KEY CONCEPTS FOR MATERIALS AND STRUCTURES

Handout for Spring Term Quizzes

Basic modeling process for 1-D structural members

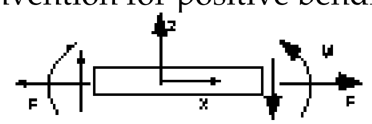
(1) Idealize/model – make assumptions on geometry, load/stress and deformations
(2) Apply governing equations (e.g. equations of elasticity)
(3) Invoke known boundary conditions to derive constitutive relations for structure (load-deformation, load-internal stress etc.)

Analytical process for 1-D structural members

(1) Idealize/model – assumptions on geometry, load/stress and deformations
(2) Draw free body diagram
(3) Apply method of sections to obtain internal force/moment resultants
(4) Apply structural constitutive relations to relate force/moment resultants to
    a) internal stresses
    b) deformations (usually requires integration – invoking boundary conditions)

Elastic bending formulae

Based on convention for positive bending moments and shear forces:



for continuous loading, q, $q = \frac{dS}{dx}, \quad S = \frac{dM}{dx}$

Bending of a symmetric cross section about its neutral axis (mid plane for a cross-section with two orthogonal axes of symmetry).

$$\sigma_{xx} = -\frac{Mz}{I} \qquad M = EI\frac{d^2w}{dx^2} \qquad \sigma_{xz} = -\frac{SQ}{Ib}$$

where σ_{xx} is the axial (bending) stress, M is the bending moment at a particular cross-section, I is the second moment of area about the neutral axis, z is the distance from the neutral axis, E is the Young's modulus of the material, w is the deflection, x is the axial coordinate along the beam, σ_{xz} is the shear stress at a distance z above the neutral axis, S is the shear force at a particular cross, section, Q is the first moment of area of the cross-section from z to the outer ligament, b is the width of the beam at a height b above the neutral axis.

Second moment of area $I = \int_A z^2 dA$

Standard solutions:

Rectangular area, breadth b, depth h: $I = \frac{bh^3}{12}$ Solid circular cross-section, radius R: $I = \frac{\pi R^4}{4}$

Isosceles Triangle, depth h, base b: $I = \frac{bh^3}{36}$ (note centroid is at h/3 above the base)

Parallel axis theorem:

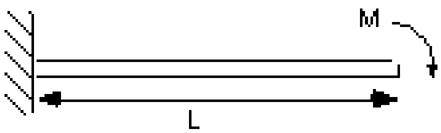
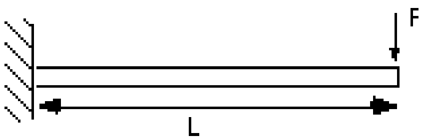
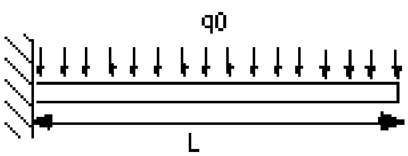
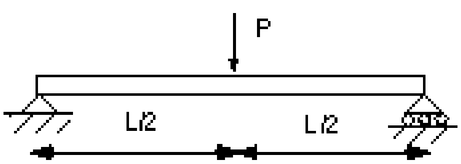
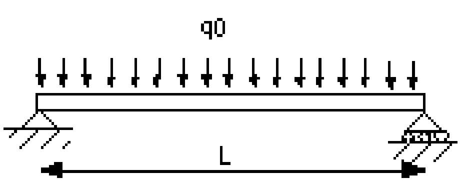
If the second moment of area of a section, area A, about an axis is I then the second moment of area I′ about a parallel axis, a perpendicular distance d away from the original axis is given by:

$$I' = I + Ad^2$$

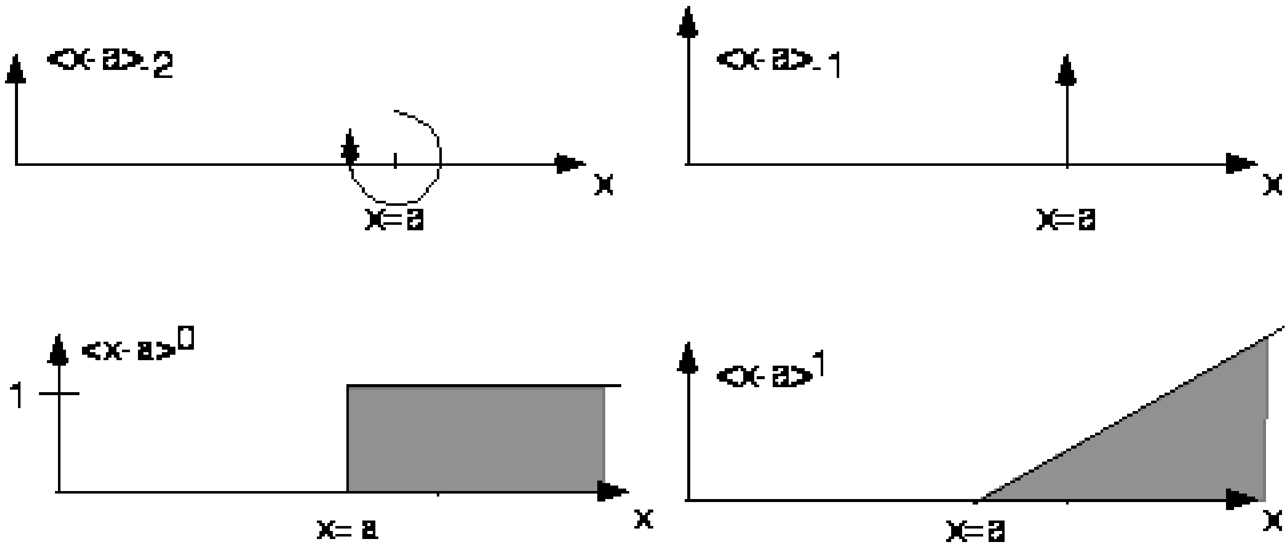
First moment of area
The first moment of area of a section between a height z from the neutral plane and the top surface (outer ligament) of the section is given by:

$$Q = \int_{A,z}^{h/2} z\,dA$$

Standard solutions for deflections of beams under commonly encountered loading

Configuration	End slope dw/dx (x=L)	End deflection, w(L)	Central deflection, w(L/2)
	$\frac{ML}{EI}$	$\frac{ML^2}{2EI}$	
	$\frac{PL^2}{2EI}$	$\frac{PL^3}{3EI}$	
	$\frac{q_0 L^3}{6EI}$	$\frac{q_0 L^4}{8EI}$	
	$\frac{PL^2}{16EI}$		$\frac{PL^3}{48EI}$
	$\frac{q_0 L^3}{24EI}$		$\frac{5q_0 L^4}{384EI}$

Singularity functions



Integration of singularity functions: $\int_{-\infty}^{x} \langle x-a \rangle^n dx = \frac{\langle x-a \rangle^{n+1}}{n+1}, \quad n \geq 0$

$$\int_{-\infty}^{x} \langle x-a \rangle_{-2} dx = \langle x-a \rangle_{-1} \qquad \int_{-\infty}^{x} \langle x-a \rangle_{-1} dx = \langle x-a \rangle^0$$

Torsion of round shafts

An internal torque resultant, T generates a circumferential shear stress, τ, at a radius r, and twist per unit length, $\frac{d\phi}{dx}$, where:

$$\tau = \frac{Tr}{J} \qquad T = GJ\frac{d\phi}{dx}$$

G is the shear modulus of the material and J is the second polar moment of area given by:

$$J = \int_A r^2 dA$$

For a solid circular cross section, radius R: $J = \frac{\pi R^4}{2}$

For a thin walled circular tube, radius R, thickness t: $J = 2\pi R^3 t$

Elastic buckling of columns

The general governing equation for the transverse (buckling) , w, of a uniform column of bending stiffness EI, under an axial load P is: $\frac{d^2w}{dx^2} + \frac{P}{EI}x = M_0$. Where M_0 is a constant.
General solutions are of the form:

$$w = A\sin\left(\sqrt{\frac{P}{EI}}x\right) + B\cos\left(\sqrt{\frac{P}{EI}}x\right) + Cx + D.$$

In general the elastic critical load, $P_{cr} = cP_E$, where the factor c depends on the boundary conditions and the order of the buckling mode, and P_E is the Euler Load for a perfect, pin ended column of length, L buckling into a half sine wave given by:

$$P_E = \frac{\pi^2 EI}{L^2}$$

Yield and Plasticity of Metals

Uniaxial loading of a bar, initial length ℓ_0, cross-sectional area A_0 past yield point: Define nominal, true stress and nominal and true strain:

$$\sigma_n = \frac{P}{A_0}, \qquad \sigma_t = \frac{P}{A}, \qquad \varepsilon_n = \frac{\Delta\ell}{\ell_0} = \frac{\ell - \ell_0}{\ell_0}, \qquad \varepsilon_t = \int_{\ell_0}^{\ell} \frac{d\ell}{\ell} = \ln\left(\frac{\ell}{\ell_0}\right)$$

Since volume is conserved: $A_0\ell_0 = A\ell$ obtain: $\sigma_t = \sigma_n(1+\varepsilon_n)$ and $\varepsilon_t = \ln(1+\varepsilon_n)$

Work of deformation per unit volume: $U = \int_{\varepsilon_{n1}}^{\varepsilon_{n2}} \sigma_n d\varepsilon_n = \int_{\varepsilon_{t1}}^{\varepsilon_{t2}} \sigma_t d\varepsilon_t$

Elastic Strain Energy (for linear elastic deformation): $U = \frac{\sigma_n^2}{2E}$

For multiaxial stress states models for yield:

Tresca: $\max\{|\sigma_I - \sigma_{II}|, |\sigma_{II} - \sigma_{III}|, |\sigma_{III} - \sigma_I|\} \geq \sigma_y$

Von Mises: $(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2 \geq 2\sigma_y^2$

Where σ_I etc are the principal stresses and σ_y is the uniaxial yield strength

Hardness $H = \frac{F_{indentation}}{A_{indentation}} \approx 3\sigma_y$

In a uniaxial tension test, necking occurs when: $\frac{d\sigma_t}{d\varepsilon_t} = \sigma_t$

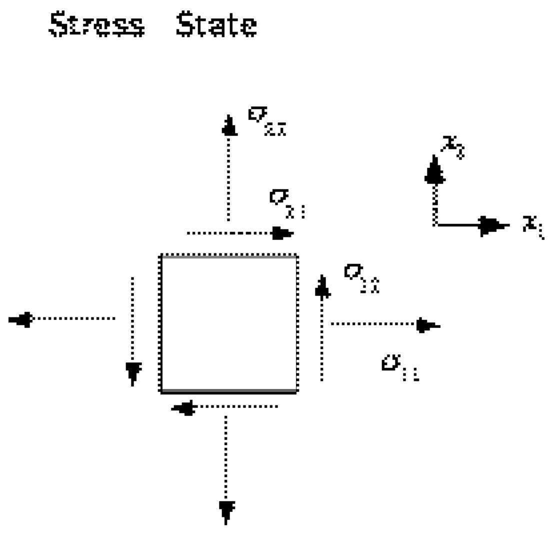
Transformation of Stress and Strain via Mohr's Circle:

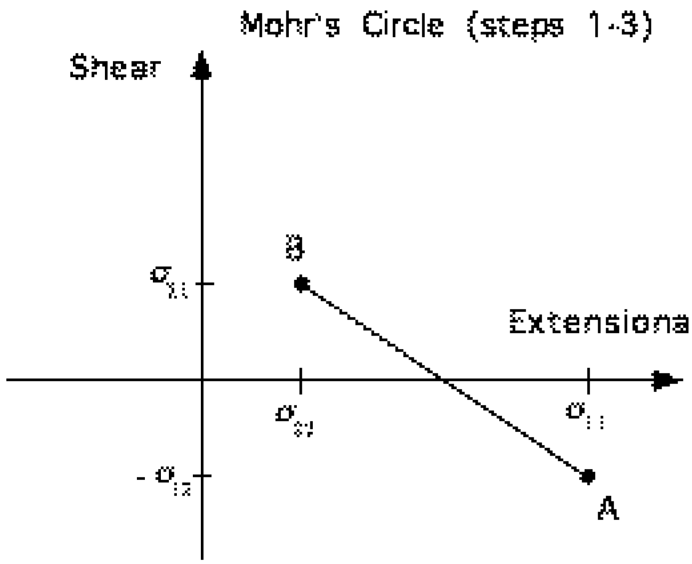
Mohr's circle is a geometric representation of the 2-D transformation of stresses.

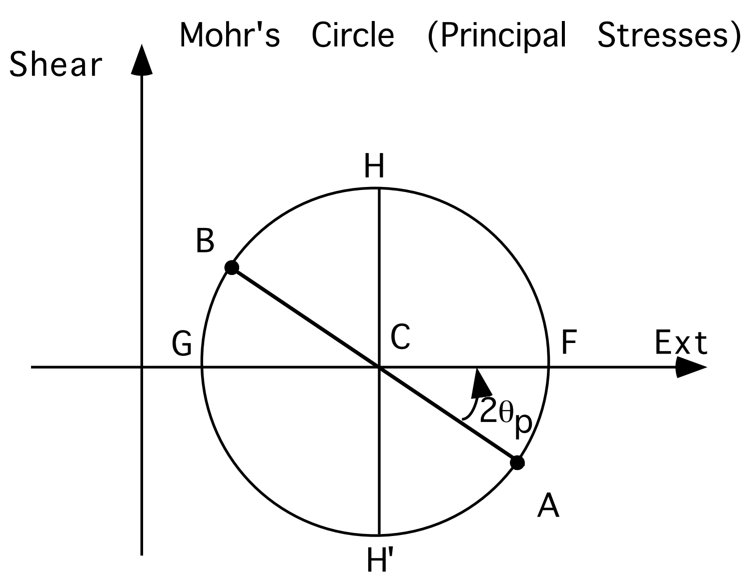
<u>Construction</u>:   Given the state of stress shown below for an infinitessimal element, with the following definition (by Mohr) of positive and negative shear:

"Positive shear would cause a clockwise rotation of the element about the element center."

Thus: σ_{21} (*below*) is plotted positive σ_{12} (*below*) is plotted negative:







Principal stresses correspond to points G, F. Max shear at H, H'.

Note that angles are doubled on the Mohr's circle relative to the physical problem.
Note that a Mohr's circle can only be drawn stresses in a plane perpendicular to a principal direction.

Strengthening Mechanisms

Precipitate Strengthening: $\Delta\tau_y \approx \frac{Gb}{L}$ where G= shear modulus, b=Burgers vector, L = particle spacing

Solid Solution strengthening: $\Delta\tau_y \propto \sqrt{c}$ where c = concentration of alloying elements

Work Hardening: $\Delta\tau_y \propto \gamma^m$ where γ = shear strain, m = exponent (0.01-0.5)

Grain Boundary Effect: $\Delta\tau_y \propto 1/\sqrt{d}$ where d = grain size

Fracture and Fatigue

Fast fracture occurs when: $dW \geq dU^{el} + G_c dA$

where W = external work, U_{el} = elastic strain energy, G_c is the material's toughness and A is the area of crack surface.

Can also be written: $K \geq K_c$ Where K_c is the fracture toughness and K is the stress intensity factor given by:

$$K = Y\sigma\sqrt{\pi a}$$

where Y is a factor which depends on the crack and component shape (≈1), a is the crack length and σ the applied stress

For many metals fatigue crack growth is of the form: $\frac{da}{dN} = A\Delta K^n$

where A and n are empirically determined constants.