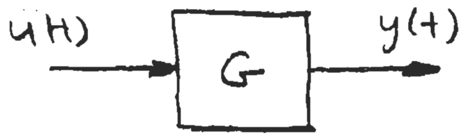
Lecture S2

Linear, Time-Invariant Systems

Any system with input $u(t)$, output $y(t)$ can be represented as



The output $y(t)$ is a functional of the input signal $u(t)$:

$$y(t) = \mathcal{G}[u(t)]$$

[Note: We should really write that

$$y(\cdot) = \mathcal{G}[u(\cdot)]$$

meaning that $y(t)$ at each t depends on $u(t)$ at all values of t.]

We will be most interested in $\mathcal{G}[\cdot]$ which is linear, time-invariant.

Linearity

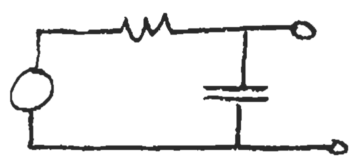
A system is linear if

$$\mathcal{G}[a\,u_1(t) + b\,u_2(t)] = a\,\mathcal{G}[u_1(t)] + b\,\mathcal{G}[u_2(t)]$$

for <u>all</u> $a, b, u_1(t), u_2(t)$

System is linear $\Longleftrightarrow$ superposition always holds

Linear system:



Nonlinear systems: real circuits, airplane, helicopter, spacecraft.

Almost linear system: real circuit, airplane, helicopter, spacecraft

Message: Although all physical systems are nonlinear, many can be <u>modeled</u> as linear for <u>some</u> purposes.

Since linear systems are <u>much</u> simpler than nonlinear systems, do this whenever possible.

Time Invariance

A system is <u>time-invariant</u> if

$$y(t) = \mathcal{G}[u(t)] \Rightarrow y(t-T) = \mathcal{G}[u(t-T)]$$

for all T, $u(t)$.

In words, shifting the input in time shifts the output in time the same amount

<u>Example</u> - An aircraft is <u>nonlinear</u>, because lift is a nonlinear function of speed ($\sim V^2$) and attitude (stall).

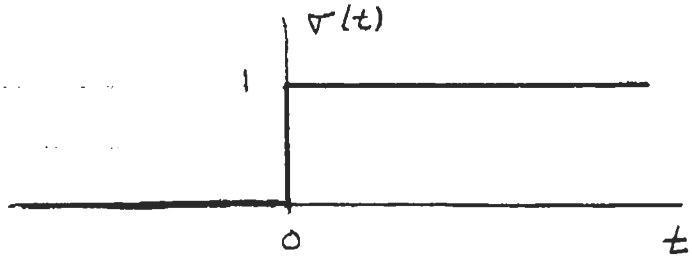
An aircraft is time-varying, because configuration, weight, etc., change over time.

In Unified, will consider only linear, time-invariant (LTI) systems, because we know so much about them.

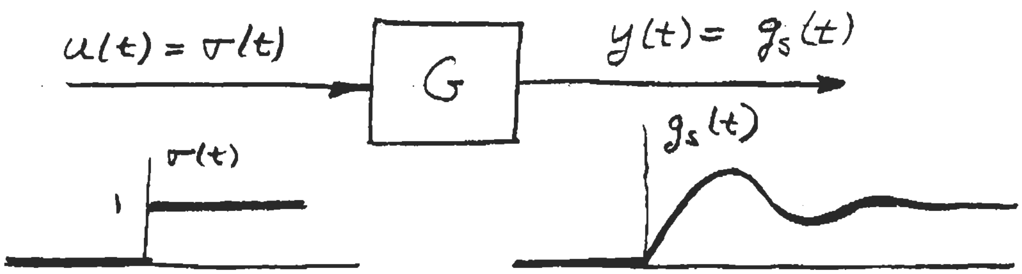
The step response

The *unit step* is defined by

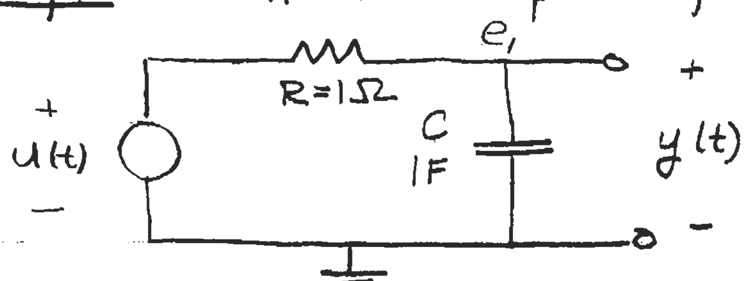
$$\sigma(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



The *step response* of an LTI system is the output of the system when the input to the system is a unit step.



Example What is step response of



?

<u>Solution</u> Use node method:

At e_1,
$$\frac{e_1 - u}{R} + C\frac{d}{dt}(e_1 - 0) = 0$$

$$\Longrightarrow \frac{d}{dt}e_1(t) + \frac{1}{RC}e_1(t) = \frac{1}{RC}u(t)$$

For $t \geq 0$, $u(t) = 1$. So solve

$$\frac{d}{dt}e_1(t) + \frac{1}{RC}e_1(t) = \frac{1}{RC}$$

subject to initial condition

$$e_1(0) = 0$$

Express solution as

$$e_1(t) = \underbrace{e_p(t)}_{\text{particular}} + \underbrace{e_h(t)}_{\text{homogeneous}}$$

Find particular, then homogeneous.

Particular Solution –

Since input is constant, assume

$$e_p(t) = E = \text{constant}$$

Plug into equation:

$$\underbrace{\cancel{\dot{E}}}_{0} + \frac{1}{RC} E = \frac{1}{RC}$$

$$\Rightarrow \quad E = 1 \quad \Longrightarrow \quad e_p(t) = 1$$

Homogeneous Solution –

Assume solution is of the form

$$e_h(t) = E e^{st}$$

Plug into <u>homogeneous</u> equation:

$$\frac{d}{dt}\left(E e^{st}\right) + \frac{1}{RC}\left(E e^{st}\right) = 0$$

$$\Rightarrow \quad E s e^{st} + \frac{1}{RC} E e^{st} = 0$$

$$\Rightarrow \quad s + \frac{1}{RC} = 0 \quad \Longrightarrow \quad s = -1/RC$$

$$\Rightarrow \quad e_h(t) = E e^{-t/RC}$$

Total Solution —

$$e_1(t) = 1 + E e^{-t/RC}$$

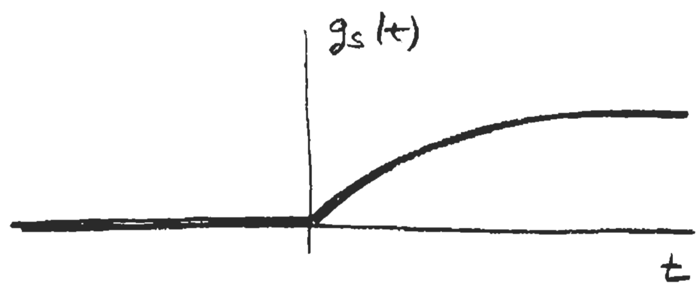
Initial condition is $e_1(0) = 0$

$$\Rightarrow \quad 1 + E e^{-0/RC} = 1 + E = 0$$

$$\Rightarrow \quad E = -1$$

So

$$g_s(t) = y(t) = e_1(t) = \begin{cases} 1 - e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

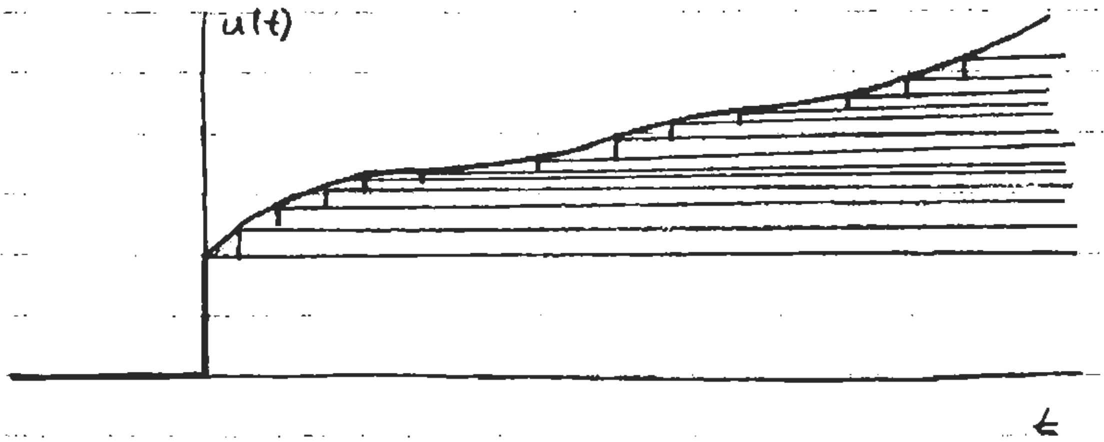


Note — could have found particular solution using impedance methods. Later, will find <u>total</u> solution using impedance methods.

Amazing fact:

The step response of an LTI system completely characterizes the system!

That is, if you know the step response, you can find the response to any input.



Can represent u(t) arbitrarily well by summing a series of scaled, delayed steps

⇒ response is sum of scaled, delayed step responses, by superposition.

Next time, will do the summation, to find $y(t)$ in terms of $g_s(t)$, $u(t)$.