Fluids – Lecture 20 Notes

1. Laval Nozzle Flows

Reading: Anderson 10.3

Laval Nozzle Flows

Subsonic flow and choking

Consider a duct with a throat, connected at its inlet to a very large still air reservoir with total pressure and enthalpy p_r, h_r. The duct exit is now subjected to an adjustable *exit static pressure* p_e, sometimes also called the *back pressure*. As p_e is gradually reduced from p_r, air will flow from the reservoir to the exit with a mass flow $\dot{m}$. We first note that the stagnation conditions are known from the reservoir values all along the duct.

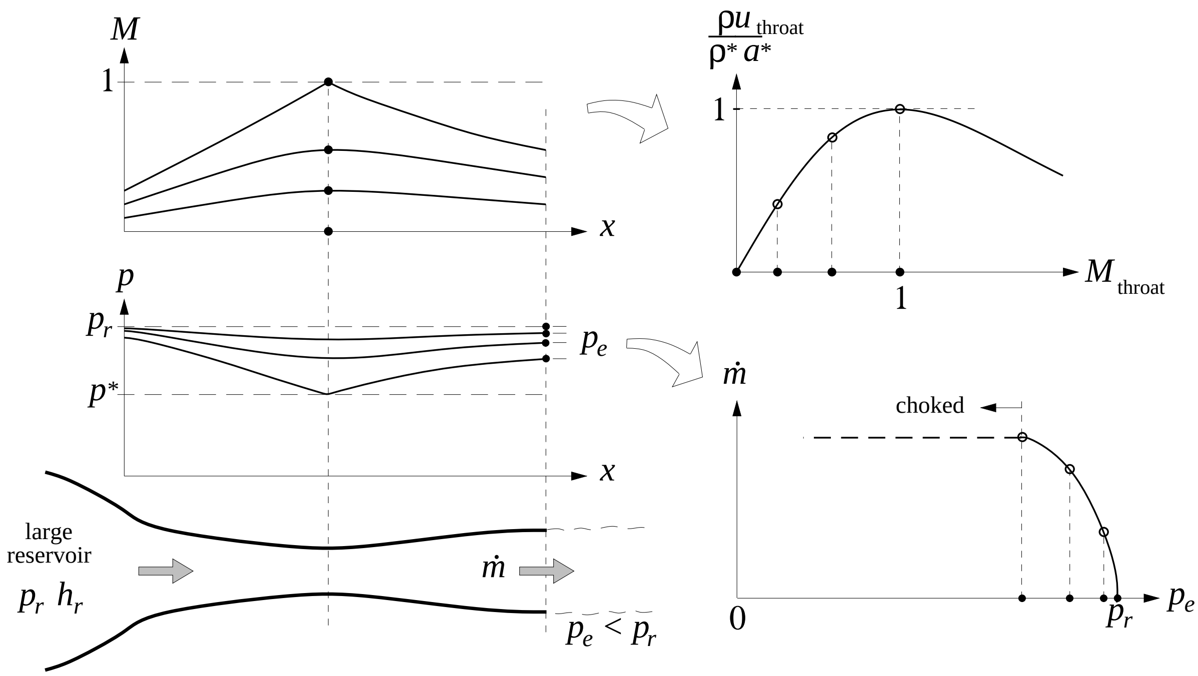
$$p_o = p_r \quad , \qquad a_o^2 = (\gamma - 1) h_o = (\gamma - 1) h_r \quad , \qquad \rho_o = \frac{\gamma p_o}{(\gamma - 1) h_o} = \frac{\gamma p_r}{(\gamma - 1) h_r}$$

If we assume isentropic flow, $\dot{m}$ can be computed with the isentropic relations applied at the exit, using the known exit pressure p_e and known exit area A_e.

$$M_e^2 = \frac{2}{\gamma - 1} \left[(p_o/p_e)^{\frac{\gamma-1}{\gamma}} - 1 \right]$$

$$\dot{m} = \rho_e u_e A_e = \frac{\gamma p_o}{\sqrt{(\gamma - 1) h_o}} \, M_e \left(1 + \frac{\gamma - 1}{2} M_e^2 \right)^{-\frac{\gamma+1}{2(\gamma-1)}} A_e \qquad (1)$$

The observed relation between p_e and $\dot{m}$ is shown on the bottom right in the figure. As p_e is reduced, $\dot{m}$ will first increase, but at some point it will level off and remain constant even if p_e is reduced all the way to zero (vacuum). When $\dot{m}$ no longer increases with a reduction in p_e, the duct is said to be *choked*.



If we examine the various flow properties along the duct, it is evident that the onset of choking cooincides with the *throat* reaching $M = 1$ locally. This also corresponds to the mass flux ρu at the throat reaching its maximum possible value $\rho^* a^*$, which is given by

$$\rho^* a^* \;=\; \rho_o a_o \, \frac{\rho^*}{\rho_o} \frac{a^*}{a_o} \;=\; \frac{\gamma p_r}{\sqrt{(\gamma-1) h_r}} \; \left(1 \,+\, \frac{\gamma-1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \tag{2}$$

Therefore, the only way to change the mass flow of a choked duct is to change the reservoir's total properties p_r and/or h_r.

Choked flow with normal shock

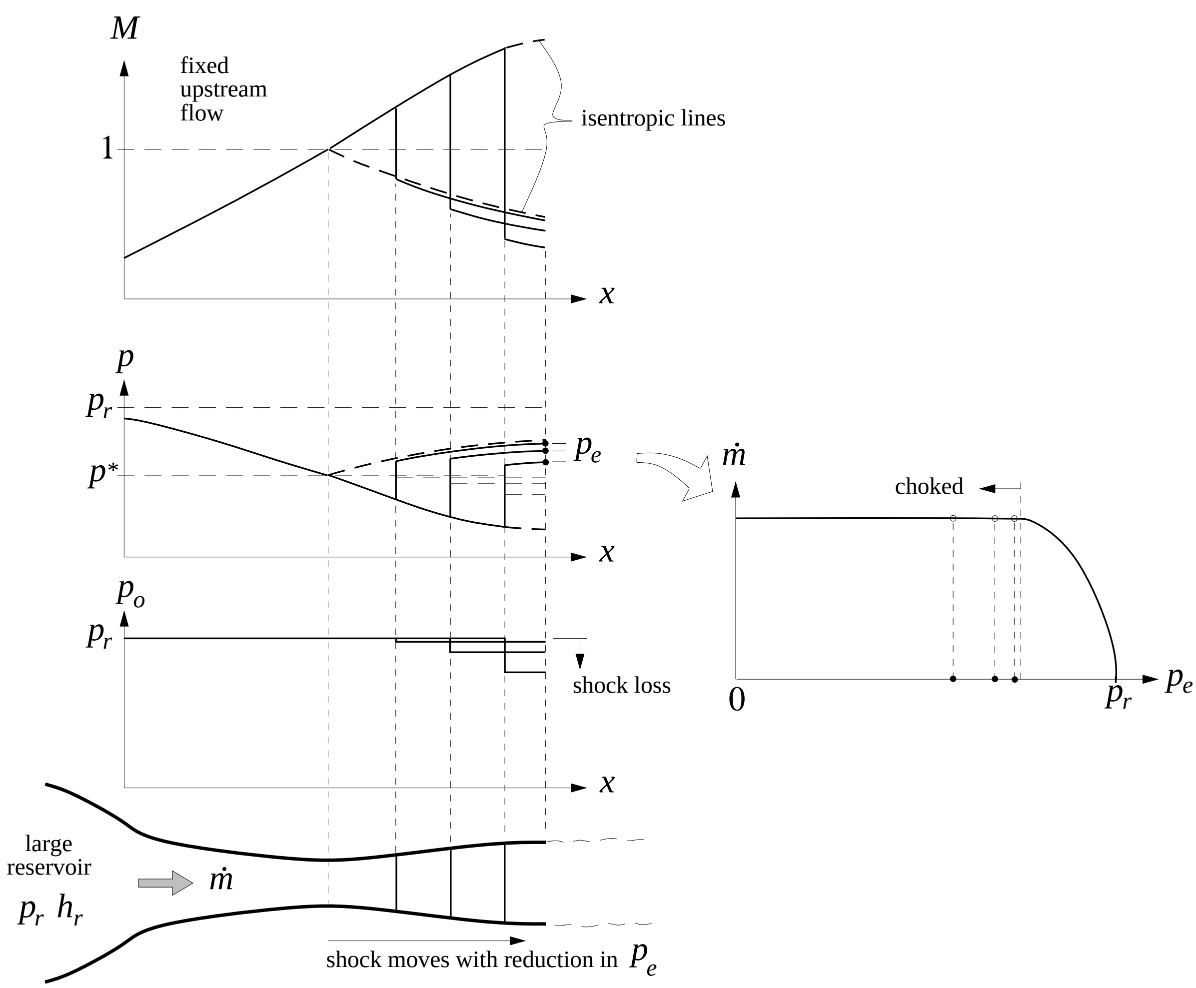
When the back pressure is reduced below the level required to reach choking, a new flow pattern emerges, called a *Laval nozzle flow*, with the following important features:

1. The flow upstream of the throat no longer changes with p_e, but remains the same as at the choking-onset condition. This is consistent with the mass flow being fixed.

2. The flow behind the throat becomes supersonic. The Mach number continues to increase and pressure to decrease as the area increases downstream.

3. A *normal shock* forms in the duct, and the flow behind the shock returns to subsonic. The Mach number then decreases and pressure increases towards p_e as the area increases.

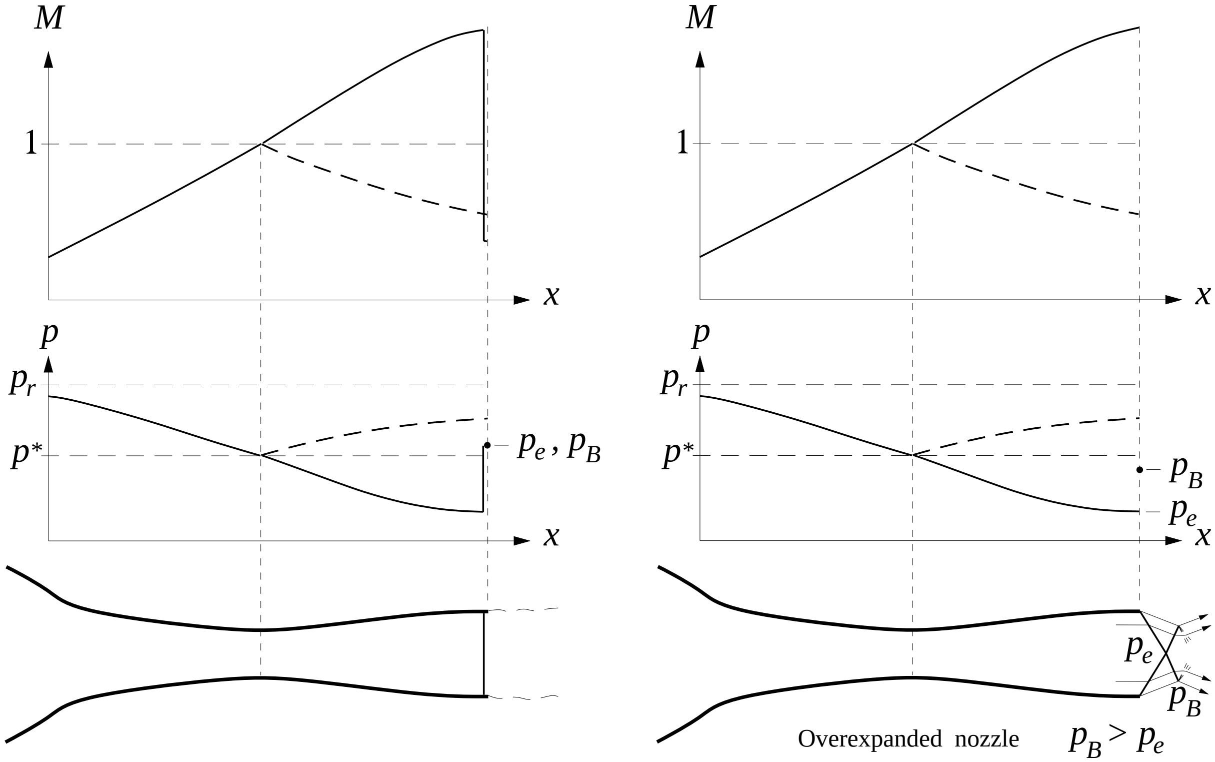
4. The shock incurs a total pressure loss, so that $p_o < p_r$ behind the shock all the way to the exit. Both $p(x)$ and $M(x)$ behind the shock are then lower than what they would be with isentropic flow at the onset of choking.



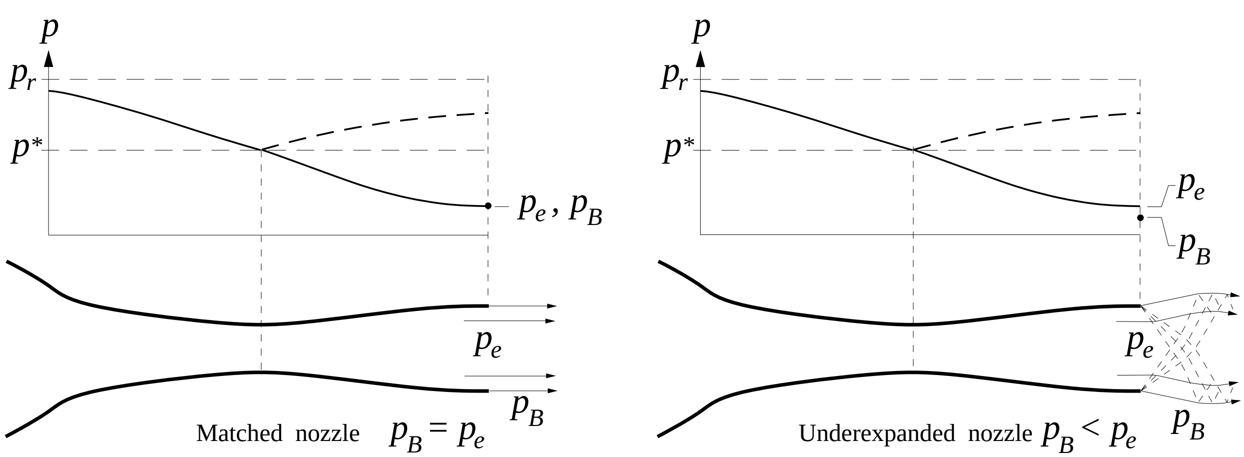
Supersonic-exit flows

With sufficiently low back pressure, the shock can be moved back to nearly the exit plane. If the back pressure is reduced further, below the sonic pressure p^*, the exit flow becomes supersonic, leading to three possible types of exit flow. In these cases it is necessary to distinguish between the exit pressure p_e of the duct flow, and the back pressure p_B of the surrounding air, since these two pressures will in general no longer be the same.

Overexpanded nozzle flow. In this case, $p_B < p^*$, so the exit flow is supersonic, but $p_B > p_e$, so the flow must adjust to a higher pressure. This is done through oblique shocks attached to the duct nozzle edges. The streamline at the edge of the jet behaves much like a solid wall, whose turning angle adjusts itself so that the post-shock pressure is equal to p_B.



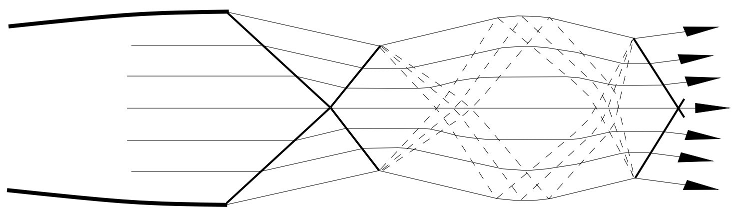
Matched nozzle flow. In this case, the back pressure is reduced further until $p_B = p_e$. The duct nozzle flow comes out at the same pressure as the surrounding air, and hence no turning takes place.



Underexpanded nozzle flow. In this case, the back pressure is reduced below the isentropic exit pressure, so that $p_B < p_e$. The duct nozzle flow must now expand to reach p_B, which is done through expansion fans attached to the duct nozzle edges.

Jet shock diamonds

In the underexpanded and overexpanded nozzle flows, each initial oblique shock or expansion fan impinges on the opposite edge of the jet, turning the flow away or towards the centerline. The shock or expansion fan reflects off the edge, and propagates back to the other side, repeating the cycle until the jet dissipates though mixing. These flow patterns are known as *shock diamonds*, which are often visible in the exhaust of rocket or jet engines.



Determination of Choked Nozzle Flows

A common flow problem is to determine the exit conditions and losses of a given choked nozzle with prescribed reservoir stagnation conditions p_r, h_r, and a prescribed exit pressure p_e. We first note that the mass flow in this situation is known, and given by combining relation (2) with the fact that $A^* = A_t$ for a choked throat.

$$\dot{m} \;=\; \rho^* a^* A_t \;=\; \frac{\gamma p_r}{\sqrt{(\gamma-1)h_r}} \left(1 + \frac{\gamma-1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} A_t \qquad \text{(choked)} \qquad (3)$$

To then determine the exit conditions corresponding to this mass flow, we use the mass flow expression (1), but recast it in terms of the (known) exit static pressure rather than the (unknown) exit total pressure. We can also set $h_o = h_r$ for adiabatic flow.

$$\dot{m} \;=\; \frac{\gamma p_e}{\sqrt{(\gamma-1)h_r}} \, M_e \left(1 + \frac{\gamma-1}{2} M_e^2\right)^{1/2} A_e \qquad \text{(choked)} \qquad (4)$$

Equating (3) and (4), and squaring the result, gives

$$M_e^2 \left(1 + \frac{\gamma-1}{2} M_e^2\right) \;=\; \left(\frac{p_r}{p_e}\frac{A_t}{A_e}\right)^2 \left(1 + \frac{\gamma-1}{2}\right)^{-\frac{\gamma+1}{\gamma-1}} \qquad \text{(choked)} \qquad (5)$$

This is a quadratic equation for M_e^2, which can be solved with a specified righthand side. The exit total pressure is then obtained via its definition.

$$p_{oe} \;=\; p_e \left(1 + \frac{\gamma-1}{2} M_e^2\right)^{\frac{\gamma}{\gamma-1}}$$

The overall nozzle total pressure ratio p_{oe}/p_r is due to the loss across the shock, so that

$$\frac{p_{oe}}{p_r} \;=\; \left(\frac{p_{o_2}}{p_{o_1}}\right)_{\text{shock}} \;=\; f(M_1)$$

where $f(M_1)$ is the shock total pressure ratio function, also available in tabulated form. Equation (6) therefore implicitly determines M_1 just in front of the shock, which together with the universal flow area function $A/A^* = f(M)$ determines the nozzle area at the shock.