}[\cos \omega_o t] = \frac{s}{s^2 + \omega_o^2}, \quad \mathrm{Re}[s] > 0$$



6. Sine: $g(t) = \sin \omega_o t$ (similar to cosine)

$$\mathcal{L}[\sin \omega_o t] = \frac{\omega_o}{s^2 + \omega_o^2}, \quad \mathrm{Re}[s] > 0$$

Properties of the Laplace Transform

1. Linearity:

$$\mathcal{L}[\alpha f(t) + \beta g(t)]$$

$$= \int_0^\infty (\alpha f(t) + \beta g(t)) e^{-st} dt$$

$$= \alpha \int_0^\infty f(t) e^{-st} dt + \beta \int_0^\infty g(t) e^{-st} dt$$

$$= \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)]$$

$\Rightarrow \mathcal{L}[\cdot]$ is a <u>linear</u> operation.

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2. LT of a derivative:

$$\mathcal{L}[g'(t)] = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{g'(t)\, dt}_{dv}$$

$$= uv \Big|_0^\infty - \int v\, du$$

$$= e^{-st} g(t) \Big|_0^\infty + \int_0^\infty g(t) s e^{-st} dt$$

$$= 0 - g(0) + s\mathcal{L}[g(t)]$$

if $e^{-st} g(t) \to 0$

So, $\boxed{\mathcal{L}[g'(t)] = sG(s) - g(0)}$

Higher derivatives:

$$\mathcal{L}[g''(t)] = s\,\mathcal{L}[g'(t)] - g'(0)$$

$$= s\left[s\mathcal{L}[g(t)] - g(0)\right] - g'(0)$$

$$= s^2 G(s) - sg(0) - g'(0)$$

This generalizes easily to higher derivatives.

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3. The LT of an integral:

Let $f(t) = \int_0^t g(\tau)\,d\tau$

Then $f'(t) = g(t)$

$$\Rightarrow G(s) = sF(s) - \cancel{f(0)}$$ $= 0$ automatically.

Therefore, $F(s) = \frac{1}{s}G(s)$, so

$$\mathcal{L}\left[\int^t g(t)\right] = \frac{1}{s}G(s) = \frac{1}{s}\mathcal{L}[g(t)]$$

So $\frac{1}{s} \sim$ integrator.

<u>Very important</u> – integrators are common in control.

<u>Example</u> $\int^t \sigma(t) = \int^t 1 = t$

$$\Rightarrow \mathcal{L}[t] = \frac{1}{s}\,\mathcal{L}[\sigma(t)] = \frac{1}{s}\cdot\frac{1}{s}$$

$$= \frac{1}{s^2}, \quad \text{as before}$$

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4. The LT of $e^{at} g(t)$:

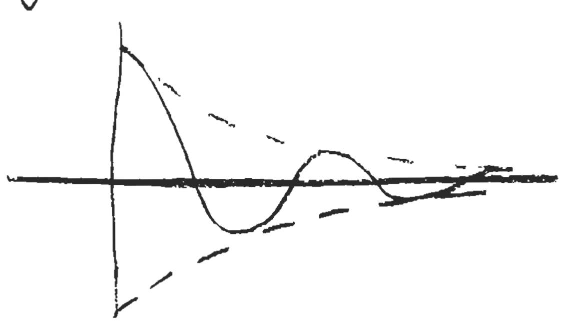
$$\mathcal{L}[e^{at} g(t)] = \int_0^\infty e^{at} g(t) e^{-st}\, dt$$

$$= \int_0^\infty g(t) e^{(-s+a)t}\, dt$$

$$= G(s-a)$$

<u>Example</u> Damped sinusoid

$$g(t) = e^{-\alpha t} \cos \omega_0 t$$



$$G(s) = \frac{s+\alpha}{(s+\alpha)^2 + \omega_0^2}, \qquad \mathrm{Re}[s] > -\alpha$$