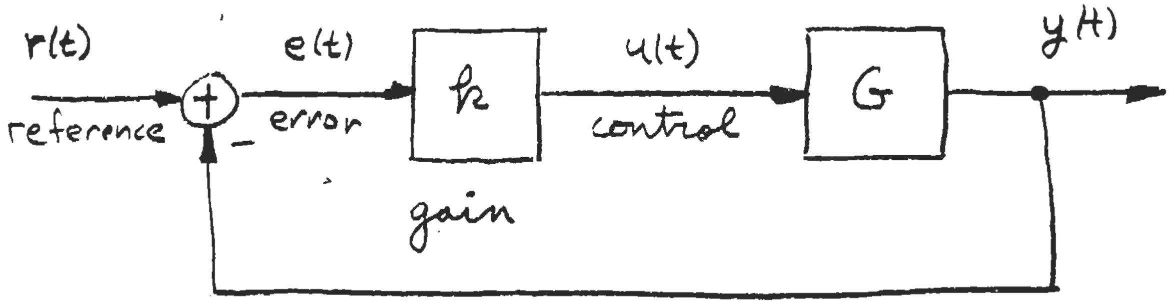
LECTURE S8

Limitations of Convolution

Convolution is a powerful concept, but is limited in practical application.

Typical control system:



Simple question: given $y(t)$, what is $r(t)$?

$$
\begin{aligned}
y(t) &= g(t) * u(t) \\
&= g(t) * (k\,e(t)) = k\,g(t) * e(t) \\
&= k\,g(t) * [r(t) - y(t)]
\end{aligned}
$$

$$\Rightarrow \underline{\underline{y(t)}} = k\,g(t) * r(t) - k\,g(t) * \underline{\underline{y(t)}}$$

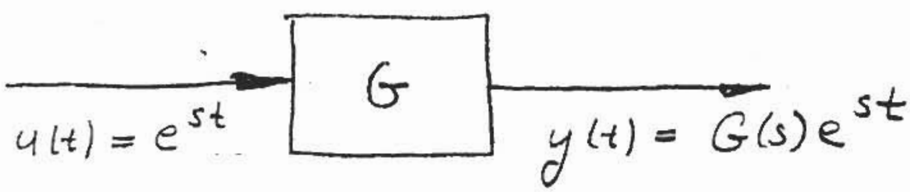
We can't solve for $y(t)$ — we can't unwrap convolution.

This is where <u>transfer</u> <u>functions</u> will come in.

Last term, looked at response to exponential inputs. This will solve our problem here!

<u>Response to Exponential Inputs</u>

Conside a system with exponential input:



$G(s)$ is the <u>transfer</u> <u>function</u> of the system G.

- s may be complex

$$s = \sigma + j\omega \quad \text{"complex frequency"}$$

- $G(s)$ is complex [when s is complex]
- Last term, found $G(s)$ by impedance methods.

Find $y(t)$ by convolution:

$$y(t) = g(t) * u(t)$$

$$= u(t) * g(t) \quad \text{[commutative!]}$$

$$= \int_{-\infty}^{\infty} u(t-\tau)\, g(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{s(t-\tau)}\, g(\tau)\, d\tau$$

$$= e^{st} \underbrace{\int_{-\infty}^{\infty} g(\tau)\, e^{-s\tau}\, d\tau}_{G(s)}$$

OR,

$$\boxed{G(s) = \int_{-\infty}^{\infty} g(t)\, e^{-st}\, dt}$$

Laplace Transform

$$= \mathcal{L}[g(t)]$$

The transfer function of a system = the Laplace Transform of its impulse response!

[Note that the LT arises <u>naturally</u> in the study of linear systems.]

Why use Laplace Transforms?

Laplace Transforms turn differential equations (and convolutions!) into algebraic equations.

LTs are central to automatic control

<u>The unilateral Laplace Transform</u>

one-sided

For causal $f(t)$, the unilateral LT is defined by

$$\mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st} dt = F(s)$$

Q: Does the integral converge (make sense)?

A: Sometimes, Sometimes not

For the integral to converge, must have

$$|f(t) e^{-st}| \to 0 \quad \text{at} \quad t \to \infty$$

$$\parallel$$

$$|f(t)| e^{-\sigma t}$$

If $|f(t)|\, e^{-\sigma_0 t} \to 0, \quad t \to \infty$

<u>and</u> $f(t)$ is "well-behaved," the integral converges for $\sigma > \sigma_0$.

Smallest σ_0 defines "region of convergence"

<u>Example</u> – $f(t) = \sigma(t) = $ unit step

$$F(s) = \int_0^{\infty} 1 \cdot e^{-st}\, dt$$

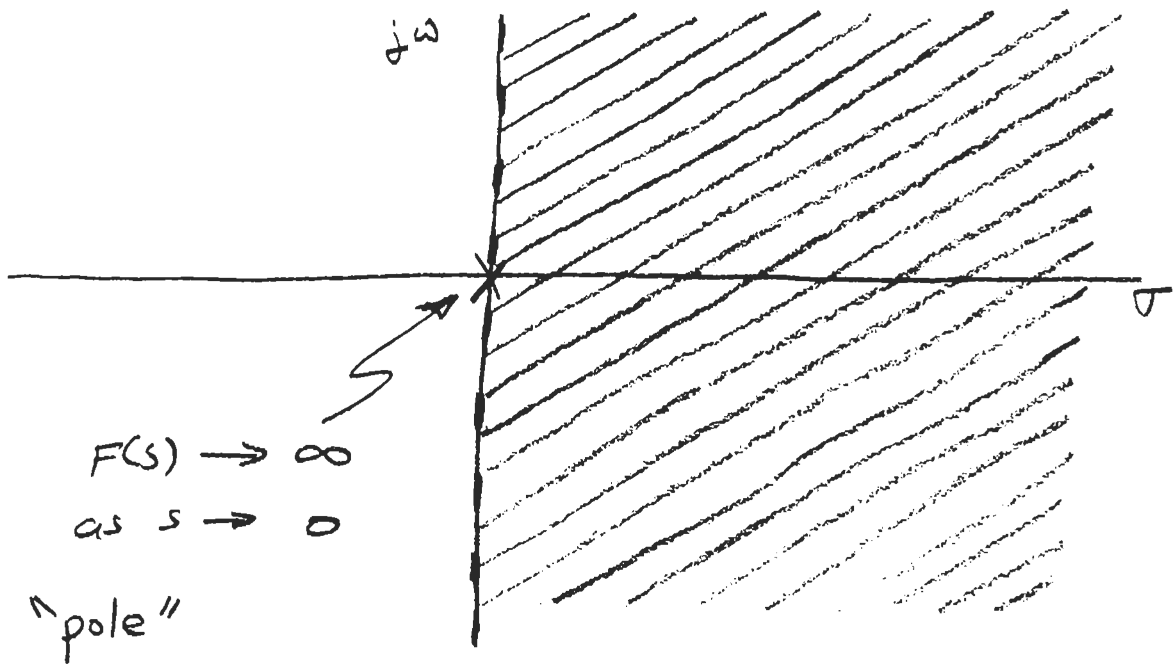
$$= \frac{-1}{s} e^{-st} \Big|_0^{\infty} = 0 - \frac{-1}{s}$$

if $\sigma > 0$

$$= \frac{1}{s}$$

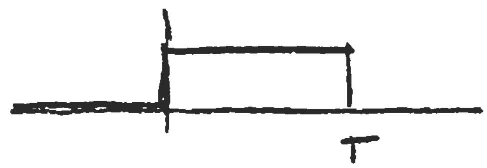
So,

$$\mathcal{L}[\sigma(t)] = \frac{1}{s}, \quad \text{Real}[s] > \sigma_0 = 0$$



Some LTs converge everywhere:

<u>Example</u> - $f(t) = \sigma(t) - \sigma(t-T)$



$$F(s) = \int_0^T 1 \cdot e^{-st}\, dt = \frac{-1}{s} e^{-st} \Big|_0^T$$

$$= \frac{1}{s} - \frac{e^{-sT}}{s} \qquad \text{for } \underline{\underline{\text{all}}}\ s!$$

$$\sigma_0 = -\infty$$

Some LTs <u>never</u> converge:

<u>Example</u> - $f(t) = e^{t^2}$

But $e^{t^2/2} e^{-st} \longrightarrow \infty$ as $t \to \infty$ for <u>all</u> s.

So $F(s) = \int_0^\infty e^{t^2/2} e^{-st}\, dt$

does not converge — $\sigma_0 = +\infty$

Nice fact: All systems described by od.e.'s have impulse responses with finite σ_0.

$\Rightarrow$ LTs always work for ordinary systems

Why worry about r.o.c.?

- For now unimportant
- Later, <u>very</u> important.