Fluids – Lecture 9 Notes

1. General Wings

Reading: Anderson 5.3.2, 5.3.3

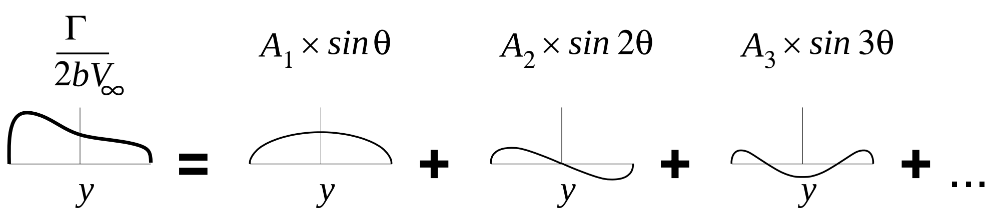
General Wings

General circulation distribution and downwash

The assumption of elliptic loading is too restrictive for the design of practical wings. A more general circulation distribution can be conveniently described by a *Fourier sine series*, in terms of the angle coordinate θ defined earlier.

$$\Gamma(\theta) \;=\; 2bV_\infty \sum_{n=1}^{N} A_n \,\sin n\theta$$

This is a superposition of individual weighted component shapes $\sin n\theta$, shown in the figure plotted versus the physical coordinate y. The induced angle for this Γ distribution is



evaluated by first noting that

$$\frac{d\Gamma}{dy}\,dy \;=\; \frac{d\Gamma}{d\theta}\,d\theta \;=\; 2bV_\infty \sum_{n=1}^{N} nA_n \,\cos n\theta \, d\theta$$

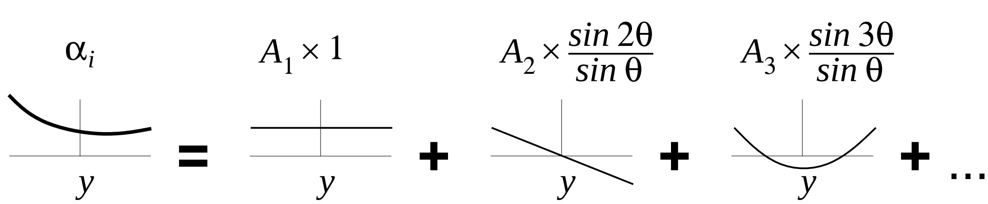
which is then substituted into the induced angle integral.

$$\alpha_i \;=\; \frac{1}{4\pi V_\infty} \int_{-b/2}^{b/2} \frac{d\Gamma}{dy} \, \frac{dy}{y_o - y} \;=\; \frac{1}{\pi} \sum_{n=1}^{N} nA_n \int_0^{\pi} \frac{\cos n\theta}{\cos\theta - \cos\theta_o}\, d\theta$$

This integral was evaluated earlier, which gives the final result.

$$\alpha_i(\theta_o) \;=\; \sum_{n=1}^{N} nA_n \, \frac{\sin n\theta_o}{\sin\theta_o}$$

Each component of $\Gamma(\theta)$ has a corresponding component of $\alpha_i(\theta)$. The leading $n = 1$ term



is the same as the elliptic loading case, with the expected uniform induced angle. The remaining terms deviate the loading away from the elliptic distribution, and deviate the downwash away from the uniform distribution.

Lift

We can now compute the lift for the general circulation distribution by integrating it across the span.

$$L = \int_{-b/2}^{b/2} \rho V_\infty \Gamma(y) \, dy$$

The integral is most easily evaluated using the θ coordinate. With the substitutions

$$y = \frac{b}{2} \cos\theta$$
$$dy = -\frac{b}{2} \sin\theta \, d\theta$$

we then have

$$L = \rho V_\infty \left[2bV_\infty \sum_{n=1}^{N} A_n \, \frac{b}{2} \int_0^\pi \sin n\theta \, \sin\theta \, d\theta \right]$$

All the integrals inside the summation are readily evaluated using the orthogonality property of the sine functions.

$$\int_0^\pi \sin n\theta \, \sin m\theta \, d\theta = \begin{cases} \pi/2 & (\text{if } n = m) \\ 0 & (\text{if } n \neq m) \end{cases}$$

For our case we have $m = 1$, and then consider $n = 1, 2, 3 \ldots$ for each term. Clearly, the $n = 1$ integral evaluates to $\pi/2$, and the rest evaluate to zero. Therefore,

$$L = \frac{\pi}{2} \rho V_\infty^2 b^2 A_1$$
$$C_L = \frac{L}{\frac{1}{2}\rho V_\infty^2 S} = \pi A_1 \frac{b^2}{S} = A_1 \pi AR$$

Only the leading $n = 1$ component of the circulation contributes to the lift. This is expected after examination of the component shapes for $\Gamma(y)$, which shows that only the $n = 1$ shape has a nonzero area under it.

Induced drag and span efficiency

The induced drag is also evaluated by spanwise integration.

$$D_i = \int_{-b/2}^{b/2} \rho V_\infty \Gamma(y) \, \alpha_i(y) \, dy$$

After switching from y to θ, and substituting for $\Gamma(\theta)$ and $\alpha_i(\theta)$, this evaluates to

$$D_i = \pi b^2 \frac{1}{2} \rho V_\infty^2 \left[A_1^2 + 2A_2^2 + 3A_3^2 + \ldots N A_N^2 \right] = \pi b^2 \frac{1}{2} \rho V_\infty^2 A_1^2 \left[1 + \sum_{n=2}^{N} n \left(\frac{A_n}{A_1} \right)^2 \right]$$

Although only the A_1 part of the circulation contributes to lift, <u>all</u> the A_n parts contribute towards increasing the induced drag. We therefore conclude that the elliptic load distribution gives the smallest induced drag for a given lift and span.

A more convenient equation for the induced drag can be obtained by replacing A_1 in terms of the lift. This gives

$$D_i = \frac{(L/b)^2}{\frac{1}{2}\rho V_\infty^2 \pi} \left[1 + \delta \right]$$

where

$$\delta \;\equiv\; \sum_{n=2}^{N} n \left(\frac{A_n}{A_1}\right)^2$$

can be thought of as a fractional induced drag penalty due to the presence of the higher $n = 2, 3\ldots$ "non-elliptic" loading terms. It is traditional to define a *span efficiency*

$$e \;\equiv\; \frac{1}{1+\delta}$$

so that the induced drag is finally given as

$$D_i \;=\; \frac{(L/b)^2}{\frac{1}{2}\rho\, V_\infty^2\, \pi\, e}$$

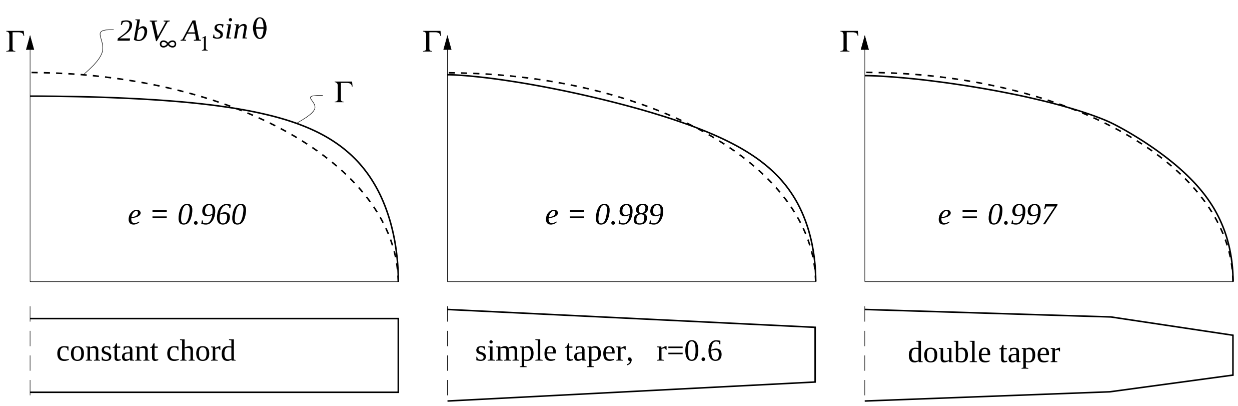
The corresponding induced drag coefficient is then easily obtained.

$$C_{Di} \;=\; \frac{C_L^2}{\pi\, e\, AR}$$

Because δ is the sum of squares and hence non-negative, the span efficiency must be $e \leq 1$, and the actual induced drag is never less than the minimum drag corresponding to elliptic loading, for which $\delta = 0$ and $e = 1$.

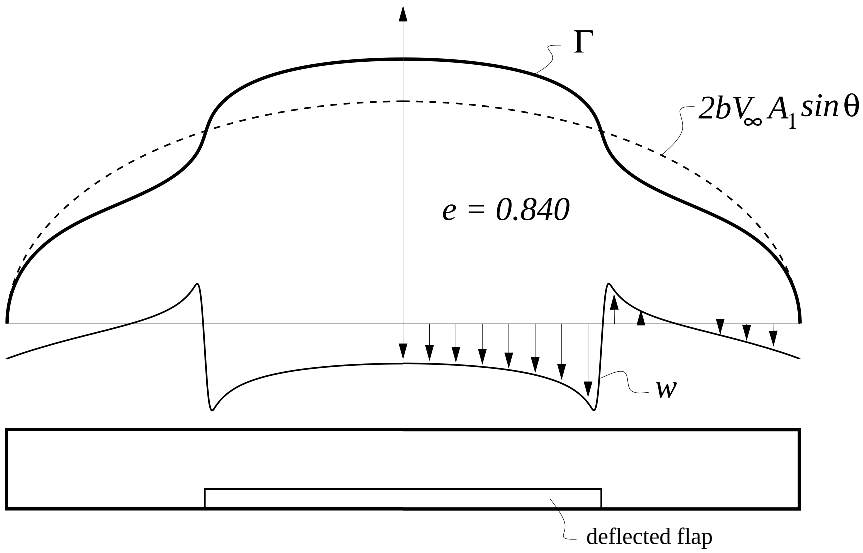
Load distributions on typical planforms

The figure shows three wing planforms with no twist (constant $\alpha_{\rm geom}$), along with their computed circulation distributions at some nonzero lift. Also shown is the elliptic component of the circulation $2bV_\infty A_1 \sin\theta$ as a dotted line. The difference between the two curves is the remaining $n = 2, 3\ldots$ terms, which produce a nonzero δ, and $e < 1$. Even the relatively crude constant-chord wing has an acceptable span efficiency, with only a 4% induced drag "penalty". The loading on the double-taper wing is very nearly elliptic, and hence $e \simeq 1$ for this case. Clearly, the complexity of a curved elliptic planform is hardly warranted.



Effects of trailing edge flaps

Deflection of a part-span trailing edge flap will usually cause a significant distortion in the load distribution, producing a significant increase in induced drag. The figure shows the constant-chord wing case, with a central flap deflected downward 15°. The loading is strongly non-elliptic, and the span efficiency has decreased to 0.840. Note also the strongly non-uniform downwash distribution resulting from this distorted loading.



Lift slope reduction

The downwash behind any finite wing modifies the wing's lift slope. Consider the c_ℓ-angle relation at a typical spanwise location.

$$c_\ell \;=\; a_0\,(\alpha + \alpha_{\rm aero} - \alpha_i)$$

For a nearly-elliptic loading, we have $\;c_\ell \simeq C_L\;$ and $\;\alpha_i \simeq C_L/\pi e A\!\!R$

$$\text{giving} \qquad C_L \;=\; a_0\left(\alpha + \alpha_{\rm aero} - \frac{C_L}{\pi e A\!\!R}\right)$$

$$\text{or} \qquad C_L \;=\; \frac{a_0}{1+\frac{a_0}{\pi e A\!\!R}}\,(\alpha + \alpha_{\rm aero}) \;=\; a\;(\alpha + \alpha_{\rm aero})$$

The lift slope is now

$$\frac{dC_L}{d\alpha} \;=\; \frac{a_0}{1+\frac{a_0}{\pi e A\!\!R}} \;\equiv\; a$$

A common approximation is to assume that $a_0 = 2\pi$ and $e \simeq 1$, in which case

$$a \;\simeq\; \frac{2\pi}{1+2/A\!\!R}$$

This slope a clearly decreases as $A\!\!R$ is reduced. Low aspect ratio wings must therefore operate at higher angles of attack than high aspect ratio wings to reach the same C_L.