20.GEM GEM4 Summer School: Cell and Molecular Biomechanics in Medicine: Cancer
Summer 2007

Basic Mechanics (II)

Elasticity, Viscoelasticity and Plasticity

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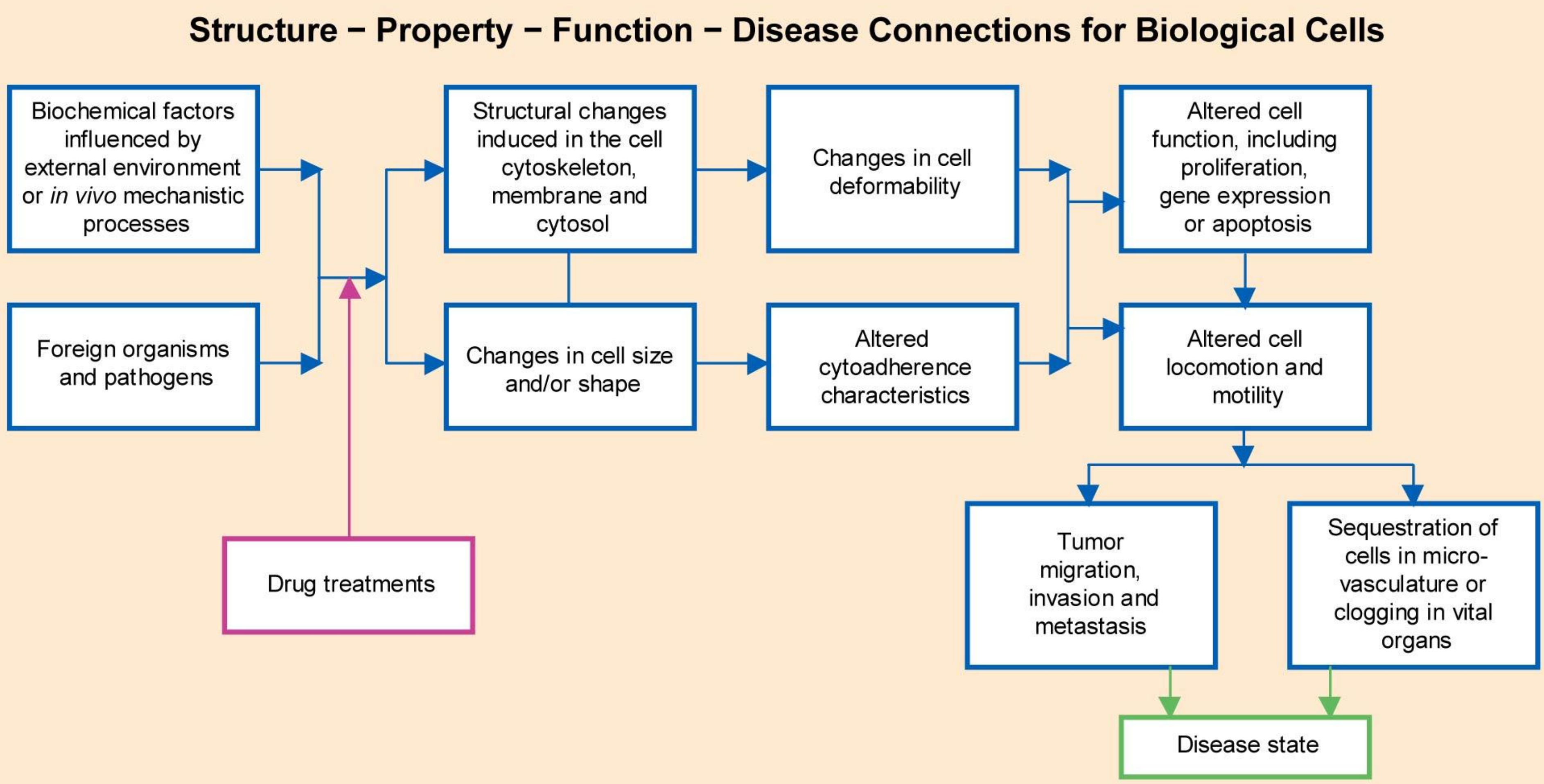
Outline

- Motivation
- Introduction of Elasticity and Viscoelasticity
- Introduction of Plasticity

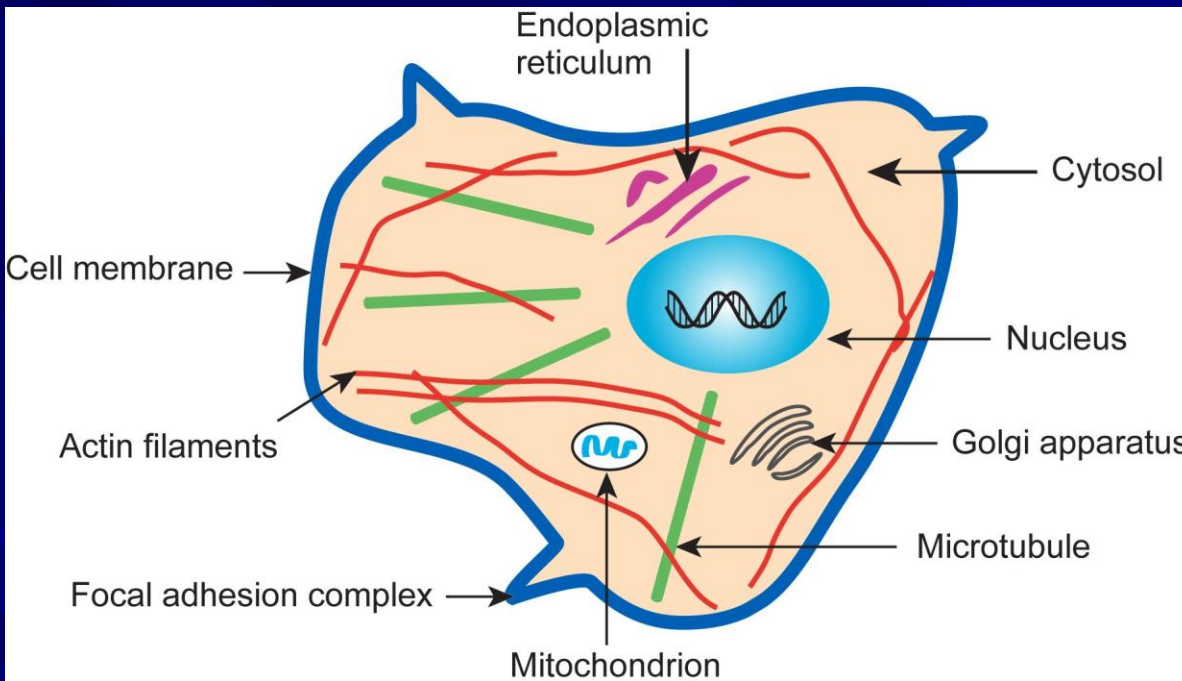
Motivation



S. Suresh, Acta Biomaterialia, 2007

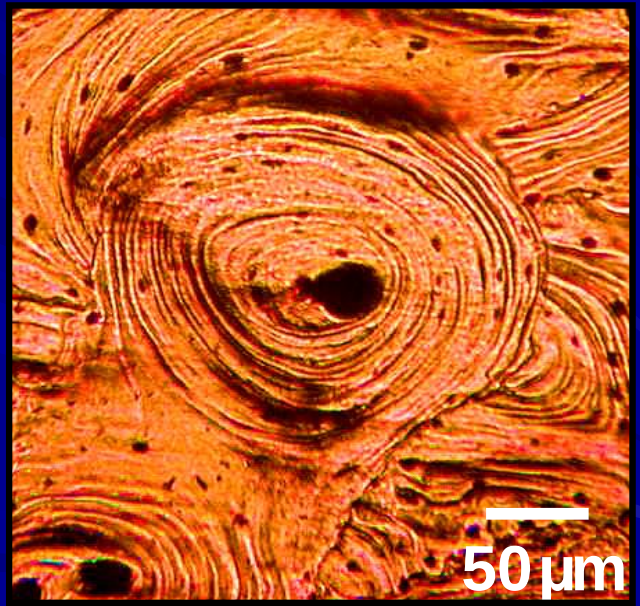
Motivation

Cell: elasticity, viscoelasticity



Suresh, Acta Biomaterialia, 2007

Bone Tissue: elasticity, plasticity, viscoelasticity

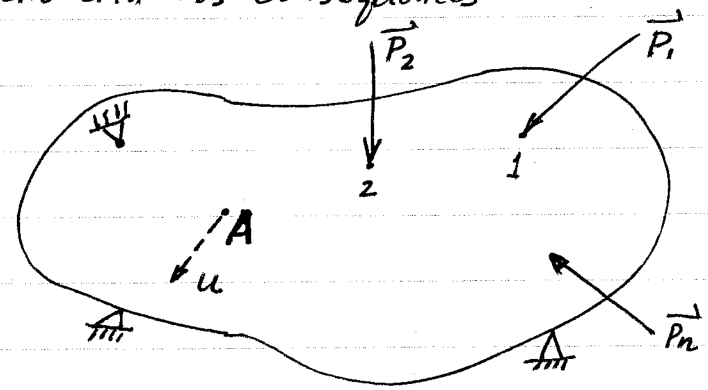


Tai, Dao, Suresh, Palazoglu & Ortiz, Nat Mater, 2007

Introduction of Elasticity and Viscoelasticity

2 Books:
Y.C. Fung Foundations of Solid mechanics
Biomechanics

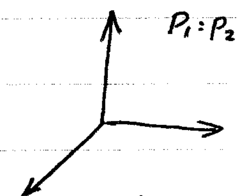
1. Hooke's Law and Its Consequences



U: displacement at an arbitrary point & in arbitrary dir.

$P_1 = |\vec{P_1}|$, $P_2 = |\vec{P_2}|$, $\cdots$

$P_1 : P_2 : \cdots : P_n \Leftrightarrow$ fixed



Cartesian space

Fig. Static equilibrium of a body under external forces

The (solid) body is supported in some manner with 3pts fixed.

To define a linear elastic solid, we need three basic hypotheses:

H1) The body is continuous and remains continuous under the external forces.

H2) Hooke's law:

$$U = a_1 P_1 + a_2 P_2 + \cdots + a_n P_n \qquad (1)$$

where $a_1, a_2, \cdots, a_n$ are constants independent of $P_1, P_2, \cdots, P_n$. (fixed ratio)

Once pts $1, 2, \cdots, n$ and A, AND directions of these pts are chosen, the $a_1, a_2, \cdots, a_n$ are constants!

H3) There exists a unique unstressed state of the body, to which the body returns whenever all the external forces

These three hypotheses complete the rigorous definition of a Hookean (or linear elastic) material.

A number of deductions can be drawn from this definition.

D1) The principle of superposition for loads at the same point.

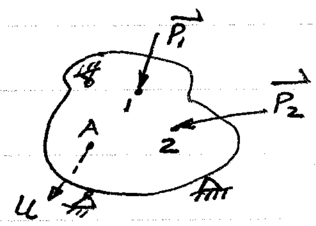
If we have $\vec{P_1}'$ acting at point 1 in the same direction as $\vec{P_1}$, then according to eq (1) (H2)

$$u = a_1(P_1 + P_2) \cancel{+a_2 P_2 + \cdots + a_n P_n}$$ regardless of the order of acting forces

D2) Principle of superposition

Combining (H2) & (H3), we can show eq (1) is valid for arbitrary set of loads $\vec{P_1}, \vec{P_2}, \ldots, \vec{P_n}$ (same points, same directions though), i.e. a_1 is independent of $\vec{P_2}, \vec{P_3}, \cdots, \vec{P_n}$; and so on.

Proof : for arbitrary pair of loads $\vec{P_1}$ & $\vec{P_2}$



$\vec{P_1}$ alone : $u_A = a_{A1} P_1$ (2)

$\vec{P_2}$ alone : $u_A = a_{A2} P_2$ (3)

When both applied : $u_A = a'_{A1} P_1 + a'_{A2} P_2$ (4)

??? $a'_{A1} = a_{A1}$, $a'_{A2} = a_{A2}$

Now take away $\vec{P_1}$: $u_A = a'_{A1} P_1 + a'_{A2} P_2 - a''_{A1} P_1$ (5)

Thus we have only $\vec{P_2}$. Let's take it off, the change obeys eq (3):

$$u_A = a'_{A1} P_1 + a'_{A2} P_2 - a''_{A1} P_1 - a_{A2} P_2 \quad (6)$$

With no load now, u_A must be zero according to (H3). After arrangements:

$$(a'_{A1} - a''_{A1}) P_1 = (a_{A2} - a'_{A2}) P_2 \quad (7)$$

For eq (7) to be valid for arbitrary P_1 & P_2 $\Leftrightarrow$ $\boxed{\begin{array}{l} a'_{A2} = a_{A2} \\ a'_{A1} = a''_{A1} = a_{A1} \end{array}}$

D3) Total work done by $\vec{P_1}, \vec{P_2}, \cdots, \vec{P_n}$ is independent of the order of application.

Measuring u_1 at point 1 & in the same direction, u_2 at pt 2 (same dir), ...,

$$
\begin{aligned}
u_1 &= a_{11}P_1 + a_{12}P_2 + \cdots + a_{1n}P_n \\
u_2 &= a_{12}P_2 + a_{22}P_2 + \cdots + a_{2n}P_n \\
&\cdots \\
u_n &= a_{n1}P_1 + a_{n2}P_2 + \cdots + a_{nn}P_n
\end{aligned}
\qquad (8)
$$

Multiply the ith eqn by P_i, and add them together:

$$
\underbrace{2\times \text{Total Work}}_{\substack{\text{keep } P_1:P_2:\cdots:P_n \\ \text{start from } 0\sim 100\% \text{ load}}} = P_1u_1 + P_2u_2 + \cdots + P_nu_n = a_{11}P_1^2 + a_{12}P_1P_2 + \cdots + a_{1n}P_1P_n
$$
$$
+ a_{21}P_1P_2 + a_{22}P_2^2 + \cdots + a_{2n}P_2P_n + \cdots
$$
$$
+ a_{n1}P_1P_n + a_{n2}P_2P_n + \cdots + a_{nn}P_nP_n \qquad (9)
$$

RHS of (9) is independent of the order of loads.

D4) Maxwell's reciprocal relation

The influence coefficients for corresponding forces & displacements are symmetric: $a_{ij} = a_{ji}$ (10)

Apply $\vec{P_i}$ first, and then $\vec{P_j}$

$$\text{Total work} = \frac{1}{2}\left(P_i^2 a_{ii} + P_j^2 a_{jj}\right) + a_{ij}P_iP_j$$

Change the order

$$\text{Total work} = \frac{1}{2}\left(P_j^2 a_{jj} + P_i^2 a_{ii}\right) + a_{ji}P_jP_i$$

Using (D3), for arbitrary P_i, P_j to hold, we have $\boxed{a_{ij} = a_{ji}}$.

(D5) Strain Energy & Principle of Virtue Work

Strain Energy :

Thermodynamics (1st law) : ΔE = work done (adiabatic, quasistatic)

$$\text{Strain } E = U = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} P_i P_j \underset{\uparrow}{=} \frac{1}{2}\sum_{i=1}^{n} a_{ii} P_i^2 + \sum_{i\neq j}\sum a_{ij} P_i P_j \quad (11)$$

Using (9) & (10)

Take derivative with respect to P_i, we have

$$\frac{\partial U}{\partial P_i} = \underbrace{a_{ii} P_i + \sum_{i\neq j}\sum a_{ij} P_j}_{u_i} \qquad i = 1, 2, \cdots, n$$

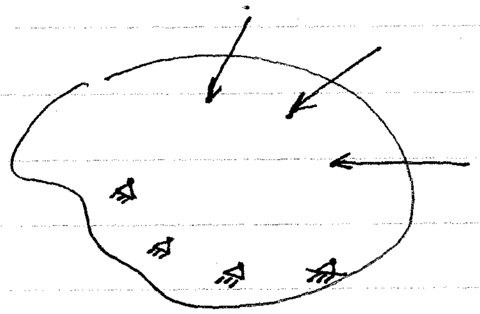
$$\Rightarrow \quad \underbrace{\frac{\partial U}{\partial P_i} = u_i}_{\text{Castigliano's theorem}} \quad , i = 1, 2, \cdots, n \qquad (12)$$

Using the principle of virtue work : If the $U = U(u_i) \longrightarrow \dfrac{\partial U}{\partial u_i} = P_i \quad (13)$, $i = 1, 2, \cdots, n$

Allowing a virtue displacement δu such that δu is continuous everywhere in the body but vanishes at all load points except P_i. Due to δu, strain energy changes by δU, and the work done is $P_i \delta u_i$. According to the priciple of virtue work : $\delta U = P_i \delta u_i \Rightarrow \dfrac{\partial U}{\partial u_i} = P_i$.

Eq (13), as long as $U = U(u_i)$, $i = 1, 2, \cdots, n$, is applicable to nonlinear elastic bodies.

Minimum Complementary Energy Theorem



S_p : points forces are ~~defined~~ applied

S_u : points displacements are specified

$$V^* = U - \sum_{S_u} u_i P_i \qquad (\text{complementary energy})$$

The exact solution should be the case where V^* is minimized.

Minimum Potential Energy Theorem

$$V = U - \sum_{S_p} u_i P_i \qquad (\text{potential energy})$$

where $U = U(u_i)$ exists. $U = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij} u_i u_j$, $k_{ij} = k_{ji}$

The exact solution should be the case where V is minimized.

2. Solving elasticity problems:

$\underset{\sim}{\sigma}$ stress tensor

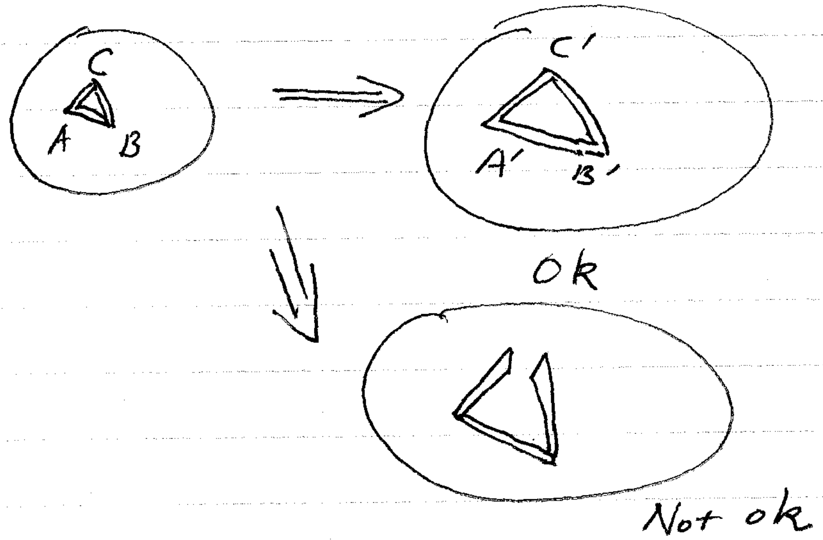
$\underset{\sim}{\varepsilon}$ strain tensor

1. Equilibrium (3 directions)

$$\sum F_i = 0 \implies \frac{\partial \sigma_{ij}}{\partial x_i} + f_j = 0 \quad (3 \text{ eqns}) \qquad (14)$$

$$\sum M_i = 0 \implies \sigma_{ij} = \sigma_{ji} \qquad (15)$$

2. Strain compatibility:



$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0 \quad (6 \text{ eqns}) \qquad (16)$$

$$,i \Leftrightarrow \frac{\partial}{\partial x_i} \qquad ,ij \Leftrightarrow \frac{\partial^2}{\partial x_i \partial x_j}$$

3. Stress - Strain Relation (linear elasticity)

$$\sigma_{ij} = \underbrace{C_{ijkl}} \varepsilon_{kl} \quad (6 \text{ eqns}) \qquad (17)$$

elasticity tensor: 81 coefficients

• Summation Convention $a_i b_i = \sum_{i=1}^{3} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$

isotropic material: 2 constants

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \qquad (18)$$

15 unknowns	15 eqn's
6 strains ε_{ij}	3 equilibrium
6 stresses σ_{ij}	6 compatibility
3 displacement u_i	6 stress-strain

Write out each one:

$$\sigma_{11} = \lambda \varepsilon_{kk} + 2\mu \varepsilon_{11}; \quad \cdots \qquad (19)$$

$$\sigma_{12} = \qquad 2\mu \varepsilon_{12}; \quad \cdots \qquad (20)$$

Note $\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \propto \dfrac{\Delta V}{V_0}$, $\varepsilon_{kk} = 0 \Rightarrow$ incompressible material

Inverse (19), (20)

$$\varepsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\right) \qquad (21)$$

$\longrightarrow \varepsilon_{11} = \dfrac{\sigma_{11}}{E}$ uniaxial tension

E : Young's modulus, $E = \dfrac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$ (22)

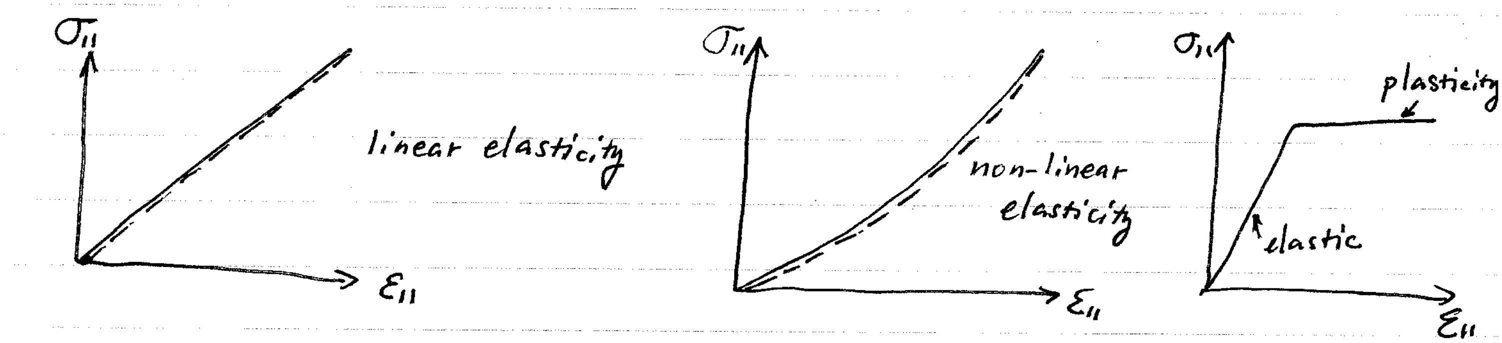
ν : Poisson's ratio, $\nu = \dfrac{\lambda}{2(\lambda + \mu)}$ (23)

$$\varepsilon_{12} = \frac{1}{2\mu}\sigma_{12} = \frac{1+\nu}{E}\sigma_{12}, \qquad (24)$$

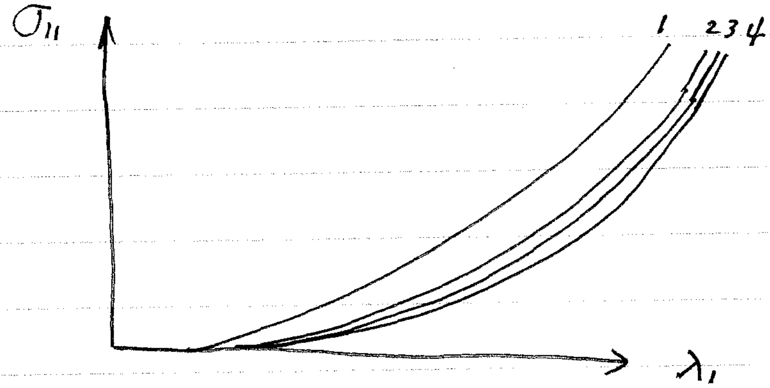
μ or G — shear modulus, $\mu = \dfrac{E}{2(1+\nu)}$ (25)

isotropic case

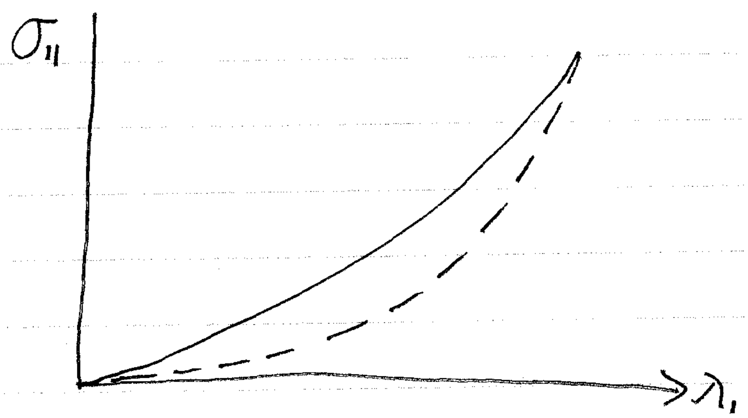
Uniaxial Stress-Strain:



In biological tissues, the elastic deformation



Soft tissue
- Non linear, large deformation.
- Preconditioning is necessary for stress-strain curve to become repeatable.



After preconditioning, unloading goes back to its starting point.

— Hysteresis

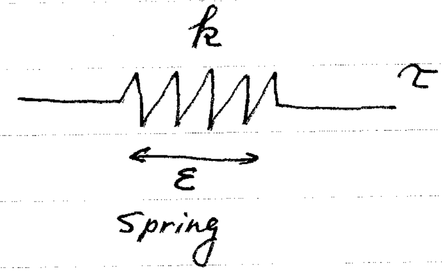
— Loop changes with loading frequency?

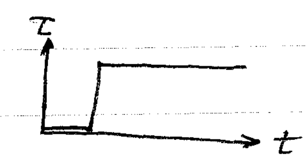
o The behavior can be modeled (approximately) by (Linear) Viscoelasticity models.

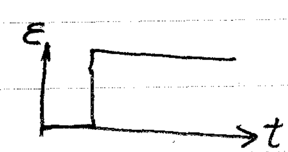
Stress τ is a function of strain ε and strain rate $\dot{\varepsilon}$

(26) $\tau = k\,\varepsilon$

linear elastic solid

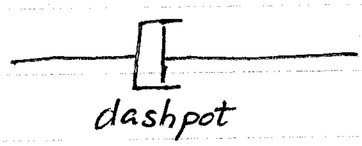


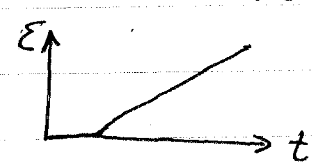




(27) $\tau = \eta\,\dot{\varepsilon}$

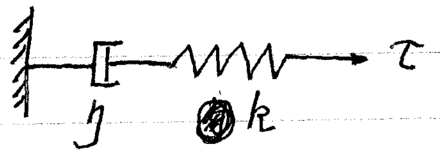
viscous fluid

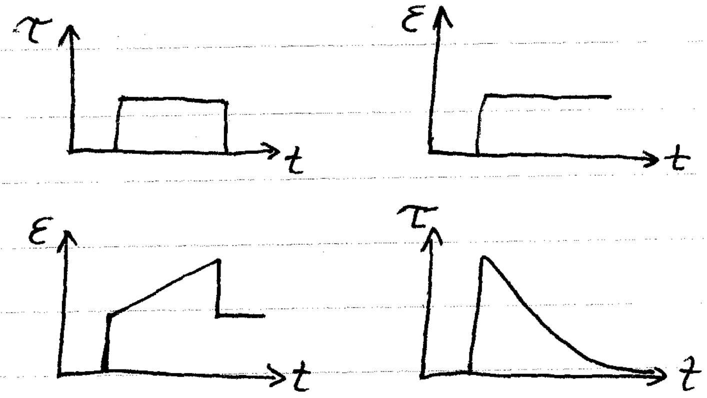




Three linear viscoelasticity models:

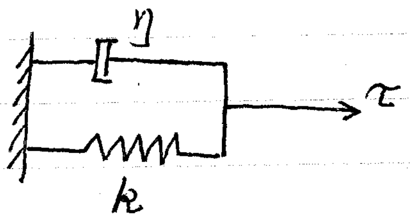
(a) Maxwell

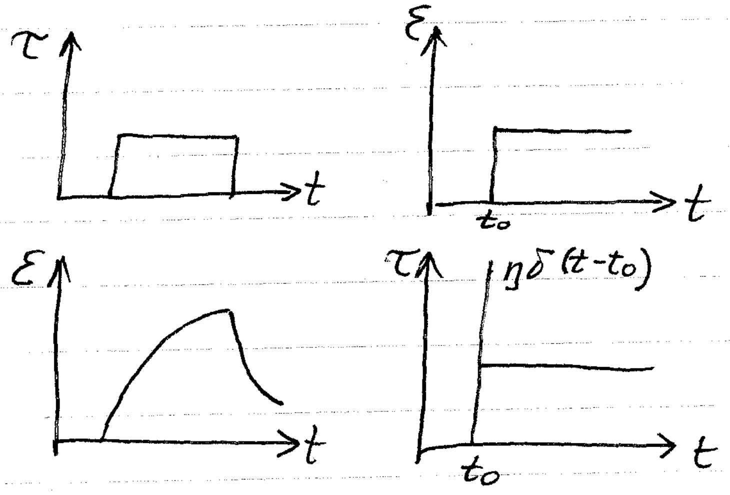




$$\frac{d\varepsilon}{dt} = \frac{1}{k}\frac{d\tau}{dt} + \frac{\tau}{\eta}$$

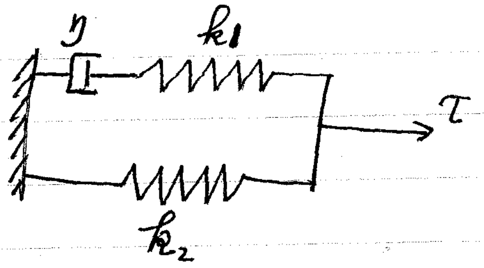
(b) Voigt

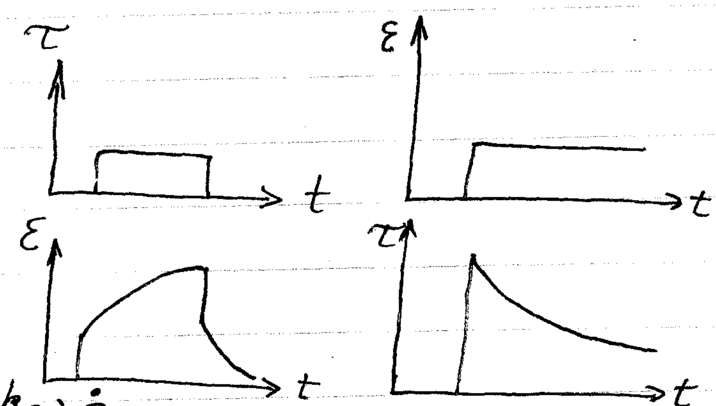




$$\tau = k\varepsilon + \eta\dot{\varepsilon}$$

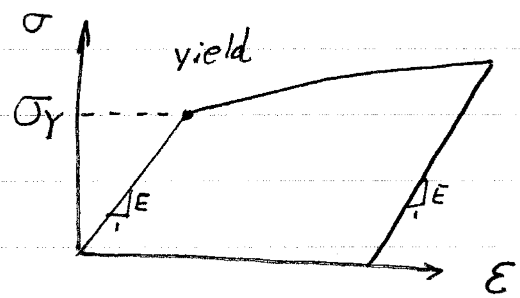
(c) Standard Linear Solid





$$\tau + \frac{\eta}{k_1}\dot{\tau} = k_2\varepsilon + \eta\left(1+\frac{k_2}{k_1}\right)\dot{\varepsilon}$$

Plasticity:



1-D

After unloading, permenent deformation remains.

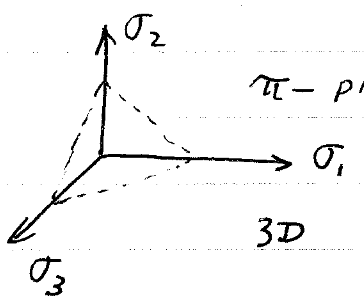
$\frac{\partial \sigma}{\partial \varepsilon} = 0$: perfect plasticity

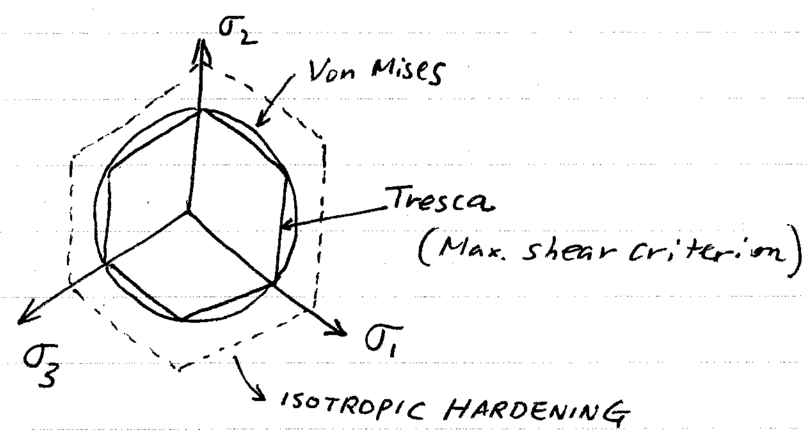
$\frac{\partial \sigma}{\partial \varepsilon} > 0$: Strain hardening

$\sigma = k' \varepsilon^{n}$ is often used to describe strain harding

n: strain hardening exponent.

Principal Stresses : $\sigma_1 \geq \sigma_2 \geq \sigma_3$





Stress Deviation Tensor : $\sigma'_{ij} = \sigma_{ij} - p\delta_{ij}$, $p = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$

$$J_2 = \frac{1}{2}\sigma'_{ij}\sigma'_{ij} \qquad I_1 = \sigma_{ii} \qquad p = \frac{1}{3} I_1$$

Mises yield function (J_2 yield criterion) : $f(\sigma'_{ij}) = J_2 - k^2 = 0$

(k = simple shear yield stress)

Example of pressure sensitive yielding:

$$f(\sigma_{ij}) = J_2 - (k + \alpha p)^2 = 0$$