Classical Mechanics: MIT 8.01 Course Notes

For Dorothea

Fundamental Physical Constants

Quantity	**Symbol**	**Value**
Avogadro's number	N_A	$6.02214129(27)\times10^{23}$ / mol
Boltzmann's constant	k_B	$1.3806488(13)\times10^{-23}$ J/K
Coulomb constant	$k_e = 1/4\pi\varepsilon_0$	$8.987551787\cdots\times10^{9}\ \mathrm{N\cdot m^2\cdot C^{-2}}$
Elementary charge	e	$1.602176565(35)\times10^{-19}$ C
Electron mass	m_e	$9.10938215(45)\times10^{-31}$ kg
Gravitational constant	G	$6.67384(80)\cdots\times10^{-11}\ \mathrm{N\cdot m^2\cdot kg^{-2}}$
Neutron mass	m_n	$1.674927351(74)\times10^{-27}$ kg
Permeability of free space	μ_0	$4\pi\times10^{-7}\ \mathrm{T\cdot m/A}$
Permittivity of free space	$\varepsilon_0 = 1/\mu_0 c^2$	$8.854187817\cdots\times10^{-12}\ \mathrm{C^2/N\cdot m^2}$
Planck's constant	h	$6.62606957(29)\times10^{-34}\ \mathrm{J\cdot s}$
Proton mass	m_p	$1.672621777(74)\times10^{-27}$ kg
Speed of light	c	$2.99792458\times10^{8}\ \mathrm{m\cdot s^{-1}}$

Astronomical Data

Earth	**Value**
Solar mass	$(1.98855 \pm 0.00025) \times 10^{30}$ kg
Earth mass	5.97219×10^{24} kg
Earth mean radius	6.371009×10^{6} m
Mean solar day	8.6400×10^{4} s
Earth-Sun Orbit	
Aphelion	$1.52098232 \times 10^{11}$ km
Perihelion	$1.470098290 \times 10^{11}$ km
Eccentricity	0.01671123
Orbital Period	3.15581495×10^{7} s
Moon	
Moon mass	7.3477×10^{22} kg
Moon mean radius	1.73710×10^{6} m
Moon orbital period (sidereal month)	2.3605847×10^{6} s
Moon synodic period	2.5514429×10^{6} s

Chapter 1 The History and Limitations of Classical Mechanics

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Chapter 1.1 Introduction

Classical mechanics is the mathematical science that studies the displacement of bodies under the action of forces. Gailieo Galilee initiated the modern era of mechanics by using mathematics to describe the motion of bodies. His *Mechanics,* published in 1623, introduced the concepts of force and described the constant accelerated motion of objects near the surface of the Earth. Sixty years later Isaac Newton formulated his Laws of Motion, which he published in 1687 under the title, *Philosophiae Naturalis Principia Mathematica* (*Mathematical Principles of Natural Philosophy*). In the third book, subtitled *De mundi systemate* (*On the system of the world*), Newton solved the greatest scientific problem of his time by applying his Universal Law of Gravitation to determine the motion of planets. Newton established a mathematical approach to the analysis of physical phenomena in which he stated that it was unnecessary to introduce final causes (hypothesis) that have no experimental basis, *"Hypotheses non fingo* (I frame no hypotheses), but that physical models are built from experimental observations and then made general by induction. This led to a great century of applications of the principles of *Newtonian mechanics* to many new problems culminating in the work of Leonhard Euler. Euler began a systematic study of the three dimensional motion of rigid bodies, leading to a set of dynamical equations now known as *Euler's equations of motion*.

Alongside this development and refinement of the concept of force and its application to the description of motion, the concept of energy slowly emerged, culminating in the middle of the nineteenth century in the discovery of the *principle of conservation of energy* and its immediate applications to the laws of *thermodynamics*. Conservation principles are now central to our study of mechanics; the conservation of momentum, energy, and angular momentum enabled a new reformulation of classical mechanics.

During this period, the experimental methodology and mathematical tools of Newtonian mechanics were applied to other non-rigid systems of particles leading to the development of *continuum mechanics.* The theories of *fluid mechanics*, *wave mechanics*, and *electromagnetism* emerged leading to the development of the wave theory of light. However there were many perplexing aspects of the wave theory of light, for example, does light propagate through a medium, the "ether"? A series of optics experiments, culminating in the Michelson-Morley experiment in 1887 ruled out the hypothesis of a stationary medium. Many attempts were made to reconcile the experimental evidence with classical mechanics but the challenges were more fundamental. The basics concepts of absolute time and absolute space, which Newton had defined in the *Principia,* were themselves inadequate to explain a host of experimental observations. Albert Einstein, by insisting on a fundamental rethinking of the concepts of space and time, and the relativity of motion, in his *special theory of relativity* (1905) was able to resolve the apparent conflicts between optics and Newtonian mechanics. In particular, special relativity provides the necessary framework for describing the motion of rapidly moving objects (speed greater than $v > 0.1\,c$).

A second limitation on the validity of Newtonian mechanics appeared at the microscopic length scale. A new theory, *statistical mechanics*, was developed relating the microscopic properties of individual atoms and molecules to the macroscopic or bulk thermodynamic properties of materials. Started in the middle of the nineteenth century, new observations at very small scales revealed anomalies in the predicted behavior of gases (heat capacity). It became increasingly clear that classical mechanics did not adequately explain a wide range of newly discovered phenomena at the atomic and sub-atomic length scales. An essential realization was that the language of classical mechanics was not even adequate to qualitatively describe certain microscopic phenomena. By the early part of the twentieth century, *quantum mechanics* provided a mathematical description of microscopic phenomena in complete agreement with our empirical knowledge of all non-relativistic phenomena.

In the twentieth century, as experimental observations led to a more detailed knowledge of the large-scale properties of the universe, Newton's Universal Law of Gravitation no longer accurately modeled the observed universe and needed to be replaced by *general relativity.* By the end of the twentieth century and beginning of the twenty-first century, many new observations, for example the accelerated expansion of the Universe, have required introduction of new concepts like *dark energy* that may lead once again to a fundamental rethinking of the basic concepts of physics in order to explain observed phenomena.

Chapter 2 Units, Dimensional Analysis, and Estimation

Chapter 2 Units, Dimensional Analysis, Problem Solving, Estimation, and Error Analysis

> *But we must not forget that all things in the world are connected with one another and depend on one another, and that we ourselves and all our thoughts are also a part of nature. It is utterly beyond our power to measure the changes of things by time. Quite the contrary, time is an abstraction, at which we arrive by means of the change of things; made because we are not restricted to any one definite measure, all being interconnected. A motion is termed uniform in which equal increments of space described correspond to equal increments of space described by some motion with which we form a comparison, as the rotation of the earth. A motion may, with respect to another motion, be uniform. But the question whether a motion is in itself uniform, is senseless. With just as little justice, also, may we speak of an "absolute time" --- of a time independent of change. This absolute time can be measured by comparison with no motion; it has therefore neither a practical nor a scientific value; and no one is justified in saying that he knows aught about it. It is an idle metaphysical conception.*[1]
>
> *Ernst Mach*

2.1 The Speed of light

When we observe and measure phenomena in the world, we try to assign numbers to the physical quantities with as much accuracy as we can possibly obtain from our measuring equipment. For example, we may want to determine the speed of light, which we can calculate by dividing the distance a known ray of light propagates over its travel time,

$$\text{speed of light} = \frac{\text{distance}}{\text{time}}. \tag{2.1.1}$$

In 1983 the General Conference on Weights and Measures defined the *speed of light* to be

$$c = 299{,}792{,}458 \text{ meters/second}. \tag{2.1.2}$$

This number was chosen to correspond to the most accurately measured value of the speed of light and is well within the experimental uncertainty.

2.2 International System of Units

The system of units most commonly used throughout science and technology today is the *Système International* (SI). It consists of seven *base quantities* and their corresponding *base units,* shown in Table 2.1.

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[1] E. Mach, *The Science of Mechanics,* translated by Thomas J. McCormack, Open Court Publishing Company, La Salle, Illinois, 1960, p. 273.

Table 2.1 International System of Units

Base Quantity	Base Unit
Length	meter (m)
Mass	kilogram (kg)
Time	second (s)
Electric Current	ampere (A)
Temperature	kelvin (K)
Amount of Substance	mole (mol)
Luminous Intensity	candela (cd)

We shall refer to the *dimension* of the base quantity by the quantity itself, for example

$$\text{dim length} \equiv \text{length} \equiv \text{L}, \text{dim mass} \equiv \text{mass} \equiv \text{M}, \text{dim time} \equiv \text{time} \equiv \text{T}. \qquad (2.2.1)$$

Mechanics is based on just the first three of these quantities, the MKS or meter-kilogram-second system. An alternative metric system, still widely used, is the CGS system (centimeter-gram-second).

2.2.1 Standard Mass

The unit of mass, the kilogram (kg), remains the only base unit in the International System of Units (SI) that is still defined in terms of a physical artifact, known as the "International Prototype of the Standard Kilogram." George Matthey (of Johnson Matthey) made the prototype in 1879 in the form of a cylinder, 39 mm high and 39 mm in diameter, consisting of an alloy of 90 % platinum and 10 % iridium. The international prototype is kept in the Bureau International des Poids et Mèsures (BIPM) at Sevres, France, under conditions specified by the 1st Conférence Générale des Poids et Mèsures (CGPM) in 1889 when it sanctioned the prototype and declared "This prototype shall henceforth be considered to be the unit of mass." It is stored at atmospheric pressure in a specially designed triple bell-jar. The prototype is kept in a vault with six official copies.

The 3rd Conférence Générale des Poids et Mèsures CGPM (1901), in a declaration intended to end the ambiguity in popular usage concerning the word "weight" confirmed that:

> *The kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram.*

There is a stainless steel one-kilogram standard that is used for comparisons with standard masses in other laboratories. In practice it is more common to quote a conventional mass value (or weight-in-air, as measured with the effect of buoyancy), than the standard mass. Standard mass is normally only used in specialized measurements wherever suitable copies of the prototype are stored.

Example 2.1 The International Prototype Kilogram

In order to minimize the effects of corrosion, the platinum-iridium prototype kilogram is a right cylinder with dimensions chosen to minimize the surface area for a given fixed volume. The standard kilogram is an alloy of 90 % platinum and 10 % iridium. The density of the alloy is $\rho = 21.56\ \mathrm{g \cdot cm^{-3}}$. Based on this information, (i) determine the radius of the prototype kilogram, and (ii) the ratio of the radius to the height.

Solution: The volume for a cylinder of radius r and height h is given by

$$V = \pi r^2 h\,. \tag{2.2}$$

The surface area can be expressed as a function of the radius r and the constant volume V according to

$$A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{2V}{r}\,. \tag{2.3}$$

To find the smallest surface area for a fixed volume, minimize the surface area with respect to the radius by setting

$$0 = \frac{dA}{dr} = 4\pi r - \frac{2V}{r^2}\,, \tag{2.4}$$

which we can solve for the radius

$$r = \left(\frac{V}{2\pi}\right)^{1/3}. \tag{2.5}$$

Because we also know that $V = \pi r^2 h$, we can rewrite Eq. (2.5) as

$$r^3 = \frac{\pi r^2 h}{2\pi}\,, \tag{2.6}$$

which implies that ratio of the radius to the height is

$$\frac{r}{h} = \frac{1}{2}\,. \tag{2.7}$$

The standard kilogram is an alloy of 90% platinum and 10% iridium. The density of platinum is $21.45\ \mathrm{g \cdot cm^{-3}}$ and the density of iridium is $22.55\ \mathrm{g \cdot cm^{-3}}$. Thus the density of the standard kilogram is

$$\rho = (0.90)(21.45\ \mathrm{g \cdot cm^{-3}}) + (0.10)(22.55\ \mathrm{g \cdot cm^{-3}}) = 21.56\ \mathrm{g \cdot cm^{-3}}\,, \tag{2.8}$$

and its volume is

$$V = m / \rho = (1000\text{ g}) / (21.56\text{ g}\cdot\text{cm}^{-3}) = 46.38\text{ cm}^3 . \tag{2.9}$$

For the standard mass, the radius is

$$r = \left(\frac{V}{2\pi}\right)^{1/3} = \left(\frac{46.38\text{ cm}^3}{2\pi}\right)^{1/3} \cong 1.95\text{ cm} . \tag{2.10}$$

Because the prototype kilogram is an artifact, there are some intrinsic problems associated with its use as a standard. It may be damaged, or destroyed. The prototype gains atoms due to environment wear and cleaning, at a rate of change of mass corresponding to approximately $1\ \mu\text{g / year}$, ($1\ \mu\text{g} \equiv 1\text{microgram} \equiv 1\times10^{-6}\text{ g}$).

Several new approaches to defining the SI unit of mass [kg] are currently being explored. One possibility is to define the kilogram as a fixed number of atoms of a particular substance, thus relating the kilogram to an atomic mass. Silicon is a good candidate for this approach because it can be grown as a large single crystal, in a very pure form.

Example 2.2 Mass of a Silicon Crystal

A given standard unit cell of silicon has a volume V_0 and contains N_0 atoms. The number of molecules in a given mole of substance is given by Avogadro's constant $N_A =$ $6.02214129(27)\times10^{23}\text{ mol}^{-1}$. The molar mass of silicon is given by M_{mol}. Find the mass m of a volume V in terms of V_0, N_0, V, M_{mol}, and N_A.

Solution: The mass m_0 of the unit cell is the density ρ of the silicon cell multiplied by the volume of the cell V_0,

$$m_0 = \rho V_0 . \tag{2.11}$$

The number of moles in the unit cell is the total mass, m_0, of the cell, divided by the molar mass M_{mol},

$$n_0 = m_0 / M_{\text{mol}} = \rho V_0 / M_{\text{mol}} . \tag{2.12}$$

The number of atoms in the unit cell is the number of moles times the Avogadro constant, N_A,

$$N_0 = n_0 N_A = \frac{\rho V_0 N_A}{M_{\text{mol}}} . \tag{2.13}$$

The density of the crystal is related to the mass m of the crystal divided by the volume V of the crystal,

$$\rho = m / V . \tag{2.14}$$

The number of atoms in the unit cell can be expressed as

$$N_0 = \frac{mV_0 N_A}{VM_{\text{mol}}} . \tag{2.15}$$

The mass of the crystal is

$$m = \frac{M_{\text{mol}}}{N_A} \frac{V}{V_0} N_0 \tag{2.16}$$

The molar mass, unit cell volume and volume of the crystal can all be measured directly. Notice that M_{mol} / N_A is the mass of a single atom, and $(V / V_0)N_0$ is the number of atoms in the volume. This accuracy of the approach depends on how accurate the Avogadro constant can be measured. Currently, the measurement of he Avogadro constant has a relative uncertainty of 1 part in 10^8, which is equivalent to the uncertainty in the present definition of the kilogram.

2.2.2 Atomic Clock and the Definition of the Second

Isaac Newton, in the *Philosophiae Naturalis Principia Mathematica* ("Mathematical Principles of Natural Philosophy"), distinguished between time as duration and an absolute concept of time,

> "*Absolute true and mathematical time, of itself and from its own nature, flows equably without relation to anything external, and by another name is called duration: relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by means of motion, which is commonly used instead of true time; such as an hour, a day, a month, a year.* "[2].

The development of clocks based on atomic oscillations allowed measures of timing with accuracy on the order of 1 part in 10^{14}, corresponding to errors of less than one microsecond (one millionth of a second) per year. Given the incredible accuracy of this measurement, and clear evidence that the best available timekeepers were atomic in nature, the *second* [s] was redefined in 1967 by the International Committee on Weights

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[2] Isaac Newton. *Mathematical Principles of Natural Philosophy*. Translated by Andrew Motte (1729). Revised by Florian Cajori. Berkeley: University of California Press, 1934. p. 6.

and Measures as a certain number of cycles of electromagnetic radiation emitted by cesium atoms as they make transitions between two designated quantum states:

> *The second is the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.*

2.2.3 Meter

The meter [m] was originally defined as 1/10,000,000 of the arc from the Equator to the North Pole along the meridian passing through Paris. To aid in calibration and ease of comparison, the meter was redefined in terms of a length scale etched into a platinum bar preserved near Paris. Once laser light was engineered, the meter was redefined by the 17th Conférence Générale des Poids et Mèsures (CGPM) in 1983 to be a certain number of wavelengths of a particular monochromatic laser beam.

> *The meter is the length of the path traveled by light in vacuum during a time interval of 1/299 792 458 of a second.*

Example 2.3 Light-Year

Astronomical distances are sometimes described in terms of *light-years* [ly]. A light-year is the distance that light will travel in one year [yr]. How far in meters does light travel in one year?

Solution: Using the relationship $\text{distance} = (\text{speed of light})\cdot(\text{time})$, one light year corresponds to a distance. Because the speed of light is given in terms of meters per second, we need to know how many seconds are in a year. We can accomplish this by converting units. We know that

$1 \text{ year} = 365.25 \text{ days},\ 1 \text{ day} = 24 \text{ hours},\ 1 \text{ hour} = 60 \text{ minutes},\ 1 \text{ minute} = 60 \text{ seconds}$

Putting this together we find that the number of seconds in a year is

$$1\text{ year} = \left(365.25\text{ day}\right)\left(\frac{24\text{ hours}}{1\text{ day}}\right)\left(\frac{60\text{ min}}{1\text{ hour}}\right)\left(\frac{60\text{ s}}{1\text{ min}}\right) = 31{,}557{,}600\text{ s}. \qquad (2.2.17)$$

The distance that light travels in a one year is

$$1\text{ ly} = \left(\frac{299{,}792{,}458\text{ m}}{1\text{ s}}\right)\left(\frac{31{,}557{,}600\text{ s}}{1\text{ yr}}\right)\left(1\text{ yr}\right) = 9.461\times 10^{15}\text{ m}. \qquad (2.2.18)$$

The distance to the nearest star, a faint red dwarf star, Proxima Centauri, is 4.24 ly.

2.2.4 Radians

Consider the triangle drawn in Figure 2.1. The basic trigonometric functions of an angle θ in a right-angled triangle ONB are $\sin(\theta) = y / r$, $\cos(\theta) = x / r$, and $\tan(\theta) = y / x$.

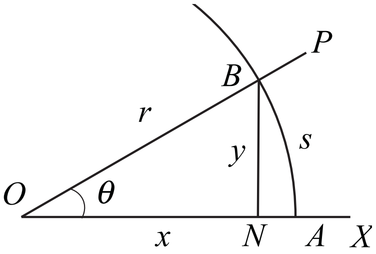

Figure 2.1 Trigonometric relations

It is very important to become familiar with using the measure of the angle θ itself as expressed in *radians* [rad]. Let θ be the angle between two straight lines OX and OP. Draw a circle of radius r centered at O. The lines OP and OX cut the circle at the points A and B where $OA = OB = r$. Denote the length of the arc AB by s, then the radian measure of θ is given by

$$\theta = s / r, \tag{2.2.19}$$

and the ratio is the same for circles of any radii centered at O -- just as the ratios y / r and y / x are the same for all right triangles with the angle θ at O. As θ approaches 360°, s approaches the complete circumference $2\pi r$ of the circle, so that $360^\circ = 2\pi \text{ rad}$.

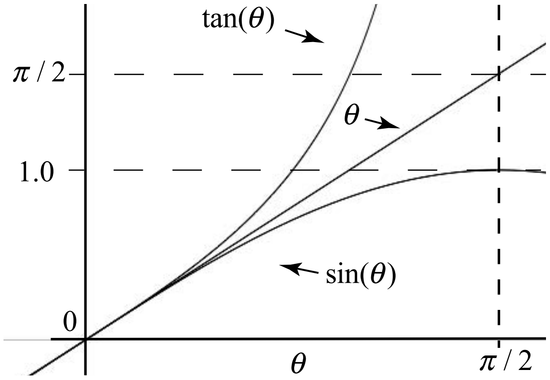

Figure 2.2 Radians compared to trigonometric functions.

Let's compare the behavior of $\sin(\theta)$, $\tan(\theta)$ and θ itself for small angles. One can see from Figure 2.1 that $s / r > y / r$. It is less obvious that $y / x > \theta$. It is very instructive to plot $\sin(\theta)$, $\tan(\theta)$, and θ as functions of θ [rad] between 0 and $\pi / 2$

on the same graph (see Figure 2.2). For small θ, the values of all three functions are almost equal. But how small is "small"? An acceptable condition is for $\theta << 1$ in radians.

We can show this with a few examples. Recall that $360^\circ = 2\pi\ \text{rad}$, $57.3^\circ = 1\ \text{rad}$, so an angle $6^\circ \cong (6^\circ)(2\pi\ \text{rad}/360^o) \cong 0.1\ \text{rad}$ when expressed in radians. In Table 2.2 we compare the value of θ (measured in radians) with $\sin(\theta)$, $\tan(\theta)$, $(\theta - \sin\theta)/\theta$, and $(\theta - \tan\theta)/\theta$, for $\theta = 0.1\ \text{rad}$, $0.2\ \text{rad}$, $0.5\ \text{rad}$, and $1.0\ \text{rad}$.

Table 2.2 Small Angle Approximation

θ [rad]	θ [deg]	$\sin(\theta)$	$\tan(\theta)$	$(\theta - \sin\theta)/\theta$	$(\theta - \tan\theta)/\theta$
0.1	5.72958	0.09983	0.10033	0.00167	-0.00335
0.2	11.45916	0.19867	0.20271	0.00665	-0.01355
0.5	28.64789	0.47943	0.54630	0.04115	-0.09260
1.0	57.29578	0.84147	1.55741	0.15853	-0.55741

The values for $(\theta - \sin\theta)/\theta$, and $(\theta - \tan\theta)/\theta$, for $\theta = 0.2\ \text{rad}$ are less than $\pm 1.4\%$. Provided that θ is not too large, the approximation that

$$\sin(\theta) \simeq \tan(\theta) \simeq \theta\,, \qquad (2.2.20)$$

called the *small angle approximation*, can be used almost interchangeably, within some small percentage error. This is the basis of many useful approximations in physics calculations.

Example 2.4 Parsec

A standard astronomical unit is the parsec. Consider two objects that are separated by a distance of one astronomical unit, $1\text{AU} = 1.50 \times 10^{11}\ \text{m}$, which is the mean distance between the earth and sun. (One astronomical unit is roughly equivalent to eight light minutes, $1\text{AU} = 8.3\text{light-minutes}$.) One parsec is the distance at which one astronomical unit subtends an angle $\theta = 1\ \text{arcsecond} = (1/3600)\ \text{degree}$. Suppose is a spacecraft is located in a space a distance 1 parsec from the Sun as shown in Figure 2.3. How far is the spacecraft in terms of light years and meters?

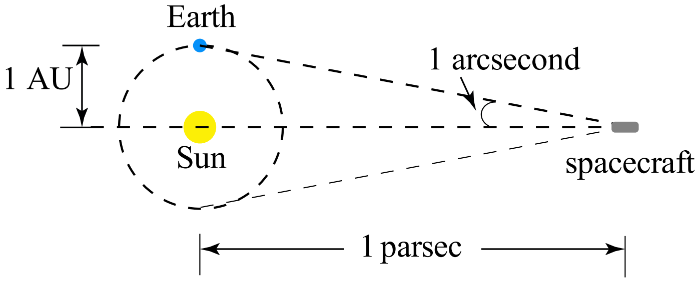

Figure 2.3 Example 2.4

Because one arc second corresponds to a very small angle, one parsec is therefore equal to distance divided by angle, hence

$$\begin{aligned}1\,\text{pc} &= \frac{(1\,\text{AU})}{(1/3600)} = (2.06\times10^{5}\,\text{AU})\left(\frac{1.50\times10^{11}\,\text{m}}{1\,\text{AU}}\right) = 3.09\times10^{16}\ \text{m} \\ &= (3.09\times10^{16}\ \text{m})\left(\frac{1\,\text{ly}}{9.46\times10^{15}\,\text{m}}\right) = 3.26\ \text{ly}\end{aligned} \quad . \ (2.2.21)$$

2.2.5 Steradians

The *steradian* [sr] is the unit of solid angle that, having its vertex in the center of a sphere, cuts off an area of the surface of the sphere equal to that of a square with sides of length equal to the radius of the sphere. The conventional symbol for steradian measure is Ω, the uppercase Greek letter "Omega." The total solid angle Ω_{sph} of a sphere is then found by dividing the surface area of the sphere by the square of the radius,

$$\Omega_{\text{sph}} = 4\pi r^2 / r^2 = 4\pi \qquad (2.2.22)$$

This result is independent of the radius of the sphere.

2.2.6 Radiant Intensity

> *"The SI unit, candela, is the luminous intensity of a source that emits monochromatic radiation of frequency* $540\times10^{12}\ \text{s}^{-1}$*, in a given direction, and that has a radiant intensity in that direction of 1/683 watts per steradian."*

Note that "in a given direction" cannot be taken too literally. The intensity is measured per steradian of spread, so if the radiation has no spread of directions, the luminous intensity would be infinite.

2.3 Dimensions of Commonly Encountered Quantities

Many physical quantities are derived from the base quantities by a set of algebraic relations defining the physical relation between these quantities. The dimension of the derived quantity is written as a power of the dimensions of the base quantities. For example velocity is a derived quantity and the dimension is given by the relationship

$$\text{dim velocity} = (\text{length})/(\text{time}) = \text{L}\cdot\text{T}^{-1}. \qquad (2.3.1)$$

where $\text{L} \equiv \text{length}$, $\text{T} \equiv \text{time}$. Force is also a derived quantity and has dimension

$$\text{dim force} = \frac{\text{(mass)(dim velocity)}}{\text{(time)}} \quad . \tag{2.3.2}$$

where $M \equiv \text{mass}$. We can also express force in terms of mass, length, and time by the relationship

$$\text{dim force} = \frac{\text{(mass)(length)}}{\text{(time)}^2} = \text{M} \cdot \text{L} \cdot \text{T}^{-2} . \tag{2.3.3}$$

The derived dimension of kinetic energy is

$$\text{dim kinetic energy} = \text{(mass)(dim velocity)}^2 , \tag{2.3.4}$$

which in terms of mass, length, and time is

$$\text{dim kinetic energy} = \frac{\text{(mass)(length)}^2}{\text{(time)}^2} = \text{M} \cdot \text{L}^2 \cdot \text{T}^{-2} . \tag{2.3.5}$$

The derived dimension of work is

$$\text{dim work} = \text{(dim force)(length)} , \tag{2.3.6}$$

which in terms of our fundamental dimensions is

$$\text{dim work} = \frac{\text{(mass)(length)}^2}{\text{(time)}^2} = \text{M} \cdot \text{L}^2 \cdot \text{T}^{-2} . \tag{2.3.7}$$

So work and kinetic energy have the same dimensions. Power is defined to be the rate of change in time of work so the dimensions are

$$\text{dim power} = \frac{\text{dim work}}{\text{time}} = \frac{\text{(dim force)(length)}}{\text{time}} = \frac{\text{(mass)(length)}^2}{\text{(time)}^3} = \text{M} \cdot \text{L}^2 \cdot \text{T}^{-3} . \tag{2.3.8}$$

In Table 2.3 we include the derived dimensions of some common mechanical quantities in terms of mass, length, and time.

2.3.1 Dimensional Analysis

There are many phenomena in nature that can be explained by simple relationships between the observed phenomena.

Table 2.3 Dimensions of Some Common Mechanical Quantities

$\mathrm{M} \equiv \text{mass},\ \mathrm{L} \equiv \text{length},\ \mathrm{T} \equiv \text{time}$

Quantity	Dimension	MKS unit
Angle	dimensionless	Dimensionless = radian
Solid Angle	dimensionless	Dimensionless = sterradian
Area	L^2	m^2
Volume	L^3	m^3
Frequency	T^{-1}	s^{-1} = hertz = Hz
Velocity	$\mathrm{L} \cdot \mathrm{T}^{-1}$	$\mathrm{m} \cdot \mathrm{s}^{-1}$
Acceleration	$\mathrm{L} \cdot \mathrm{T}^{-2}$	$\mathrm{m} \cdot \mathrm{s}^{-2}$
Angular Velocity	T^{-1}	$\mathrm{rad} \cdot \mathrm{s}^{-1}$
Angular Acceleration	T^{-2}	$\mathrm{rad} \cdot \mathrm{s}^{-2}$
Density	$\mathrm{M} \cdot \mathrm{L}^{-3}$	$\mathrm{kg} \cdot \mathrm{m}^{-3}$
Momentum	$\mathrm{M} \cdot \mathrm{L} \cdot \mathrm{T}^{-1}$	$\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-1}$
Angular Momentum	$\mathrm{M} \cdot \mathrm{L}^2 \cdot \mathrm{T}^{-1}$	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-1}$
Force	$\mathrm{M} \cdot \mathrm{L} \cdot \mathrm{T}^{-2}$	$\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}$ = newton = N
Work, Energy	$\mathrm{M} \cdot \mathrm{L}^2 \cdot \mathrm{T}^{-2}$	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$ = joule = J
Torque	$\mathrm{M} \cdot \mathrm{L}^2 \cdot \mathrm{T}^{-2}$	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$
Power	$\mathrm{M} \cdot \mathrm{L}^2 \cdot \mathrm{T}^{-3}$	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-3}$ = watt = W
Pressure	$\mathrm{M} \cdot \mathrm{L}^{-1} \cdot \mathrm{T}^{-2}$	$\mathrm{kg} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}^{-2}$ = pascal = Pa

Example 2.5 Period of a Pendulum

Consider a simple pendulum consisting of a massive bob suspended from a fixed point by a string. Let T denote the time interval (period of the pendulum) that it takes the bob to complete one cycle of oscillation. How does the period of the simple pendulum depend on the quantities that define the pendulum and the quantities that determine the motion?

Solution: What possible quantities are involved? The length of the pendulum l, the mass of the pendulum bob m, the gravitational acceleration g, and the angular amplitude of the bob θ_0 are all possible quantities that may enter into a relationship for the period of the swing. Have we included every possible quantity? We can never be sure but let's first work with this set and if we need more than we will have to think harder! Our problem is then to find a function f such that

$$T = f(l, m, g, \theta_0) \tag{2.3.9}$$

We first make a list of the dimensions of our quantities as shown in Table 2.4.

Table 2.4 Dimensions of Quantities Relevant to the Period of Pendulum

Name of Quantity	Symbol	Dimensional Formula
Time of swing	t	T
Length of pendulum	l	L
Mass of pendulum	m	M
Gravitational acceleration	g	$\mathrm{L} \cdot \mathrm{T}^{-2}$
Angular amplitude of swing	θ_0	No dimension

Our first observation is that the mass of the bob cannot enter into our relationship, as our final quantity has no dimensions of mass and no other quantity has dimensions of mass. Let's focus on the length of the string and the gravitational acceleration. In order to eliminate length, these quantities must divide each other when appearing in some functional relation for the period T. If we choose the combination l / g, the dimensions are

$$\dim[l / g] = \frac{\text{length}}{\text{length/(time)}^2} = (\text{time})^2 \tag{2.3.10}$$

It appears that the time of swing may proportional to the square root of this ratio. Thus we have a candidate formula

$$T \sim \left(\frac{l}{g}\right)^{1/2}. \tag{2.3.11}$$

(in the above expression, the symbol " $\sim$ " represents a proportionality, not an approximation). Because the angular amplitude θ_0 is dimensionless, it may or may not appear. We can account for this by introducing some function $y(\theta_0)$ into our relationship, which is beyond the limits of this type of analysis. The period is then

$$T = y(\theta_0)\left(\frac{l}{g}\right)^{1/2}. \tag{2.3.12}$$

We shall discover later on that $y(\theta_0)$ is nearly independent of the angular amplitude θ_0 for very small amplitudes and is equal to $y(\theta_0) = 2\pi$,

$$T = 2\pi\left(\frac{l}{g}\right)^{1/2} \tag{2.3.13}$$

2.4 Order of Magnitude Estimates - Fermi Problems

Counting is the first mathematical skill we learn. We came to use this skill by distinguishing elements into groups of similar objects, but counting becomes problematic when our desired objects are not easily identified, or there are too many to count. Rather than spending a huge amount of effort to attempt an exact count, we can try to estimate the number of objects. For example, we can try to estimate the total number of grains of sand contained in a bucket of sand. Because we can see individual grains of sand, we expect the number to be very large but finite. Sometimes we can try to estimate a number, which we are fairly sure but not certain is finite, such as the number of particles in the universe.

We can also assign numbers to quantities that carry dimensions, such as mass, length, time, or charge, which may be difficult to measure exactly. We may be interested in estimating the mass of the air inside a room, or the length of telephone wire in the United States, or the amount of time that we have slept in our lives. We choose some set of units, such as kilograms, miles, hours, and coulombs, and then we can attempt to estimate the number with respect to our standard quantity.

Often we are interested in estimating quantities such as speed, force, energy, or power. We may want to estimate our natural walking speed, or the force of wind acting against a bicycle rider, or the total energy consumption of a country, or the electrical power necessary to operate a university. All of these quantities have no exact, well-defined value; they instead lie within some range of values.

When we make these types of estimates, we should be satisfied if our estimate is reasonably close to the middle of the range of possible values. But what does "reasonably close" mean? Once again, this depends on what quantities we are estimating. If we are describing a quantity that has a very large number associated with it, then an estimate within an order of magnitude should be satisfactory. The number of molecules in a breath of air is close to 10^{22}; an estimate anywhere between 10^{21} and 10^{23} molecules is close enough. If we are trying to win a contest by estimating the number of marbles in a glass container, we cannot be so imprecise; we must hope that our estimate is within 1% of the real quantity. These types of estimations are called *Fermi problems*. The technique is named after the physicist Enrico Fermi, who was famous for making these sorts of "back of the envelope" calculations.

2.4.1 Methodology for Estimation Problems

Estimating is a skill that improves with practice. Here are two guiding principles that may help you get started.

(1) You must identify a set of quantities that can be estimated or calculated.

(2) You must establish an approximate or exact relationship between these quantities and the quantity to be estimated in the problem.

Estimations may be characterized by a precise relationship between an estimated quantity and the quantity of interest in the problem. When we estimate, we are drawing upon what we know. But different people are more familiar with certain things than others. If you are basing your estimate on a fact that you already know, the accuracy of your estimate will depend on the accuracy of your previous knowledge. When there is no precise relationship between estimated quantities and the quantity to be estimated in the problem, then the accuracy of the result will depend on the type of relationships you decide upon. There are often many approaches to an estimation problem leading to a reasonably accurate estimate. So use your creativity and imagination!

Example 2.6 Lining Up Pennies

Suppose you want to line pennies up, diameter to diameter, until the total length is 1 kilometer . How many pennies will you need? How accurate is this estimation?

Solution: The first step is to consider what type of quantity is being estimated. In this example we are estimating a dimensionless scalar quantity, the number of pennies. We can now give a precise relationship for the number of pennies needed to mark off 1 kilometer

$$\#\ \text{of pennies} = \frac{\text{total distance}}{\text{diameter of penny}} . \tag{2.4.1}$$

We can estimate a penny to be approximately 2 centimeters wide. Therefore the number of pennies is

$$\begin{aligned} \#\ \text{of pennies} &= \frac{\text{total distance}}{\text{length of a penny}} = \frac{(1\ \text{km})}{(2\ \text{cm})(1\ \text{km} / 10^5\ \text{cm})} \\ &= 50{,}000\ \text{pennies} = 5 \times 10^4\ \text{pennies}. \end{aligned} \tag{2.4.2}$$

When applying numbers to relationships we must be careful to convert units whenever necessary. How accurate is this estimation? If you measure the size of a penny, you will find out that the width is $1.9\ \text{cm}$, so our estimate was accurate to within 5%. This accuracy was fortuitous. Suppose we estimated the length of a penny to be 1 cm. Then our estimate for the total number of pennies would be within a factor of 2, a margin of error we can live with for this type of problem.

Example 2.7 Estimation of Mass of Water on Earth

Estimate the mass of the water on the Earth.

Solution: In this example we are estimating mass, a quantity that is a fundamental in SI units, and is measured in kg. We start by approximating that the amount of water on

Earth is approximately equal to the amount of water in all the oceans. Initially we will try to estimate two quantities: the density of water and the volume of water contained in the oceans. Then the relationship we want is

$$\text{mass} = (\text{density})(\text{volume})\,. \tag{2.4.3}$$

One of the hardest aspects of estimation problems is to decide which relationship applies. One way to check your work is to check dimensions. Density has dimensions of mass/volume, so our relationship is correct dimensionally.

The density of fresh water is $\rho = 1.0\ \text{g}\cdot\text{cm}^{-3}$; the density of seawater is slightly higher, but the difference won't matter for this estimate. You could estimate this density by estimating how much mass is contained in a one-liter bottle of water. (The density of water is a point of reference for all density problems. Suppose we need to estimate the density of iron. If we compare iron to water, we might estimate that iron is 5 to 10 times denser than water. The actual density of iron is $\rho_{\text{iron}} = 7.8\ \text{g}\cdot\text{cm}^{-3}$).

Because there is no precise relationship, estimating the volume of water in the oceans is much harder. Let's model the volume occupied by the oceans as if the water completely covers the earth, forming a spherical shell of radius R_E and thickness d (Figure 2.4, which is decidedly not to scale), where R_E is the radius of the earth and d is the average depth of the ocean. The volume of that spherical shell is

$$\text{volume} \cong 4\pi R_{\text{earth}}^2 d\,. \tag{2.4.4}$$

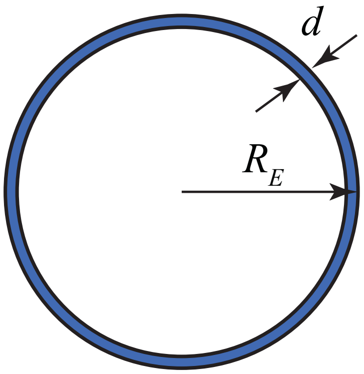

Figure 2.4 A model for estimating the mass of the water on Earth.

We also estimate that the oceans cover about 75% of the surface of the earth. So we can refine our estimate that the volume of the oceans is

$$\text{volume} \cong (0.75)(4\pi R_E{}^2 d)\,. \tag{2.4.5}$$

We therefore have two more quantities to estimate, the average depth of the ocean, which we can estimate as $d \cong 1\text{km}$, and the radius of the earth, which is approximately $R_E \cong 6\times10^3\text{km}$. (The quantity that you may remember is the circumference of the earth, about $25{,}000\,\text{miles}$. Historically the circumference of the earth was defined to be $4\times10^7\,\text{m}$). The radius R_E and the circumference s are exactly related by

$$s = 2\pi R_E\,. \tag{2.4.6}$$

Thus

$$R_E = \frac{s}{2\pi} = \frac{\left(2.5\times10^4\ \text{mi}\right)\left(1.6\ \text{km}\cdot\text{mi}^{-1}\right)}{2\pi} = 6.4\times10^3\ \text{km} \tag{2.4.7}$$

We will use $R_E \cong 6\times10^3\text{km}$; additional accuracy is not necessary for this problem, since the ocean depth estimate is clearly less accurate. In fact, the factor of 75% is not needed, but included more or less from habit. Altogether, our estimate for the mass of the oceans is

$$\begin{aligned}
&\text{mass} = (\text{density})(\text{volume}) \cong \rho(0.75)(4\pi {R_E}^2 d)\\
&\text{mass} \cong \left(\frac{1\text{g}}{\text{cm}^3}\right)\left(\frac{1\,\text{kg}}{10^3\ \text{g}}\right)\left(\frac{(10^5\ \text{cm})^3}{(1\,\text{km})^3}\right)(0.75)(4\pi)(6\times10^3\ \text{km})^2(1\text{km}) \qquad (2.4.8)\\
&\text{mass} \cong 3\times10^{20}\ \text{kg} \cong 10^{20}\ \text{kg}.
\end{aligned}$$

Chapter 3 Vectors

Chapter 3 Vectors

> *Philosophy is written in this grand book, the universe which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles and other geometric figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth.*[1]
>
> *Galileo Galilee*

3.1 Vector Analysis

3.1.1 Introduction to Vectors

Certain physical quantities such as mass or the absolute temperature at some point in space only have magnitude. A single number can represent each of these quantities, with appropriate units, which are called **scalar** quantities. There are, however, other physical quantities that have both magnitude and direction. Force is an example of a quantity that has both direction and magnitude (strength). Three numbers are needed to represent the magnitude and direction of a vector quantity in a three dimensional space. These quantities are called **vector** quantities. Vector quantities also satisfy two distinct operations, vector addition and multiplication of a vector by a scalar. We can add two forces together and the sum of the forces must satisfy the rule for vector addition. We can multiply a force by a scalar thus increasing or decreasing its strength. Position, displacement, velocity, acceleration, force, and momentum are all physical quantities that can be represented mathematically by vectors. The set of vectors and the two operations form what is called a **vector space**. There are many types of vector spaces but we shall restrict our attention to the very familiar type of vector space in three dimensions that most students have encountered in their mathematical courses. We shall begin our discussion by defining what we mean by a vector in three dimensional space, and the rules for the operations of vector addition and multiplication of a vector by a scalar.

3.1.2 Properties of Vectors

A vector is a quantity that has both direction and magnitude. Let a vector be denoted by the symbol $\vec{\mathbf{A}}$. The magnitude of $\vec{\mathbf{A}}$ is $|\vec{\mathbf{A}}| \equiv A$. We can represent vectors as geometric objects using arrows. The length of the arrow corresponds to the magnitude of the vector. The arrow points in the direction of the vector (Figure 3.1).

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[1] Galileo Galilei, *The Assayer*, tr. Stillman Drake (1957), *Discoveries and Opinions of Galileo* pp. 237-8.

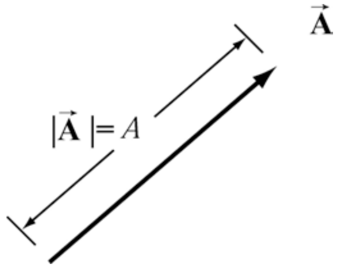


Figure 3.1 Vectors as arrows.

There are two defining operations for vectors:

(1) Vector Addition:

Vectors can be added. Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors. We define a new vector, $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$, the "vector addition" of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, by a geometric construction. Draw the arrow that represents $\vec{\mathbf{A}}$. Place the tail of the arrow that represents $\vec{\mathbf{B}}$ at the tip of the arrow for $\vec{\mathbf{A}}$ as shown in Figure 3.2a. The arrow that starts at the tail of $\vec{\mathbf{A}}$ and goes to the tip of $\vec{\mathbf{B}}$ is defined to be the "vector addition" $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$. There is an equivalent construction for the law of vector addition. The vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ can be drawn with their tails at the same point. The two vectors form the sides of a parallelogram. The diagonal of the parallelogram corresponds to the vector $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$, as shown in Figure 3.2b.

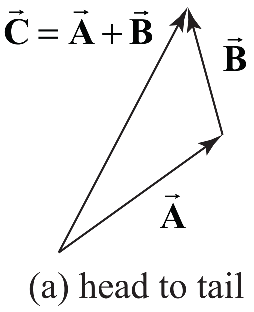


Figure 3.2a

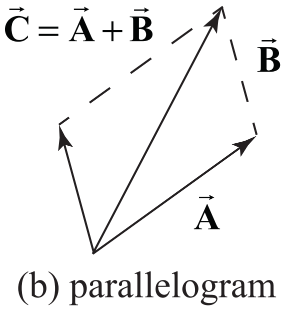


Figure 3.2b

Vector addition satisfies the following four properties:

(i) Commutativity:

The order of adding vectors does not matter;

$$\vec{\mathbf{A}} + \vec{\mathbf{B}} = \vec{\mathbf{B}} + \vec{\mathbf{A}} . \tag{3.1.1}$$

Our geometric definition for vector addition satisfies the commutative property (3.1.1). We can understand this geometrically because in the head to tail representation for the addition of vectors, it doesn't matter which vector you begin with, the sum is the same vector, as seen in Figure 3.3.

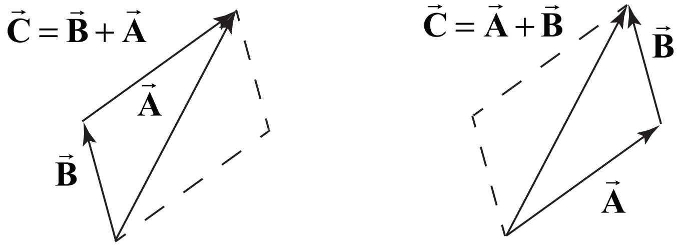


Figure 3.3 Commutative property of vector addition.

(ii) Associativity:

When adding three vectors, it doesn't matter which two you start with

$$(\vec{\mathbf{A}} + \vec{\mathbf{B}}) + \vec{\mathbf{C}} = \vec{\mathbf{A}} + (\vec{\mathbf{B}} + \vec{\mathbf{C}}) . \tag{3.1.2}$$

In Figure 3.4a, we add $(\vec{\mathbf{B}} + \vec{\mathbf{C}}) + \vec{\mathbf{A}}$, and use commutativity to get $\vec{\mathbf{A}} + (\vec{\mathbf{B}} + \vec{\mathbf{C}})$. In figure, we add $(\vec{\mathbf{A}} + \vec{\mathbf{B}}) + \vec{\mathbf{C}}$ to arrive at the same vector as in Figure 3.4a.

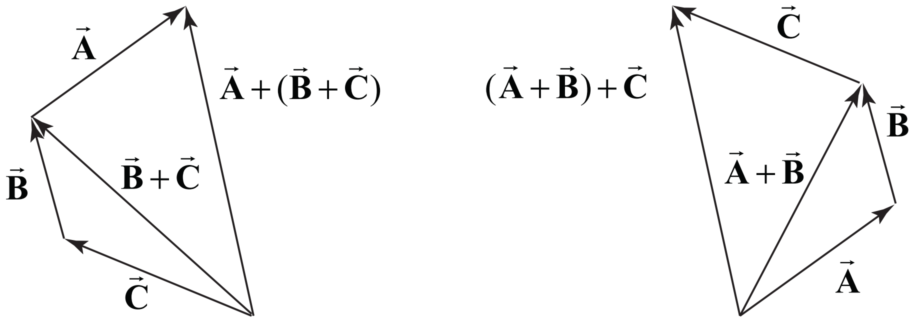


Figure 3.4a Associative law.

(iii) Identity Element for Vector Addition:

There is a unique vector, $\vec{\mathbf{0}}$, that acts as an identity element for vector addition. For all vectors $\vec{\mathbf{A}}$,

$$\vec{\mathbf{A}} + \vec{\mathbf{0}} = \vec{\mathbf{0}} + \vec{\mathbf{A}} = \vec{\mathbf{A}} . \tag{3.1.3}$$

(iv) Inverse Element for Vector Addition:

For every vector $\vec{\mathbf{A}}$, there is a unique inverse vector $-\vec{\mathbf{A}}$ such that

$$\vec{\mathbf{A}} + (-\vec{\mathbf{A}}) = \vec{\mathbf{0}} . \tag{3.1.4}$$

The vector $-\vec{\mathbf{A}}$ has the same magnitude as $\vec{\mathbf{A}}$, $|\vec{\mathbf{A}}| = |-\vec{\mathbf{A}}| = A$, but they point in opposite directions (Figure 3.5).

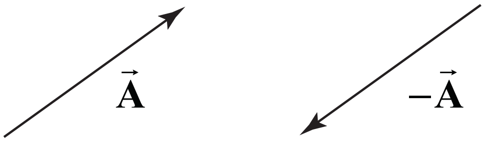


Figure 3.5 Additive inverse

(2) Scalar Multiplication of Vectors:

Vectors can be multiplied by real numbers. Let $\vec{\mathbf{A}}$ be a vector. Let c be a real positive number. Then the multiplication of $\vec{\mathbf{A}}$ by c is a new vector, which we denote by the symbol $c\vec{\mathbf{A}}$. The magnitude of $c\vec{\mathbf{A}}$ is c times the magnitude of $\vec{\mathbf{A}}$ (Figure 3.6a),

$$\left|c\vec{\mathbf{A}}\right| = c\left|\vec{\mathbf{A}}\right| . \tag{3.1.5}$$

Let $c > 0$, then the direction of $c\vec{\mathbf{A}}$ is the same as the direction of $\vec{\mathbf{A}}$. However, the direction of $-c\vec{\mathbf{A}}$ is opposite of $\vec{\mathbf{A}}$ (Figure 3.6).

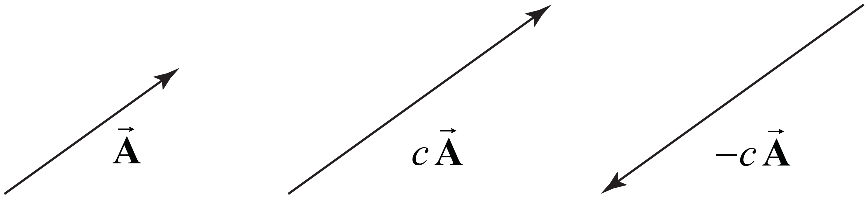


Figure 3.6 Multiplication of vector $\vec{\mathbf{A}}$ by $c > 0$, and $-c < 0$.

Scalar multiplication of vectors satisfies the following properties:

(i) Associative Law for Scalar Multiplication:

The order of multiplying numbers is doesn't matter. Let b and c be real numbers. Then

$$b(c\vec{\mathbf{A}}) = (bc)\vec{\mathbf{A}} = (cb\vec{\mathbf{A}}) = c(b\vec{\mathbf{A}}) . \tag{3.1.6}$$

(ii) Distributive Law for Vector Addition:

Vectors satisfy a distributive law for vector addition. Let c be a real number. Then

$$c(\vec{\mathbf{A}}+\vec{\mathbf{B}}) = c\,\vec{\mathbf{A}} + c\,\vec{\mathbf{B}}\,. \qquad (3.1.7)$$

Figure 3.7 illustrates this property.

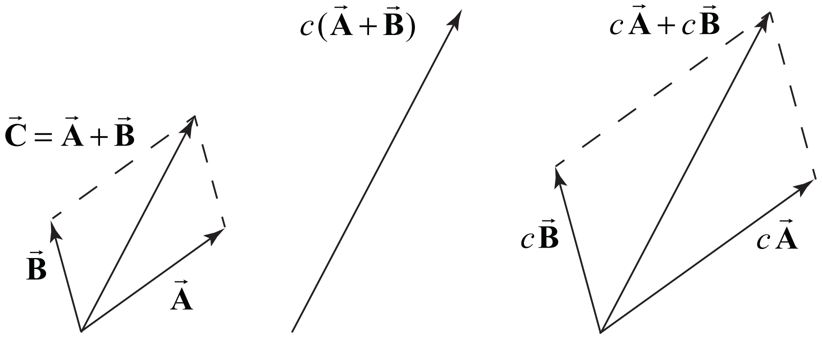


Figure 3.7 Distributive Law for vector addition.

(iii) Distributive Law for Scalar Addition:

Vectors also satisfy a distributive law for scalar addition. Let b and c be real numbers. Then

$$(b+c)\vec{\mathbf{A}} = b\,\vec{\mathbf{A}} + c\,\vec{\mathbf{A}} \qquad (3.1.8)$$

Our geometric definition of vector addition and scalar multiplication satisfies this condition as seen in Figure 3.8.

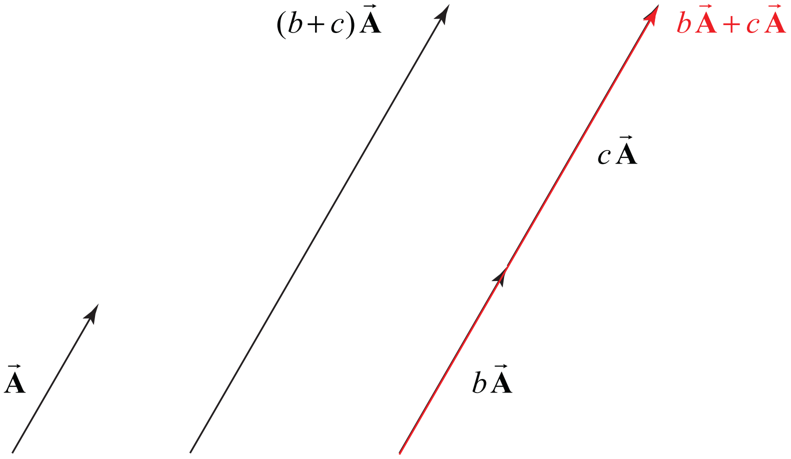


Figure 3.8 Distributive law for scalar multiplication.

(iv) Identity Element for Scalar Multiplication:

The number 1 acts as an identity element for multiplication,

$$1\,\vec{\mathbf{A}} = \vec{\mathbf{A}}\,. \qquad (3.1.9)$$

Unit vector:

Dividing a vector by its magnitude results in a vector of unit length which we denote with a caret symbol

$$\hat{\mathbf{A}} = \frac{\vec{\mathbf{A}}}{\left|\vec{\mathbf{A}}\right|}. \tag{3.1.10}$$

Note that $\left|\hat{\mathbf{A}}\right| = \left|\vec{\mathbf{A}}\right| / \left|\vec{\mathbf{A}}\right| = 1$.

3.2 Coordinate Systems

Physics involve the study of phenomena that we observe in the world. In order to connect the phenomena to mathematics we begin by introducing the concept of a coordinate system. A coordinate system consists of four basic elements:

(1) Choice of origin

(2) Choice of axes

(3) Choice of positive direction for each axis

(4) Choice of unit vectors at every point in space

There are three commonly used coordinate systems: Cartesian, cylindrical and spherical. In this chapter we will describe a Cartesian coordinate system and a cylindrical coordinate system.

3.2.1 Cartesian Coordinate System

Cartesian coordinates consist of a set of mutually perpendicular axes, which intersect at a common point, the origin O. We live in a three-dimensional spatial world; for that reason, the most common system we will use has three axes.

(1) Choice of Origin: Choose an origin O at any point that is most convenient.

(2) Choice of Axes: The simplest set of axes is known as the Cartesian axes, x-axis, y-axis, and the z-axis, that are at right angles with respect to each other. Then each point P in space can be assigned a triplet of values (x_P, y_P, z_P), the Cartesian coordinates of the point P. The ranges of these values are: $-\infty < x_P < +\infty$, $-\infty < y_P < +\infty$, $-\infty < z_P < +\infty$.

(3) Choice of Positive Direction: Our third choice is an assignment of positive direction for each coordinate axis. We shall denote this choice by the symbol + along the positive axis. In physics problems we are free to choose our axes and positive directions any way that we decide best fits a given problem. Problems that are very difficult using the conventional choices may turn out to be much easier to solve by making a thoughtful choice of axes.

(4) Choice of Unit Vectors: We now associate to each point P in space, a set of three **unit vectors** $(\hat{\mathbf{i}}_P, \hat{\mathbf{j}}_P, \hat{\mathbf{k}}_P)$. A unit vector has magnitude one: $|\hat{\mathbf{i}}_P| = 1$, $|\hat{\mathbf{j}}_P| = 1$, and $|\hat{\mathbf{k}}_P| = 1$. We assign the direction of $\hat{\mathbf{i}}_P$ to point in the direction of the increasing x-coordinate at the point P. We define the directions for $\hat{\mathbf{j}}_P$ and $\hat{\mathbf{k}}_P$ in the direction of the increasing y-coordinate and z-coordinate respectively, (Figure 3.10). If we choose a different point S, and define a similar set of unit vectors $(\hat{\mathbf{i}}_S, \hat{\mathbf{j}}_S, \hat{\mathbf{k}}_S)$, the unit vectors at S and P satisfy the equalities

$$\hat{\mathbf{i}}_S = \hat{\mathbf{i}}_P,\ \hat{\mathbf{j}}_S = \hat{\mathbf{j}}_P, \text{and } \hat{\mathbf{k}}_S = \hat{\mathbf{k}}_P\ , \qquad (3.2.1)$$

because vectors are equal if they have the same direction and magnitude regardless of where they are located in space.

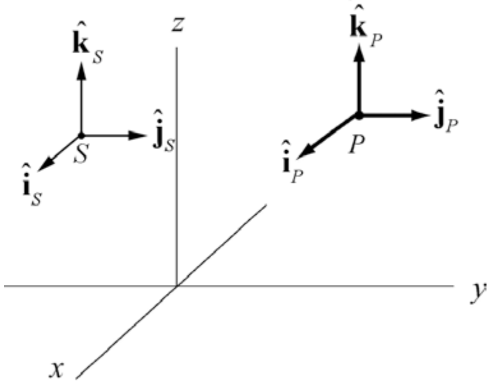


Figure 3.10 Choice of unit vectors at points P and S.

A Cartesian coordinate system is the only coordinate system in which Eq. (3.2.1) holds for all pair of points. We therefore drop the reference to the point P and use $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ to represent the unit vectors in a Cartesian coordinate system (Figure 3.11).

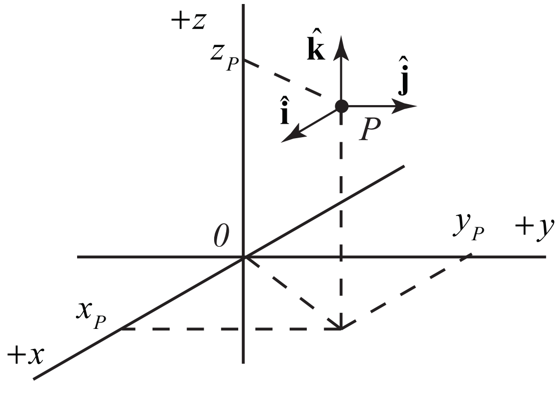


Figure 3.11 Unit vectors in a Cartesian coordinate system

3.2.2 Cylindrical Coordinate System

Many physical objects demonstrate some type of symmetry. For example if you rotate a uniform cylinder about the longitudinal axis (symmetry axis), the cylinder appears unchanged. The operation of rotating the cylinder is called a symmetry operation, and the object undergoing the operation, the cylinder, is exactly the same as before the operation was performed. This symmetry property of cylinders suggests a coordinate system, called a cylindrical coordinate system, that makes the symmetrical property under rotations transparent.

First choose an origin O and axis through O, which we call the z-axis. The ***cylindrical coordinates*** for a point P are the three numbers (r,θ,z) (Figure 3.12). The number z represents the familiar coordinate of the point P along the z-axis. The nonnegative number r represents the distance from the z-axis to the point P. The points in space corresponding to a constant positive value of r lie on a circular cylinder. The locus of points corresponding to $r = 0$ is the z-axis. In the plane $z = 0$, define a reference ray through O, which we shall refer to as the positive x-axis. Draw a line through the point P that is parallel to the z-axis. Let D denote the point of intersection between that line PD and the plane $z = 0$. Draw a ray OD from the origin to the point D. Let θ denote the directed angle from the reference ray to the ray OD. The angle θ is positive when measured counterclockwise and negative when measured clockwise.

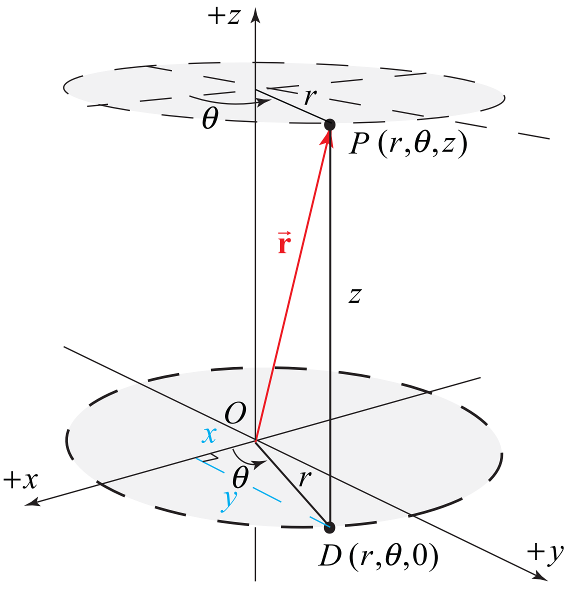


Figure 3.12 Cylindrical Coordinates

The coordinates (r,θ) are called ***polar coordinates***. The coordinate transformations between (r,θ) and the Cartesian coordinates (x, y) are given by

$$x = r\cos\theta\,, \tag{3.2.2}$$
$$y = r\sin\theta\,. \tag{3.2.3}$$

Conversely, if we are given the Cartesian coordinates (x, y), the coordinates (r,θ) can be determined from the coordinate transformations

$$r = +(x^2 + y^2)^{1/2}\,, \tag{3.2.4}$$
$$\theta = \tan^{-1}(y / x)\,. \tag{3.2.5}$$

We choose a set of unit vectors $(\hat{\mathbf{r}}_P, \hat{\boldsymbol{\theta}}_P, \hat{\mathbf{k}}_P)$ at the point P as follows. We choose $\hat{\mathbf{k}}_P$ to point in the direction of increasing z. We choose $\hat{\mathbf{r}}_P$ to point in the direction of increasing r, directed radially away from the z-axis. We choose $\hat{\boldsymbol{\theta}}_P$ to point in the direction of increasing θ. This unit vector points in the counterclockwise direction, tangent to the circle (Figure 3.13a). One crucial difference between cylindrical coordinates and Cartesian coordinates involves the choice of unit vectors. Suppose we consider a different point S in the plane. The unit vectors $(\hat{\mathbf{r}}_S, \hat{\boldsymbol{\theta}}_S, \hat{\mathbf{k}}_S)$ at the point S are also shown in Figure 3.13. Note that $\hat{\mathbf{r}}_P \neq \hat{\mathbf{r}}_S$ and $\hat{\boldsymbol{\theta}}_P \neq \hat{\boldsymbol{\theta}}_S$ because their direction differ. We shall drop the subscripts denoting the points at which the unit vectors are defined at and simple refer to the set of unit vectors at a point as $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{k}})$, with the understanding that the directions of the set $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}})$ depend on the location of the point in question.

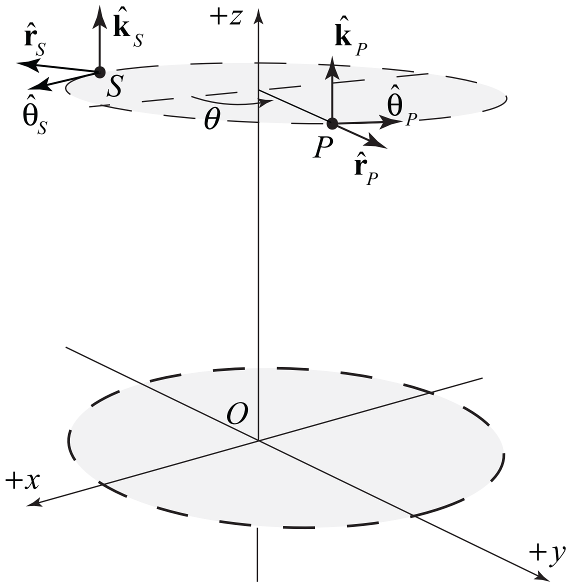


Figure 3.13a Unit vectors at two different points in cylindrical coordinates.

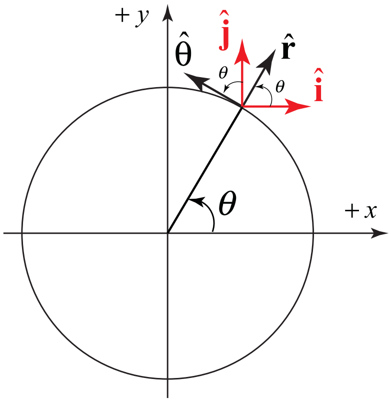


Figure 3.13b Unit vectors in polar coordinates and Cartesian coordinates.

The unit vectors $(\hat{\mathbf{r}},\hat{\boldsymbol{\theta}})$ at the point P also are related to the Cartesian unit vectors $(\hat{\mathbf{i}},\hat{\mathbf{j}})$ by the transformations

$$\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}, \tag{3.2.6}$$

$$\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}. \tag{3.2.7}$$

Similarly the inverse transformations are given by

$$\hat{\mathbf{i}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}, \tag{3.2.8}$$

$$\hat{\mathbf{j}} = \sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}. \tag{3.2.9}$$

A cylindrical coordinate system is also a useful choice to describe the motion of an object moving in a circle about a central point. Consider a vertical axis passing perpendicular to the plane of motion passing through that central point. Then any rotation about this vertical axis leaves circles unchanged.

3.3 Vectors

3.3.1 The Use of Vectors in Physics

From the last section we have three important ideas about vectors, (1) vectors can exist at any point P in space, (2) vectors have direction and magnitude, and (3) any two vectors that have the same direction and magnitude are equal no matter where in space they are located. When we apply vectors to physical quantities it's nice to keep in the back of our minds all these formal properties. However from the physicist's point of view, we are interested in representing physical quantities such as displacement, velocity, acceleration, force, impulse, and momentum as vectors. We can't add force to velocity or subtract momentum from force. We must always understand the physical context for the vector quantity. Thus, instead of approaching vectors as formal mathematical objects we shall instead consider the following essential properties that enable us to represent physical quantities as vectors.

3.3.2 Vectors in Cartesian Coordinates

(1) Vector Decomposition: Choose a coordinate system with an origin, axes, and unit vectors. We can decompose a vector into component vectors along each coordinate axis (Figure 3.14).

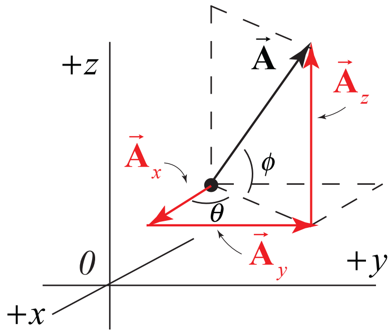


Figure 3.14 Component vectors in Cartesian coordinates.

A vector $\vec{\mathbf{A}}$ at P can be decomposed into the vector sum,

$$\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y + \vec{\mathbf{A}}_z , \tag{3.3.1}$$

where $\vec{\mathbf{A}}_x$ is the x-component vector pointing in the positive or negative x-direction, $\vec{\mathbf{A}}_y$ is the y-component vector pointing in the positive or negative y-direction, and $\vec{\mathbf{A}}_z$ is the z-component vector pointing in the positive or negative z-direction.

(2) Vector Components: Once we have defined unit vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$, we then define the **components** of a vector. Recall our vector decomposition, $\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y + \vec{\mathbf{A}}_z$. We define the x-component vector, $\vec{\mathbf{A}}_x$, as

$$\vec{\mathbf{A}}_x = A_x \, \hat{\mathbf{i}} . \tag{3.3.2}$$

In this expression the term A_x, (without the arrow above) is called the x-component of the vector $\vec{\mathbf{A}}$. The x-component A_x can be positive, zero, or negative. It is not the magnitude of $\vec{\mathbf{A}}_x$ which is given by $(A_x^{\,2})^{1/2}$. The x-component A_x is a scalar quantity and the x-component vector, $\vec{\mathbf{A}}_x$ is a vector. In a similar fashion we define the y-component, A_y, and the z-component, A_z, of the vector $\vec{\mathbf{A}}$ according to

$$\vec{\mathbf{A}}_y = A_y \, \hat{\mathbf{j}}, \quad \vec{\mathbf{A}}_z = A_z \, \hat{\mathbf{k}} . \tag{3.3.3}$$

A vector $\vec{\mathbf{A}}$ is represented by its three components (A_x, A_y, A_z). Thus we need three numbers to describe a vector in three-dimensional space. We write the vector $\vec{\mathbf{A}}$ as

$$\vec{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}\,. \tag{3.3.4}$$

(3) Magnitude: Using the Pythagorean theorem, the magnitude of $\vec{\mathbf{A}}$ is,

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}\,. \tag{3.3.5}$$

(4) Direction: Let's consider a vector $\vec{\mathbf{A}} = (A_x, A_y, 0)$. Because the z-component is zero, the vector $\vec{\mathbf{A}}$ lies in the x-y plane. Let θ denote the angle that the vector $\vec{\mathbf{A}}$ makes in the counterclockwise direction with the positive x-axis (Figure 3.15).

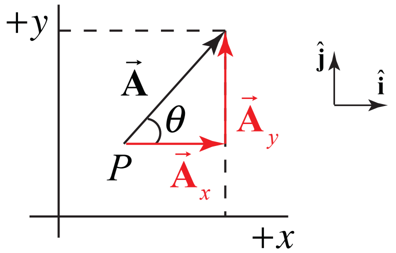


Figure 3.15 Components of a vector in the xy-plane.

Then the x-component and y-component are

$$A_x = A\cos(\theta), \quad A_y = A\sin(\theta)\,. \tag{3.3.6}$$

We now write a vector in the xy-plane as

$$\vec{\mathbf{A}} = A\cos(\theta)\;\hat{\mathbf{i}} + A\sin(\theta)\;\hat{\mathbf{j}} \tag{3.3.7}$$

Once the components of a vector are known, the tangent of the angle θ can be determined by

$$\frac{A_y}{A_x} = \frac{A\sin(\theta)}{A\cos(\theta)} = \tan(\theta)\,, \tag{3.3.8}$$

and hence the angle θ is given by

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right). \tag{3.3.9}$$

Clearly, the direction of the vector depends on the sign of A_x and A_y. For example, if both $A_x > 0$ and $A_y > 0$, then $0 < \theta < \pi / 2$. If $A_x < 0$ and $A_y > 0$ then $\pi / 2 < \theta < \pi$. If $A_x < 0$ and $A_y < 0$ then $\pi < \theta < 3\pi / 2$. If $A_x > 0$ and $A_y < 0$, then $3\pi / 2 < \theta < 2\pi$. Note that $\tan(\theta)$ is a double valued function because

$$\frac{-A_y}{-A_x} = \frac{A_y}{A_x}, \text{and } \frac{A_y}{-A_x} = \frac{-A_y}{A_x}. \tag{3.3.10}$$

(5) Unit Vectors: Unit vector in the direction of $\vec{\mathbf{A}}$: Let $\vec{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}$. Let $\hat{\mathbf{A}}$ denote a unit vector in the direction of $\vec{\mathbf{A}}$. Then

$$\hat{\mathbf{A}} = \frac{\vec{\mathbf{A}}}{\left|\vec{\mathbf{A}}\right|} = \frac{A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}}{(A_x{}^2 + A_y{}^2 + A_z{}^2)^{1/2}}. \tag{3.3.11}$$

(6) Vector Addition: Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors in the *x*-*y* plane. Let θ_A and θ_B denote the angles that the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ make (in the counterclockwise direction) with the positive x-axis. Then

$$\vec{\mathbf{A}} = A\cos(\theta_A)\,\hat{\mathbf{i}} + A\sin(\theta_A)\,\hat{\mathbf{j}}, \tag{3.3.12}$$

$$\vec{\mathbf{B}} = B\cos(\theta_B)\,\hat{\mathbf{i}} + B\sin(\theta_B)\,\hat{\mathbf{j}} \tag{3.3.13}$$

In Figure 3.16, the vector addition $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ is shown. Let θ_C denote the angle that the vector $\vec{\mathbf{C}}$ makes with the positive *x*-axis.

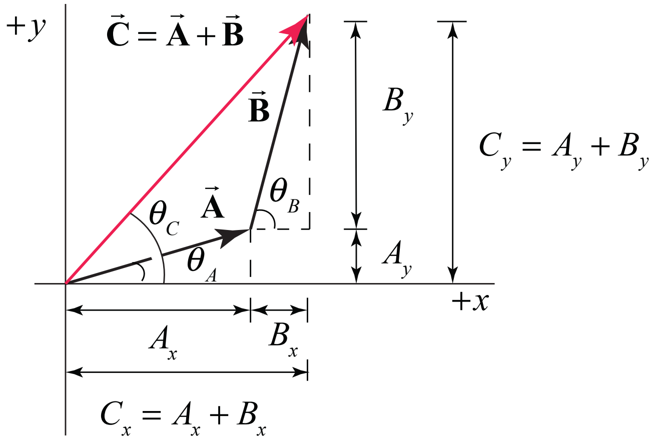


Figure 3.16 Vector addition using components.

From Figure 3.16, the components of $\vec{\mathbf{C}}$ are

$$C_x = A_x + B_x, \quad C_y = A_y + B_y. \tag{3.3.14}$$

In terms of magnitudes and angles, we have

$$\begin{aligned} C_x &= C\cos(\theta_C) = A\cos(\theta_A) + B\cos(\theta_B) \\ C_y &= C\sin(\theta_C) = A\sin(\theta_A) + B\sin(\theta_B). \end{aligned} \tag{3.3.15}$$

We can write the vector $\vec{\mathbf{C}}$ as

$$\vec{\mathbf{C}} = (A_x + B_x)\,\hat{\mathbf{i}} + (A_y + B_y)\,\hat{\mathbf{j}} = C\cos(\theta_C)\,\hat{\mathbf{i}} + C\sin(\theta_C)\,\hat{\mathbf{j}}, \tag{3.3.16}$$

Example 3.1 Vector Addition

Given two vectors, $\vec{\mathbf{A}} = 2\,\hat{\mathbf{i}} + -3\,\hat{\mathbf{j}} + 7\,\hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = 5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, find: (a) $\left|\vec{\mathbf{A}}\right|$; (b) $\left|\vec{\mathbf{B}}\right|$; (c) $\vec{\mathbf{A}} + \vec{\mathbf{B}}$; (d) $\vec{\mathbf{A}} - \vec{\mathbf{B}}$; (e) a unit vector $\hat{\mathbf{A}}$ pointing in the direction of $\vec{\mathbf{A}}$; (f) a unit vector $\hat{\mathbf{B}}$ pointing in the direction of $\vec{\mathbf{B}}$.

Solution:

(a) $\left|\vec{\mathbf{A}}\right| = \left(2^2 + (-3)^2 + 7^2\right)^{1/2} = \sqrt{62} = 7.87$. (b) $\left|\vec{\mathbf{B}}\right| = \left(5^2 + 1^2 + 2^2\right)^{1/2} = \sqrt{30} = 5.48$.

(c)
$$\begin{aligned} \vec{\mathbf{A}} + \vec{\mathbf{B}} &= (A_x + B_x)\,\hat{\mathbf{i}} + (A_y + B_y)\,\hat{\mathbf{j}} + (A_z + B_z)\,\hat{\mathbf{k}} \\ &= (2+5)\,\hat{\mathbf{i}} + (-3+1)\,\hat{\mathbf{j}} + (7+2)\,\hat{\mathbf{k}} \\ &= 7\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} + 9\,\hat{\mathbf{k}}. \end{aligned}$$

(d)
$$\begin{aligned} \vec{\mathbf{A}} - \vec{\mathbf{B}} &= (A_x - B_x)\,\hat{\mathbf{i}} + (A_y - B_y)\,\hat{\mathbf{j}} + (A_z - B_z)\,\hat{\mathbf{k}} \\ &= (2-5)\,\hat{\mathbf{i}} + (-3-1)\,\hat{\mathbf{j}} + (7-2)\,\hat{\mathbf{k}} \\ &= -3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}} + 5\,\hat{\mathbf{k}}. \end{aligned}$$

(e) A unit vector $\hat{\mathbf{A}}$ in the direction of $\vec{\mathbf{A}}$ can be found by dividing the vector $\vec{\mathbf{A}}$ by the magnitude of $\vec{\mathbf{A}}$. Therefore

$$\hat{\mathbf{A}} = \vec{\mathbf{A}} / \left|\vec{\mathbf{A}}\right| = \left(2\,\hat{\mathbf{i}} + -3\,\hat{\mathbf{j}} + 7\,\hat{\mathbf{k}}\right) / \sqrt{62}.$$

(f) In a similar fashion, $\hat{\mathbf{B}} = \vec{\mathbf{B}} / \left|\vec{\mathbf{B}}\right| = \left(5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}\right) / \sqrt{30}$.

Example 3.2 Sinking Sailboat

A Coast Guard ship is located 35 km away from a checkpoint in a direction 52° north of west. A distressed sailboat located in still water 24 km from the same checkpoint in a direction 18° south of east is about to sink. Draw a diagram indicating the position of both ships. In what direction and how far must the Coast Guard ship travel to reach the sailboat?

Solution: The diagram of the set-up is Figure 3.17.

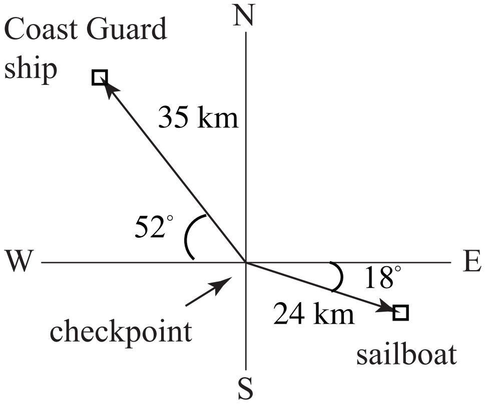


Figure 3.17 Example 3.2

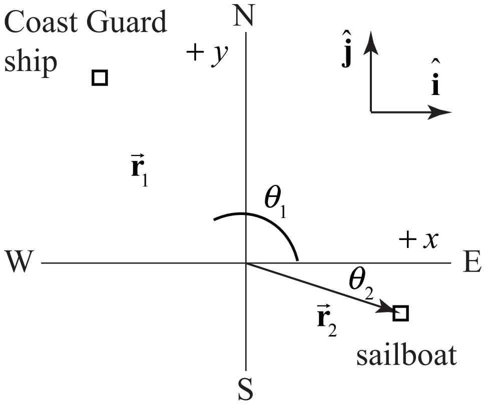


Figure 3.18 Coordinate system for sailboat and ship

Choose the checkpoint as the origin of a Cartesian coordinate system with the positive x-axis in the East direction and the positive y–axis in the North direction. Choose the corresponding unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ as shown in Figure 3.18. The Coast Guard ship is then a distance $r_1 = 35\text{ km}$ at an angle $\theta_1 = 180^{\circ} - 52^{\circ} = 128^{\circ}$ from the positive x-axis, and the sailboat is at a distance $r_2 = 24\text{ km}$ at an angle $\theta_2 = -18^{\circ}$ from the positive x-axis. The position of the Coast Guard ship is then

$$\vec{\mathbf{r}}_1 = r_1(\cos\theta_1\,\hat{\mathbf{i}} + \sin\theta_1\,\hat{\mathbf{j}})$$
$$\vec{\mathbf{r}} = -21.5\,\text{km}\,\hat{\mathbf{i}} + 27.6\,\text{km}\,\hat{\mathbf{j}}$$

and the position of the sailboat is

$$\vec{\mathbf{r}}_2 = r_2(\cos\theta_2\,\hat{\mathbf{i}} + \sin\theta_2\,\hat{\mathbf{j}})$$
$$\vec{\mathbf{r}}_2 = 22.8\,\text{km}\,\hat{\mathbf{i}} - 7.4\,\text{km}\;\hat{\mathbf{j}}.$$

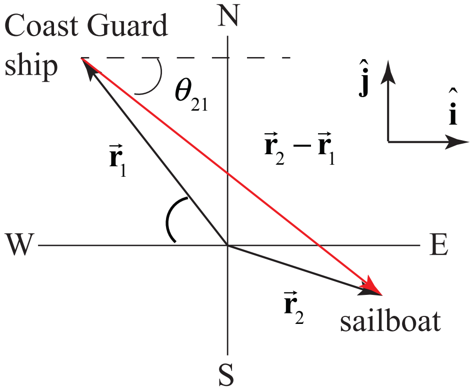


Figure 3.19 Relative position vector from ship to sailboat

The relative position vector from the Coast Guard ship to the sailboat is (Figure 3.19)

$$\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1 = (22.8\,\text{km}\,\hat{\mathbf{i}} - 7.4\,\text{km}\;\hat{\mathbf{j}}) - (-21.5\,\text{km}\,\hat{\mathbf{i}} + 27.6\,\text{km}\,\hat{\mathbf{j}})$$
$$\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1 = 44.4\,\text{km}\,\hat{\mathbf{i}} - 35.0\,\text{km}\,\hat{\mathbf{j}}.$$

The distance between the ship and the sailboat is

$$\left|\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1\right| = ((44.4\,\text{km})^2 + (-35.0\,\text{km})^2)^{1/2} = 56.5\,\text{km}$$

The rescue ship's heading would be the inverse tangent of the ratio of the y - and x - components of the relative position vector,

$$\theta_{21} = \tan^{-1}(-35.0\,\text{km}/44.4\,\text{km}) = -38.3^\circ.$$

or 38.3° South of East.

Example 3.3 Vector Addition

Two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, such that $\left|\vec{\mathbf{B}}\right| = 2\left|\vec{\mathbf{A}}\right|$, have a resultant $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ of magnitude 26.5. The vector $\vec{\mathbf{C}}$ makes an angle $\theta_C = 41^\circ$ with respect to vector $\vec{\mathbf{A}}$. Find the magnitude of each vector and the angle between vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

Solution: We begin by making a sketch of the three vectors, choosing $\vec{\mathbf{A}}$ to point in the positive x-direction (Figure 3.20).

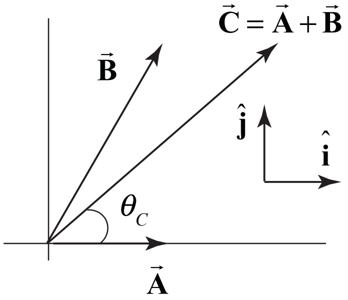


Figure 3.20 Choice of coordinates system for Example 3.3

Denote the magnitude of $\vec{\mathbf{C}}$ by $C \equiv \left|\vec{\mathbf{C}}\right| = \sqrt{(C_x)^2 + (C_y)^2} = 26.5$. The components of $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ are given by

$$C_x = A_x + B_x = C\cos\theta_C = (26.5)\cos(41^\circ) = 20 \tag{3.3.17}$$

$$C_y = B_y = C\sin\theta_C = (26.5)\sin(41^\circ) = 17.4\,. \tag{3.3.18}$$

From the condition that $\left|\vec{\mathbf{B}}\right| = 2\left|\vec{\mathbf{A}}\right|$, the square of their magnitudes satisfy

$$(B_x)^2 + (B_y)^2 = 4(A_x)^2\,. \tag{3.3.19}$$

Using Eqs. (3.3.17) and (3.3.18), Eq. (3.3.19)becomes

$$(C_x - A_x)^2 + (C_y)^2 = 4(A_x)^2$$
$$(C_x)^2 - 2C_xA_x + (A_x)^2 + (C_y)^2 = 4(A_x)^2\,.$$

This is a quadratic equation

$$0 = 3(A_x)^2 + 2C_xA_x - C^2$$

which we solve for the component A_x:

$$A_x = \frac{-2C_x \pm \sqrt{(2C_x)^2 + (4)(3)(C^2)}}{6} = \frac{-2(20) \pm \sqrt{(40))^2 + (4)(3)(26.5)^2}}{6} = 10.0,$$

where we choose the positive square root because we originally chose $A_x > 0$. The components of $\vec{\mathbf{B}}$ are then given by Eqs. (3.3.17) and (3.3.18):

$$B_x = C_x - A_x = 20.0 - 10.0 = 10.0$$
$$B_y = 17.4\,.$$

The magnitude of $\left|\vec{\mathbf{B}}\right| = \sqrt{(B_x)^2 + (B_y)^2} = 20.0$ which is equal to two times the magnitude of $\left|\vec{\mathbf{A}}\right| = 10.0$. The angle between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is given by

$$\theta = \sin^{-1}(B_y / \left|\vec{\mathbf{B}}\right|) = \sin^{-1}(17.4 / 20.0\ \mathrm{N}) = 60^\circ\,.$$

Example 3.4 Vector Description of a Point on a Line

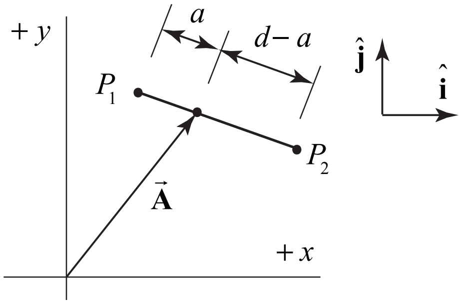


Figure 3.21 Example 3.4

Consider two points, P_1 with coordinates (x_1, y_1) and P_2 with coordinates (x_1, y_1), that are separated by distance d. Find a vector $\vec{\mathbf{A}}$ from the origin to the point on the line connecting P_1 and P_2 that is located a distance a from the point P_1 (Figure 3.21).

Solution: Let $\vec{\mathbf{r}}_1 = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}}$ be the position vector of P_1 and $\vec{\mathbf{r}}_2 = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}}$ the position vector of P_2. Let $\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$ be the vector from P_2 to P_1 (Figure 3.22a). The unit vector pointing from P_2 to P_1 is given by $\hat{\mathbf{r}}_{21} = (\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2) / \left|\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\right| = (\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2) / d$, where $d = ((x_2 - x_1)^2 + (y_2 - y_1)^2)^{1/2}$

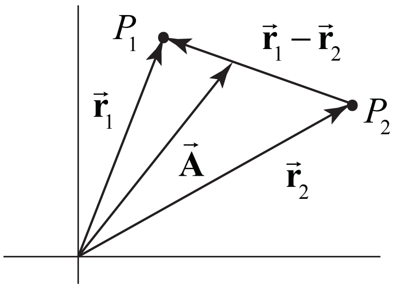


Figure 3.22a: Relative position vector

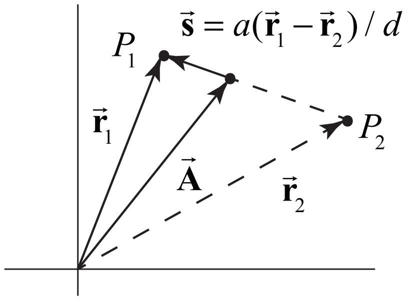


Figure 3.22b: Relative position vector

The vector $\vec{\mathbf{s}}$ in Figure 3.22b connects $\vec{\mathbf{A}}$ to the point at $\vec{\mathbf{r}}_1$, points in the direction of $\hat{\mathbf{r}}_{12}$, and has length a. Therefore $\vec{\mathbf{s}} = a\hat{\mathbf{r}}_{21} = a(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2)/d$. The vector $\vec{\mathbf{r}}_1 = \vec{\mathbf{A}} + \vec{\mathbf{s}}$. Therefore

$$\vec{\mathbf{A}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{s}} = \vec{\mathbf{r}}_1 - a(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2)/d = (1 - a/d)\vec{\mathbf{r}}_1 + (a/d)\vec{\mathbf{r}}_2$$
$$\vec{\mathbf{A}} = (1 - a/d)(x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}}) + (a/d)(x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}})$$
$$\vec{\mathbf{A}} = \left(x_1 + \frac{a(x_2 - x_1)}{((x_2 - x_1)^2 + (y_2 - y_1)^2)^{1/2}}\right)\hat{\mathbf{i}} + \left(y_1 + \frac{a(y_2 - y_1)}{((x_2 - x_1)^2 + (y_2 - y_1)^2)^{1/2}}\right)\hat{\mathbf{j}}.$$

3.3.2 Transformation of Vectors in Rotated Coordinate Systems

Consider two Cartesian coordinate systems S and S' such that the (x', y') coordinate axes in S' are rotated by an angle θ with respect to the (x, y) coordinate axes in S, (Figure 3.23).

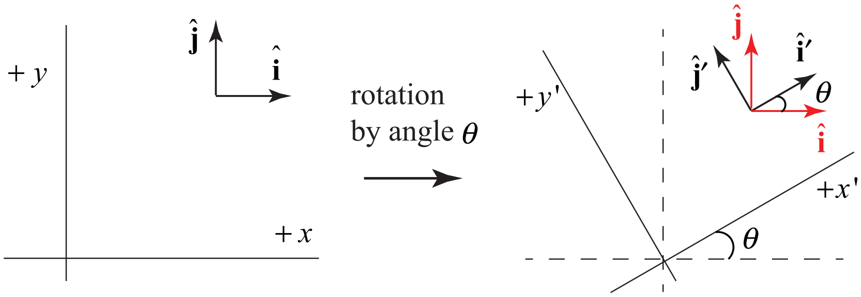


Figure 3.23 Rotated coordinate systems

The components of the unit vector $\hat{\mathbf{i}}'$ in the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ direction are given by $i'_x = |\hat{\mathbf{i}}'|\cos\theta = \cos\theta$ and $i'_y = |\hat{\mathbf{i}}'|\sin\theta = \sin\theta$. Therefore

$$\hat{\mathbf{i}}' = i'_x\,\hat{\mathbf{i}} + i'_y\,\hat{\mathbf{j}} = \hat{\mathbf{i}}\cos\theta + \hat{\mathbf{j}}\sin\theta\,. \qquad (3.3.20)$$

A similar argument holds for the components of the unit vector $\hat{\mathbf{j}}'$. The components of $\hat{\mathbf{j}}'$ in the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ direction are given by $j'_x = -\left|\hat{\mathbf{j}}'\right|\sin\theta = -\sin\theta$ and $j'_y = \left|\hat{\mathbf{j}}'\right|\cos\theta = \cos\theta$. Therefore

$$\hat{\mathbf{j}}' = j'_x\,\hat{\mathbf{i}} + j'_y\,\hat{\mathbf{j}} = \hat{\mathbf{j}}\cos\theta - \hat{\mathbf{i}}\sin\theta\,. \tag{3.3.21}$$

Conversely, from Figure 3.23 and similar vector decomposition arguments, the components of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ in S' are given by

$$\hat{\mathbf{i}} = \hat{\mathbf{i}}'\cos\theta - \hat{\mathbf{j}}'\sin\theta\,, \tag{3.3.22}$$

$$\hat{\mathbf{j}} = \hat{\mathbf{i}}'\sin\theta + \hat{\mathbf{j}}'\cos\theta\,. \tag{3.3.23}$$

Consider a fixed vector $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ with components (x, y) in coordinate system S. In coordinate system S', the vector is given by $\vec{\mathbf{r}} = x'\,\hat{\mathbf{i}}' + y'\hat{\mathbf{j}}'$, where (x', y') are the components in S', (Figure 3.24).

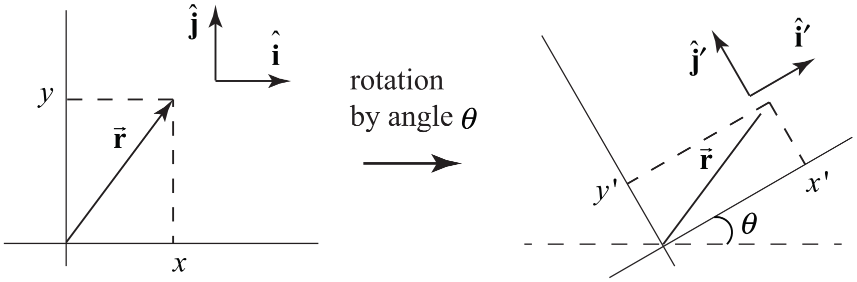


Figure 3.24 Transformation of vector components

Using the Eqs. (3.3.20) and (3.3.21), we have that

$$\begin{aligned}\vec{\mathbf{r}} &= x\,\hat{\mathbf{i}} + y\hat{\mathbf{j}} = x(\hat{\mathbf{i}}'\cos\theta - \hat{\mathbf{j}}'\sin\theta) + y(\hat{\mathbf{j}}'\cos\theta + \hat{\mathbf{i}}'\sin\theta)\\ \vec{\mathbf{r}} &= (x\cos\theta + y\sin\theta)\hat{\mathbf{i}}' + (x\sin\theta - y\cos\theta)\hat{\mathbf{j}}'.\end{aligned} \tag{3.3.24}$$

Therefore the components of the vector transform according to

$$x' = x\cos\theta + y\sin\theta\ , \tag{3.3.25}$$

$$y' = x\sin\theta - y\cos\theta\ . \tag{3.3.26}$$

We now consider an alternate approach to understanding the transformation laws for the components of the position vector of a fixed point in space. In coordinate system S, suppose the position vector $\vec{\mathbf{r}}$ has length $r = \left|\vec{\mathbf{r}}\right|$ and makes an angle ϕ with respect to the positive x-axis (Figure 3.25).

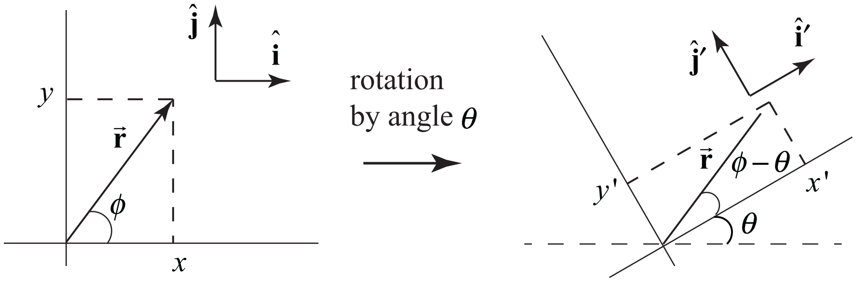


Figure 3.25 Transformation of vector components of the position vector

Then the components of $\vec{\mathbf{r}}$ in S are given by

$$x = r\cos\phi \ , \tag{3.3.27}$$
$$y = r\sin\phi \ . \tag{3.3.28}$$

In coordinate system S', the components of $\vec{\mathbf{r}}$ are given by

$$x' = r\cos(\phi-\theta) \ , \tag{3.3.29}$$
$$y' = r\sin(\phi-\theta) \ . \tag{3.3.30}$$

Apply the addition of angle trigonometric identities to Eqs. (3.3.29) and (3.3.30) yielding

$$x' = r\cos(\phi-\theta) = r\cos\phi\cos\theta + r\sin\phi\sin\theta = x\cos\theta + y\sin\theta \ , \tag{3.3.31}$$
$$y' = r\sin(\phi-\theta) = r\sin\phi\cos\theta - r\cos\phi\sin\theta = y\cos\theta - x\sin\theta \ , \tag{3.3.32}$$

in agreement with Eqs. (3.3.25) and (3.3.26).

Example 3.5 Vector Decomposition in Rotated Coordinate Systems

With respect to a given Cartesian coordinate system S, a vector $\vec{\mathbf{A}}$ has components $A_x = 5$, $A_y = -3$, $A_z = 0$. Consider a second coordinate system S' such that the (x', y') coordinate axes in S' are rotated by an angle $\theta = 60^\circ$ with respect to the (x, y) coordinate axes in S, (Figure 3.26). (a) What are the components $A_{x'}$ and $A_{y'}$ of vector $\vec{\mathbf{A}}$ in coordinate system S'? (b) Calculate the magnitude of the vector using the (A_x, A_y) components and using the $(A_{x'}, A_{y'})$ components. Does your result agree with what you expect?

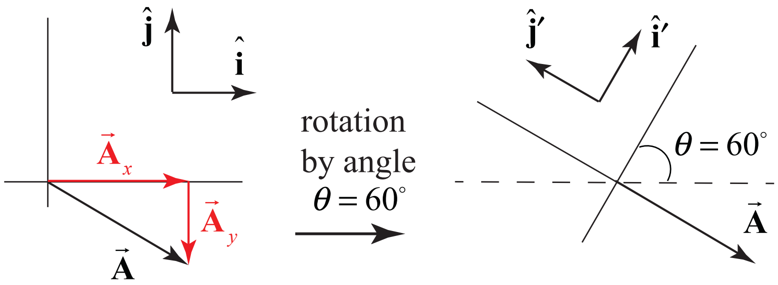


Figure 3.26 Example 3.4

Solution: a) We begin by considering the vector decomposition of $\vec{\mathbf{A}}$ with respect to the coordinate system S,

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}. \tag{3.3.33}$$

Now we can use our results for the transformation of unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ in terms of $\hat{\mathbf{i}}'$ and $\hat{\mathbf{j}}'$, (Eqs. (3.3.22) and (3.3.23)) in order decompose the vector $\vec{\mathbf{A}}$ in coordinate system S'

$$\begin{aligned}\vec{\mathbf{A}} &= A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} = A_x(\cos\theta\hat{\mathbf{i}}' - \sin\theta\hat{\mathbf{j}}') + A_y(\sin\theta\hat{\mathbf{i}}' + \cos\theta\hat{\mathbf{j}}')\\ &= (A_x\cos\theta + A_y\sin\theta)\hat{\mathbf{i}}' + (-A_x\sin\theta + A_y\cos\theta)\hat{\mathbf{j}}'\\ &= A_{x'}\hat{\mathbf{i}} + A_{y'}\hat{\mathbf{j}},\end{aligned} \tag{3.3.34}$$

where

$$A_{x'} = A_x\cos\theta + A_y\sin\theta \tag{3.3.35}$$

$$A_{y'} = -A_x\sin\theta + A_y\cos\theta. \tag{3.3.36}$$

We now use the given information that $A_x = 5$, $A_y = -3$, and $\theta = 60^\circ$ to solve for the components of $\vec{\mathbf{A}}$ in coordinate system S'

$$\begin{aligned}A_{x'} &= A_x\cos\theta + A_y\sin\theta = (1/2)(5 - 3\sqrt{3}),\\ A_{y'} &= -A_x\sin\theta + A_y\cos\theta = (1/2)(-5\sqrt{3} - 3).\end{aligned}$$

b) The magnitude can be calculated in either coordinate system

$$\left|\vec{\mathbf{A}}\right| = \sqrt{(A_x)^2 + (A_y)^2} = \sqrt{(5)^2 + (-3)^2} = \sqrt{34}$$

$$\left|\vec{\mathbf{A}}\right| = \sqrt{(A_{x'})^2 + (A_{y'})^2} = \sqrt{\left((1/2)(5 - 3\sqrt{3})\right)^2 + \left((1/2)(-5\sqrt{3} - 3)\right)^2} = \sqrt{34}.$$

This result agrees with what I expect because the length of vector $\vec{\mathbf{A}}$ is independent of the choice of coordinate system.

3.4 Vector Product (Cross Product)

Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors. Because any two non-parallel vectors form a plane, we denote the angle θ to be the angle between the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ as shown in Figure 3.27. The **magnitude of the vector product** $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is defined to be product of the magnitude of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with the sine of the angle θ between the two vectors,

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|\sin(\theta). \qquad (3.3.37)$$

The angle θ between the vectors is limited to the values $0 \le \theta \le \pi$ ensuring that $\sin(\theta) \ge 0$.

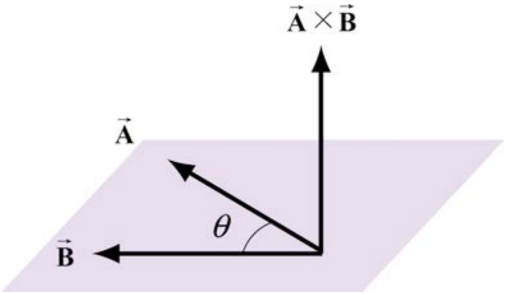


Figure 3.27 Vector product geometry.

The direction of the vector product is defined as follows. The vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ form a plane. Consider the direction perpendicular to this plane. There are two possibilities: we shall choose one of these two (the one shown in Figure 3.27) for the direction of the vector product $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ using a convention that is commonly called the "**right-hand rule**".

3.4.1 Right-hand Rule for the Direction of Vector Product

The first step is to redraw the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ so that the tails are touching. Then draw an arc starting from the vector $\vec{\mathbf{A}}$ and finishing on the vector $\vec{\mathbf{B}}$. Curl your right fingers the same way as the arc. Your right thumb points in the direction of the vector product $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ (Figure 3.28).

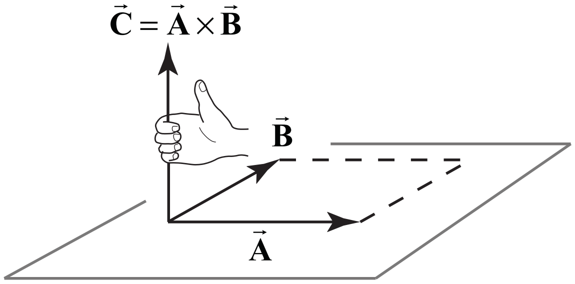


Figure 3.28 Right-Hand Rule.

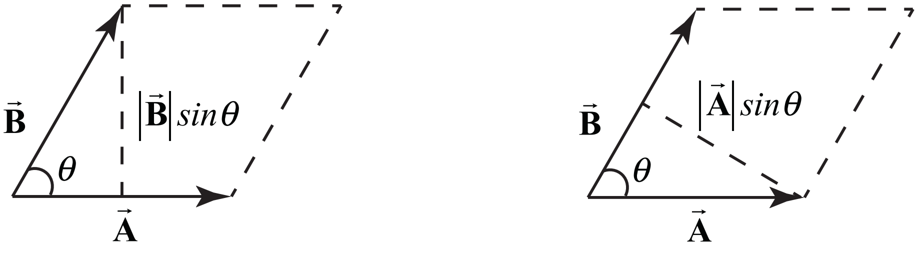
You should remember that the direction of the vector product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is perpendicular to the plane formed by $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. We can give a geometric interpretation to the magnitude of the vector product by writing the magnitude as

$$\left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\right|\left(\left|\vec{\mathbf{B}}\right|\sin\theta\right). \tag{3.3.38}$$

The vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ form a parallelogram. The area of the parallelogram is equal to the height times the base, which is the magnitude of the vector product. In Figure 3.29, two different representations of the height and base of a parallelogram are illustrated. As depicted in Figure 3.29(a), the term $\left|\vec{\mathbf{B}}\right|\sin\theta$ is the projection of the vector $\vec{\mathbf{B}}$ in the direction perpendicular to the vector $\vec{\mathbf{B}}$. We could also write the magnitude of the vector product as

$$\left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = \left(\left|\vec{\mathbf{A}}\right|\sin\theta\right)\left|\vec{\mathbf{B}}\right|. \tag{3.3.39}$$

The term $\left|\vec{\mathbf{A}}\right|\sin\theta$ is the projection of the vector $\vec{\mathbf{A}}$ in the direction perpendicular to the vector $\vec{\mathbf{B}}$ as shown in Figure 3.29(b). The vector product of two vectors that are parallel (or anti-parallel) to each other is zero because the angle between the vectors is 0 (or π) and $\sin(0) = 0$ (or $\sin(\pi) = 0$). Geometrically, two parallel vectors do not have a unique component perpendicular to their common direction.



(a) (b)

Figure 3.29 Projection of (a) $\vec{\mathbf{B}}$ perpendicular to $\vec{\mathbf{A}}$, (b) of $\vec{\mathbf{A}}$ perpendicular to $\vec{\mathbf{B}}$

3.4.2 Properties of the Vector Product

(1) The vector product is anti-commutative because changing the order of the vectors changes the direction of the vector product by the right hand rule:

$$\vec{\mathbf{A}}\times\vec{\mathbf{B}} = -\vec{\mathbf{B}}\times\vec{\mathbf{A}}\,. \qquad (3.3.40)$$

(2) The vector product between a vector $c\,\vec{\mathbf{A}}$ where c is a scalar and a vector $\vec{\mathbf{B}}$ is

$$c\,\vec{\mathbf{A}}\times\vec{\mathbf{B}} = c\,(\vec{\mathbf{A}}\times\vec{\mathbf{B}})\,. \qquad (3.3.41)$$

Similarly,

$$\vec{\mathbf{A}}\times c\,\vec{\mathbf{B}} = c\,(\vec{\mathbf{A}}\times\vec{\mathbf{B}})\,. \qquad (3.3.42)$$

(3) The vector product between the sum of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with a vector $\vec{\mathbf{C}}$ is

$$(\vec{\mathbf{A}}+\vec{\mathbf{B}})\times\vec{\mathbf{C}} = \vec{\mathbf{A}}\times\vec{\mathbf{C}}+\vec{\mathbf{B}}\times\vec{\mathbf{C}} \qquad (3.3.43)$$

Similarly,

$$\vec{\mathbf{A}}\times(\vec{\mathbf{B}}+\vec{\mathbf{C}}) = \vec{\mathbf{A}}\times\vec{\mathbf{B}}+\vec{\mathbf{A}}\times\vec{\mathbf{C}}\,. \qquad (3.3.44)$$

3.4.3 Vector Decomposition and the Vector Product: Cartesian Coordinates

We first calculate that the magnitude of vector product of the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$:

$$|\,\hat{\mathbf{i}}\times\hat{\mathbf{j}}\,| = |\,\hat{\mathbf{i}}\,|\,|\,\hat{\mathbf{j}}\,|\sin(\pi/2) = 1\,, \qquad (3.3.45)$$

because the unit vectors have magnitude $|\,\hat{\mathbf{i}}\,| = |\,\hat{\mathbf{j}}\,| = 1$ and $\sin(\pi/2) = 1$. By the right hand rule, the direction of $\hat{\mathbf{i}}\times\hat{\mathbf{j}}$ is in the $+\hat{\mathbf{k}}$ as shown in Figure 3.30. Thus $\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}$.

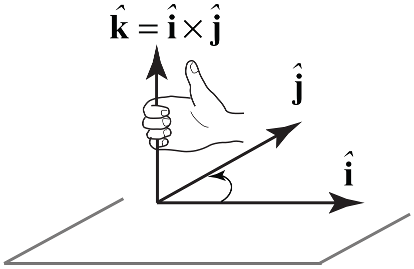


Figure 3.30 Vector product of $\hat{\mathbf{i}}\times\hat{\mathbf{j}}$

We note that the same rule applies for the unit vectors in the *y* and *z* directions,

$$\hat{\mathbf{j}}\times\hat{\mathbf{k}}=\hat{\mathbf{i}},\ \ \hat{\mathbf{k}}\times\hat{\mathbf{i}}=\hat{\mathbf{j}}\,. \tag{3.3.46}$$

By the anti-commutatively property (1) of the vector product,

$$\hat{\mathbf{j}}\times\hat{\mathbf{i}}=-\hat{\mathbf{k}},\ \ \ \hat{\mathbf{i}}\times\hat{\mathbf{k}}=-\hat{\mathbf{j}} \tag{3.3.47}$$

The vector product of the unit vector $\hat{\mathbf{i}}$ with itself is zero because the two unit vectors are parallel to each other, ($\sin(0)=0$),

$$|\,\hat{\mathbf{i}}\times\hat{\mathbf{i}}\,|=|\,\hat{\mathbf{i}}\,||\,\hat{\mathbf{i}}\,|\sin(0)=0\,. \tag{3.3.48}$$

The vector product of the unit vector $\hat{\mathbf{j}}$ with itself and the unit vector $\hat{\mathbf{k}}$ with itself are also zero for the same reason,

$$\left|\hat{\mathbf{j}}\times\hat{\mathbf{j}}\right|=0,\ \ \left|\hat{\mathbf{k}}\times\hat{\mathbf{k}}\right|=0\,. \tag{3.3.49}$$

With these properties in mind we can now develop an algebraic expression for the vector product in terms of components. Let's choose a Cartesian coordinate system with the vector $\vec{\mathbf{B}}$ pointing along the positive *x*-axis with positive *x*-component B_x. Then the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ can be written as

$$\vec{\mathbf{A}}=A_x\,\hat{\mathbf{i}}+A_y\,\hat{\mathbf{j}}+A_z\,\hat{\mathbf{k}} \tag{3.3.50}$$

$$\vec{\mathbf{B}}=B_x\,\hat{\mathbf{i}}\,, \tag{3.3.51}$$

respectively. The vector product in vector components is

$$\vec{\mathbf{A}}\times\vec{\mathbf{B}}=(A_x\,\hat{\mathbf{i}}+A_y\,\hat{\mathbf{j}}+A_z\,\hat{\mathbf{k}})\times B_x\,\hat{\mathbf{i}}\,. \tag{3.3.52}$$

This becomes,

$$\begin{aligned}\vec{\mathbf{A}}\times\vec{\mathbf{B}}&=(A_x\,\hat{\mathbf{i}}\times B_x\,\hat{\mathbf{i}})+(A_y\,\hat{\mathbf{j}}\times B_x\,\hat{\mathbf{i}})+(A_z\,\hat{\mathbf{k}}\times B_x\,\hat{\mathbf{i}})\\&=A_xB_x\,(\hat{\mathbf{i}}\times\hat{\mathbf{i}})+A_yB_x\,(\hat{\mathbf{j}}\times\hat{\mathbf{i}})+A_zB_x\,(\hat{\mathbf{k}}\times\hat{\mathbf{i}})\quad.\\&=-A_yB_x\,\hat{\mathbf{k}}+A_zB_x\,\hat{\mathbf{j}}\end{aligned} \tag{3.3.53}$$

The vector component expression for the vector product easily generalizes for arbitrary vectors

$$\vec{\mathbf{A}}=A_x\,\hat{\mathbf{i}}+A_y\,\hat{\mathbf{j}}+A_z\,\hat{\mathbf{k}} \tag{3.3.54}$$

$$\vec{\mathbf{B}}=B_x\,\hat{\mathbf{i}}+B_y\,\hat{\mathbf{j}}+B_z\,\hat{\mathbf{k}}\,, \tag{3.3.55}$$

to yield

$$\vec{\mathbf{A}}\times\vec{\mathbf{B}} = (A_yB_z - A_zB_y)\,\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\,\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\,\hat{\mathbf{k}}\,. \qquad (3.3.56)$$

3.4.4 Vector Decomposition and the Vector Product: Cylindrical Coordinates

Recall the cylindrical coordinate system, which we show in Figure 3.31. We have chosen two directions, radial and tangential in the plane, and a perpendicular direction to the plane.

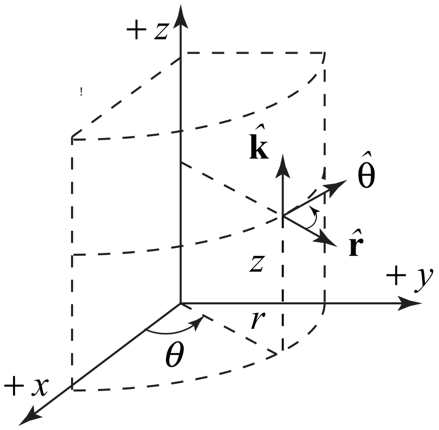


Figure 3.31 Cylindrical coordinates

The unit vectors are at right angles to each other and so using the right hand rule, the vector product of the unit vectors are given by the relations

$$\hat{\mathbf{r}}\times\hat{\mathbf{\theta}} = \hat{\mathbf{k}} \qquad (3.3.57)$$

$$\hat{\mathbf{\theta}}\times\hat{\mathbf{k}} = \hat{\mathbf{r}} \qquad (3.3.58)$$

$$\hat{\mathbf{k}}\times\hat{\mathbf{r}} = \hat{\mathbf{\theta}}\,. \qquad (3.3.59)$$

Because the vector product satisfies $\vec{\mathbf{A}}\times\vec{\mathbf{B}} = -\vec{\mathbf{B}}\times\vec{\mathbf{A}}$, we also have that

$$\hat{\mathbf{\theta}}\times\hat{\mathbf{r}} = -\hat{\mathbf{k}} \qquad (3.3.60)$$

$$\hat{\mathbf{k}}\times\hat{\mathbf{\theta}} = -\hat{\mathbf{r}} \qquad (3.3.61)$$

$$\hat{\mathbf{r}}\times\hat{\mathbf{k}} = -\hat{\mathbf{\theta}}\,. \qquad (3.3.62)$$

Finally

$$\hat{\mathbf{r}}\times\hat{\mathbf{r}} = \hat{\mathbf{\theta}}\times\hat{\mathbf{\theta}} = \hat{\mathbf{k}}\times\hat{\mathbf{k}} = \vec{\mathbf{0}}\,. \qquad (3.3.63)$$

Example 3.6 Vector Products

Given two vectors, $\vec{\mathbf{A}} = 2\,\hat{\mathbf{i}} + -3\,\hat{\mathbf{j}} + 7\,\hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = 5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, find $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$.

Solution:

$$\begin{aligned}\vec{\mathbf{A}}\times\vec{\mathbf{B}} &= (A_yB_z - A_zB_y)\,\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\,\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\,\hat{\mathbf{k}}\\ &= ((-3)(2)-(7)(1))\,\hat{\mathbf{i}} + ((7)(5)-(2)(2))\,\hat{\mathbf{j}} + ((2)(1)-(-3)(5))\,\hat{\mathbf{k}}\\ &= -13\,\hat{\mathbf{i}} + 31\,\hat{\mathbf{j}} + 17\,\hat{\mathbf{k}}.\end{aligned}$$

Example 3.7 Law of Sines

For the triangle shown in Figure 3.32(a), prove the law of sines, $\left|\vec{\mathbf{A}}\right| / \sin\alpha = \left|\vec{\mathbf{B}}\right| / \sin\beta = \left|\vec{\mathbf{C}}\right| / \sin\gamma$, using the vector product.

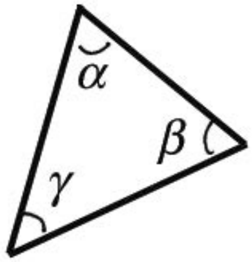


Figure 3.32(a) Example 3.6

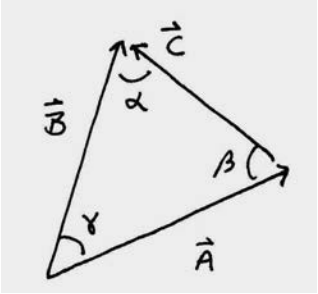


Figure 3.32(b) Vector analysis

Solution: Consider the area of a triangle formed by three vectors $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, and $\vec{\mathbf{C}}$, where $\vec{\mathbf{A}}+\vec{\mathbf{B}}+\vec{\mathbf{C}}=0$ (Figure 3.32(b)). Because $\vec{\mathbf{A}}+\vec{\mathbf{B}}+\vec{\mathbf{C}}=0$, we have that $0=\vec{\mathbf{A}}\times(\vec{\mathbf{A}}+\vec{\mathbf{B}}+\vec{\mathbf{C}})=\vec{\mathbf{A}}\times\vec{\mathbf{B}}+\vec{\mathbf{A}}\times\vec{\mathbf{C}}$. Thus $\vec{\mathbf{A}}\times\vec{\mathbf{B}}=-\vec{\mathbf{A}}\times\vec{\mathbf{C}}$ or $\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right|=\left|\vec{\mathbf{A}}\times\vec{\mathbf{C}}\right|$. From Figure 17.7b we see that $\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right|=\left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|\sin\gamma$ and $\left|\vec{\mathbf{A}}\times\vec{\mathbf{C}}\right|=\left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{C}}\right|\sin\beta$. Therefore $\left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|\sin\gamma=\left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{C}}\right|\sin\beta$, and hence $\left|\vec{\mathbf{B}}\right|/\sin\beta=\left|\vec{\mathbf{C}}\right|/\sin\gamma$. A similar argument shows that $\left|\vec{\mathbf{B}}\right|/\sin\beta=\left|\vec{\mathbf{A}}\right|/\sin\alpha$ proving the law of sines.

Example 3.8 Unit Normal

Find a unit vector perpendicular to $\vec{\mathbf{A}}=\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $\vec{\mathbf{B}}=-2\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}}$.

Solution: The vector product $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ is perpendicular to both $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. Therefore the unit vectors $\hat{\mathbf{n}}=\pm\vec{\mathbf{A}}\times\vec{\mathbf{B}}/\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right|$ are perpendicular to both $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. We first calculate

$$\begin{aligned}\vec{\mathbf{A}}\times\vec{\mathbf{B}} &= (A_yB_z - A_zB_y)\,\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\,\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\,\hat{\mathbf{k}}\\ &= ((1)(3)-(-1)(-1))\,\hat{\mathbf{i}} + ((-1)(2)-(1)(3))\,\hat{\mathbf{j}} + ((1)(-1)-(1)(2))\,\hat{\mathbf{k}}\\ &= 2\,\hat{\mathbf{i}} - 5\,\hat{\mathbf{j}} - 3\,\hat{\mathbf{k}}.\end{aligned}$$

We now calculate the magnitude

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = (2^2+5^2+3^2)^{1/2} = (38)^{1/2} \ .$$

Therefore the perpendicular unit vectors are

$$\hat{\mathbf{n}} = \pm\vec{\mathbf{A}}\times\vec{\mathbf{B}} / \left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \pm(2\,\hat{\mathbf{i}} - 5\,\hat{\mathbf{j}} - 3\,\hat{\mathbf{k}}) / (38)^{1/2} .$$

Example 3.9 Volume of Parallelepiped

Show that the volume of a parallelepiped with edges formed by the vectors $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, and $\vec{\mathbf{C}}$ is given by $\vec{\mathbf{A}}\cdot(\vec{\mathbf{B}}\times\vec{\mathbf{C}})$.

Solution: The volume of a parallelepiped is given by area of the base times height. If the base is formed by the vectors $\vec{\mathbf{B}}$ and $\vec{\mathbf{C}}$, then the area of the base is given by the magnitude of $\vec{\mathbf{B}}\times\vec{\mathbf{C}}$. The vector $\vec{\mathbf{B}}\times\vec{\mathbf{C}} = \left|\vec{\mathbf{B}}\times\vec{\mathbf{C}}\right|\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the base (Figure 3.33).

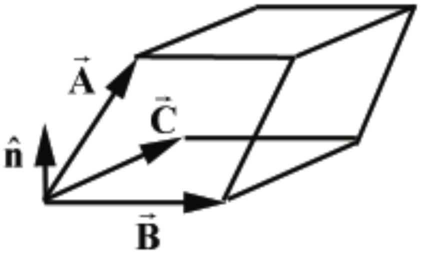


Figure 3.33 Example 3.9

The projection of the vector $\vec{\mathbf{A}}$ along the direction $\hat{\mathbf{n}}$ gives the height of the parallelepiped. This projection is given by taking the dot product of $\vec{\mathbf{A}}$ with a unit vector and is equal to $\vec{\mathbf{A}}\cdot\hat{\mathbf{n}} = \mathit{height}$. Therefore

$$\vec{\mathbf{A}}\cdot(\vec{\mathbf{B}}\times\vec{\mathbf{C}}) = \vec{\mathbf{A}}\cdot(\left|\vec{\mathbf{B}}\times\vec{\mathbf{C}}\right|)\hat{\mathbf{n}} = (\left|\vec{\mathbf{B}}\times\vec{\mathbf{C}}\right|)\vec{\mathbf{A}}\cdot\hat{\mathbf{n}} = (\mathit{area})(\mathit{height}) = (\mathit{volume}) .$$

Example 3.10 Vector Decomposition

Let $\vec{\mathbf{A}}$ be an arbitrary vector and let $\hat{\mathbf{n}}$ be a unit vector in some fixed direction. Show that $\vec{\mathbf{A}} = (\vec{\mathbf{A}}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + (\hat{\mathbf{n}}\times\vec{\mathbf{A}})\times\hat{\mathbf{n}}$.

Solution: Let $\vec{\mathbf{A}} = A_{\|}\hat{\mathbf{n}} + A_{\perp}\hat{\mathbf{e}}$ where $A_{\|}$ is the component $\vec{\mathbf{A}}$ in the direction of $\hat{\mathbf{n}}$, $\hat{\mathbf{e}}$ is the direction of the projection of $\vec{\mathbf{A}}$ in a plane perpendicular to $\hat{\mathbf{n}}$, and $A_{\perp}$ is the component of $\vec{\mathbf{A}}$ in the direction of $\hat{\mathbf{e}}$. Because $\hat{\mathbf{e}}\cdot\hat{\mathbf{n}}=0$, we have that $\vec{\mathbf{A}}\cdot\hat{\mathbf{n}}=A_{\parallel}$. Note that

$$\hat{\mathbf{n}}\times\vec{\mathbf{A}}=\hat{\mathbf{n}}\times(A_{\parallel}\hat{\mathbf{n}}+A_{\perp}\hat{\mathbf{e}})=\hat{\mathbf{n}}\times A_{\perp}\hat{\mathbf{e}}=A_{\perp}(\hat{\mathbf{n}}\times\hat{\mathbf{e}})\,.$$

The unit vector $\hat{\mathbf{n}}\times\hat{\mathbf{e}}$ lies in the plane perpendicular to $\hat{\mathbf{n}}$ and is also perpendicular to $\hat{\mathbf{e}}$. Therefore $(\hat{\mathbf{n}}\times\hat{\mathbf{e}})\times\hat{\mathbf{n}}$ is also a unit vector that is parallel to $\hat{\mathbf{e}}$ (by the right hand rule. So $(\hat{\mathbf{n}}\times\vec{\mathbf{A}})\times\hat{\mathbf{n}}=A_{\perp}\hat{\mathbf{e}}$. Thus

$$\vec{\mathbf{A}}=A_{\parallel}\hat{\mathbf{n}}+A_{\perp}\hat{\mathbf{e}}=(\vec{\mathbf{A}}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}+(\hat{\mathbf{n}}\times\vec{\mathbf{A}})\times\hat{\mathbf{n}}\,.$$

Chapter 4 One Dimensional Kinematics

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> *In the first place, what do we mean by time and space? It turns out that these deep philosophical questions have to be analyzed very carefully in physics, and this is not easy to do. The theory of relativity shows that our ideas of space and time are not as simple as one might imagine at first sight. However, for our present purposes, for the accuracy that we need at first, we need not be very careful about defining things precisely. Perhaps you say, "That's a terrible thing—I learned that in science we have to define everything precisely." We cannot define anything precisely! If we attempt to, we get into that paralysis of thought that comes to philosophers, who sit opposite each other, one saying to the other, "You don't know what you are talking about!" The second one says. "What do you mean by know? What do you mean by talking? What do you mean by you?", and so on. In order to be able to talk constructively, we just have to agree that we are talking roughly about the same thing. You know as much about time as you need for the present, but remember that there are some subtleties that have to be discussed; we shall discuss them later.*[1]
>
> *Richard Feynman*

4.1 Introduction

Kinematics is the mathematical description of motion. The term is derived from the Greek word *kinema,* meaning movement. In order to quantify motion, a mathematical coordinate system, called a *reference frame*, is used to describe space and time. Once a reference frame has been chosen, we shall introduce the physical concepts of position, velocity and acceleration in a mathematically precise manner. Figure 4.1 shows a Cartesian coordinate system in one dimension with unit vector $\hat{\mathbf{i}}$ pointing in the direction of increasing x-coordinate.

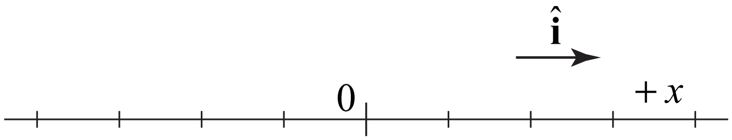

Figure 4.1 A one-dimensional Cartesian coordinate system.

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[1] Richard P. Feynman, Robert B. Leighton, Matthew Sands, *The Feynman Lectures on Physics*, Addison-Wesley, Reading, Massachusetts, (1963), p. 12-2.

4.2 Position, Time Interval, Displacement

4.2.1 Position

Consider a point-like object moving in one dimension. We denote the ***position coordinate*** of the object *with respect to the choice of origin* by $x(t)$. The position coordinate is a function of time and can be positive, zero, or negative, depending on the location of the object. The position of the object with respect to the origin has both direction and magnitude, and hence is a vector (Figure 4.2), which we shall denote as the **position vector** (or simply position) and write as

$$\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}}\,. \tag{4.2.1}$$

We denote the position coordinate at $t = 0$ by the symbol $x_0 \equiv x(t = 0)$. The SI unit for position is the meter [m].

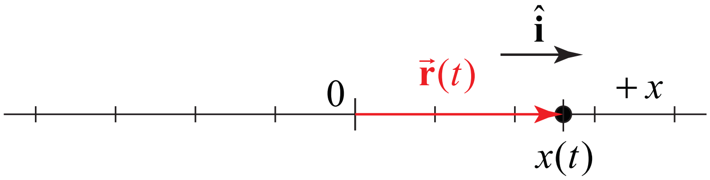

Figure 4.2 The position vector, with reference to a chosen origin.

4.2.2 Time Interval

Consider a closed interval of time $[t_1, t_2]$. We characterize this time interval by the difference in endpoints of the interval,

$$\Delta t = t_2 - t_1\,. \tag{4.2.2}$$

The SI units for time intervals are seconds [s].

4.2.3 Displacement

The ***displacement*** of a body during a time interval $[t_1, t_2]$ (Figure 4.3) is defined to be the change in the position of the body

$$\Delta\vec{\mathbf{r}} \equiv \vec{\mathbf{r}}(t_2) - \vec{\mathbf{r}}(t_1) = (x(t_2) - x(t_1))\,\hat{\mathbf{i}} \equiv \Delta x(t)\,\hat{\mathbf{i}}\,. \tag{4.2.3}$$

Displacement is a vector quantity.

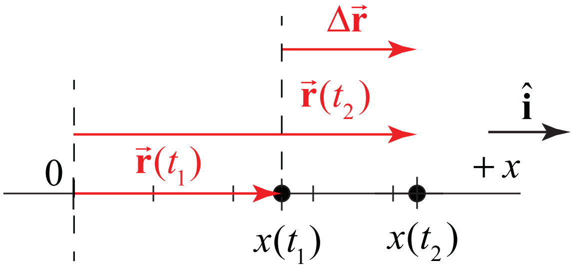

Figure 4.3 The displacement vector of an object over a time interval is the vector difference between the two position vectors

4.3 Velocity

When describing the motion of objects, words like "speed" and "velocity" are used in natural language; however when introducing a mathematical description of motion, we need to define these terms precisely. Our procedure will be to define average quantities for finite intervals of time and then examine what happens in the limit as the time interval becomes infinitesimally small. This will lead us to the mathematical concept that velocity at an instant in time is the derivative of the position with respect to time.

4.3.1 Average Velocity

The x-component of the ***average velocity***, $v_{x,ave}$, for a time interval Δt is defined to be the displacement Δx divided by the time interval Δt,

$$v_{x,ave} \equiv \frac{\Delta x}{\Delta t}. \tag{4.3.1}$$

Because we are describing one-dimensional motion we shall drop the subscript x and denote

$$v_{ave} = v_{x,ave}. \tag{4.3.2}$$

When we introduce two-dimensional motion we will distinguish the components of the velocity by subscripts. The average velocity vector is then

$$\vec{\mathbf{v}}_{ave} \equiv \frac{\Delta x}{\Delta t}\,\hat{\mathbf{i}} = v_{ave}\,\hat{\mathbf{i}}. \tag{4.3.3}$$

The SI units for average velocity are meters per second $\left[\mathrm{m \cdot s^{-1}}\right]$. The average velocity is not necessarily equal to the distance in the time interval Δt traveled divided by the time interval Δt. For example, during a time interval, an object moves in the positive x-direction and then returns to its starting position, the displacement of the object is zero, but the distance traveled is non-zero.

4.3.3 Instantaneous Velocity

Consider a body moving in one direction. During the time interval $[t, t+\Delta t]$, the average velocity corresponds to the slope of the line connecting the points $(t, x(t))$ and $(t+\Delta t, x(t+\Delta t))$. The slope, the rise over the run, is the change in position divided by the change in time, and is given by

$$v_{ave} \equiv \frac{\text{rise}}{\text{run}} = \frac{\Delta x}{\Delta t} = \frac{x(t+\Delta t) - x(t)}{\Delta t}. \tag{4.3.4}$$

As $\Delta t \to 0$, the slope of the lines connecting the points $(t, x(t))$ and $(t+\Delta t, x(t+\Delta t))$, approach slope of the tangent line to the graph of the function $x(t)$ at the time t (Figure 4.4).

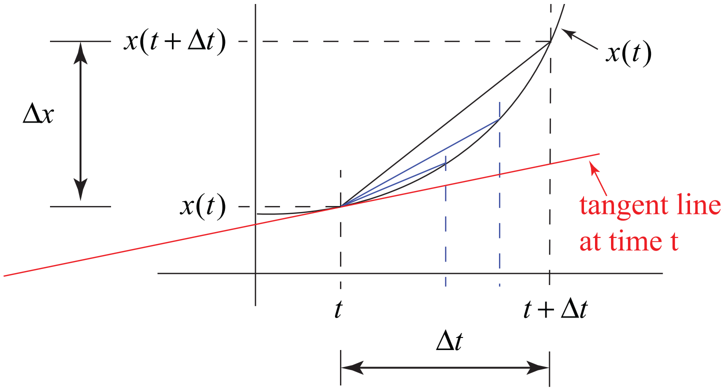

Figure 4.4 Plot of position *vs*. time showing the tangent line at time t.

The limiting value of this sequence is defined to be the x-component of the instantaneous velocity at the time t.

> The x-component of ***instantaneous velocity*** at time t is given by the slope of the tangent line to the graph of the position function at time t:

$$v(t) \equiv \lim_{\Delta t \to 0} v_{ave} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \equiv \frac{dx}{dt}. \tag{4.3.5}$$

The instantaneous velocity vector is then

$$\vec{\mathbf{v}}(t) = v(t)\,\hat{\mathbf{i}}. \tag{4.3.6}$$

The component of the velocity, $v(t)$, can be positive, zero, or negative, depending on whether the object is travelling in the positive x-direction, instantaneously at rest, or the negative x-direction.

Example 4.1 Determining Velocity from Position

Consider an object that is moving along the x-coordinate axis with the position function given by

$$x(t) = x_0 + \frac{1}{2}bt^2 \tag{4.3.7}$$

where x_0 is the initial position of the object at $t = 0$. We can explicitly calculate the x-component of instantaneous velocity from Equation (4.3.5) by first calculating the displacement in the x-direction, $\Delta x = x(t + \Delta t) - x(t)$. We need to calculate the position at time $t + \Delta t$,

$$x(t + \Delta t) = x_0 + \frac{1}{2}b(t + \Delta t)^2 = x_0 + \frac{1}{2}b(t^2 + 2t\Delta t + \Delta t^2)\,. \tag{4.3.8}$$

Then the x-component of instantaneous velocity is

$$v(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\left(x_0 + \frac{1}{2}b(t^2 + 2t\,\Delta t + \Delta t^2)\right) - \left(x_0 + \frac{1}{2}bt^2\right)}{\Delta t}\,. \tag{4.3.9}$$

This expression reduces to

$$v(t) = \lim_{\Delta t \to 0}\left(bt + \frac{1}{2}b\Delta t\right). \tag{4.3.10}$$

The first term is independent of the interval Δt and the second term vanishes because in the limit as $\Delta t \to 0$, the term $(1/2)b\Delta t \to 0$ is zero. Therefore the x-component of instantaneous velocity at time *t* is

$$v(t) = bt\,. \tag{4.3.11}$$

In Figure 4.5 we plot the instantaneous velocity, $v(t)$, as a function of time t.

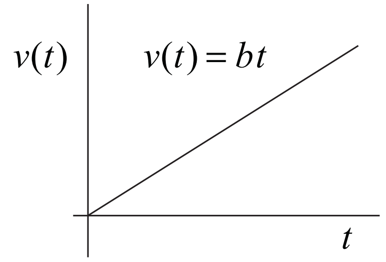

Figure 4.5 Plot of instantaneous velocity instantaneous velocity as a function of time.

Example 4.2 Mean Value Theorem

Consider an object that is moving along the x-coordinate axis with the position function given by

$$x(t) = x_0 + v_0 t + \frac{1}{2} b t^2 . \tag{4.3.12}$$

The graph of $x(t)$ vs. t is shown in Figure 4.6.

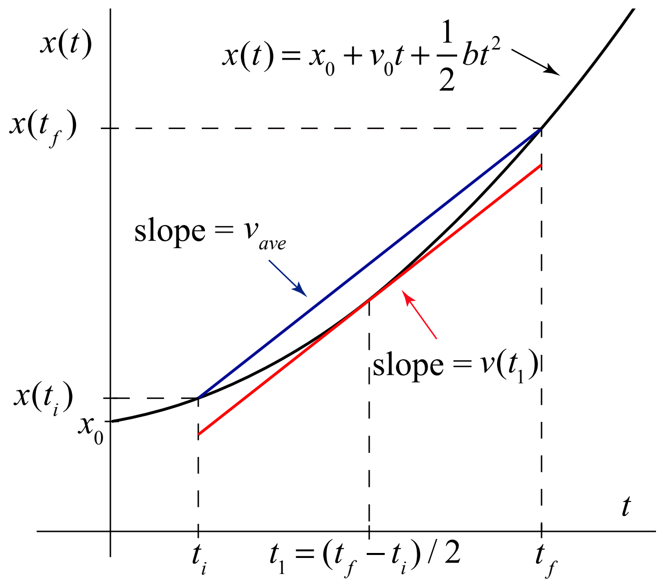

Figure 4.6 Intermediate Value Theorem

The x-component of the instantaneous velocity is

$$v(t) = \frac{dx(t)}{dt} = v_0 + bt . \tag{4.3.13}$$

For the time interval $[t_i, t_f]$, the displacement of the object is

$$x(t_f) - x(t_i) = \Delta x = v_0 (t_f - t_i) + \frac{1}{2} b (t_f{}^2 - t_i^2) = v_0 (t_f - t_i) + \frac{1}{2} b (t_f - t_i)(t_f + t_i) . \tag{4.3.14}$$

Recall that the x-component of the average velocity is defined by the condition that

$$\Delta x = v_{ave} (t_f - t_i) . \tag{4.3.15}$$

We can determine the average velocity by substituting Eq. (4.3.15) into Eq. (4.3.14) yielding

$$v_{ave} = v_0 + \frac{1}{2}b(t_f + t_i)\,. \qquad (4.3.16)$$

The Mean Value Theorem from calculus states that there exists an instant in time t_1, with $t_i < t_1 < t_f$, such that the x-component of the instantaneously velocity, $v(t_1)$, satisfies

$$\Delta x = v(t_1)(t_f - t_i)\ . \qquad (4.3.17)$$

Geometrically this means that the slope of the straight line (blue line in Figure 4.6) connecting the points $(t_i, x(t_i))$ to $(t_f, x(t_f))$ is equal to the slope of the tangent line (red line in Figure 4.6) to the graph of $x(t)$ vs. t at the point $(t_1, x(t_1))$ (Figure 4.6),

$$v(t_1) = v_{ave}\ . \qquad (4.3.18)$$

We know from Eq. (4.3.13) that

$$v(t_1) = v_0 + bt_1\,. \qquad (4.3.19)$$

We can solve for the time t_1 by substituting Eqs. (4.3.19) and (4.3.16) into Eq. (4.3.18) yielding

$$t_1 = (t_f + t_i)/2 \qquad (4.3.20)$$

This intermediate value $v(t_1)$ is also equal to one-half the sum of the initial velocity and final velocity

$$v(t_1) = \frac{v(t_i) + v(t_f)}{2} = \frac{(v_0 + bt_i) + (v_0 + bt_f)}{2} = v_0 + \frac{1}{2}b(t_f + t_i) = v_0 + bt_1\ . \qquad (4.3.21)$$

For any time interval, the quantity $(v(t_i) + v(t_f))/2$, is the arithmetic mean of the initial velocity and the final velocity (but unfortunately is also sometimes referred to as the average velocity). The average velocity, which we defined as $v_{ave} = (x_f - x_i)/\Delta t$, and the arithmetic mean, $(v(t_i) + v(t_f))/2$, are only equal in the special case when the velocity is a linear function in the variable t as in this example, (Eq. (4.3.13)). We shall only use the term average velocity to mean displacement divided by the time interval.

4.4 Acceleration

We shall apply the same physical and mathematical procedure for defining acceleration, as the rate of change of velocity with respect to time. We first consider how the instantaneous velocity changes over a fixed time interval of time and then take the limit as the time interval approaches zero.

4.4.1 Average Acceleration

Average acceleration is the quantity that measures a change in velocity over a particular time interval. Suppose during a time interval Δt a body undergoes a change in velocity

$$\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}(t+\Delta t) - \vec{\mathbf{v}}(t)\,. \tag{4.3.22}$$

The change in the x-component of the velocity, Δv, for the time interval $[t, t+\Delta t]$ is then

$$\Delta v = v(t+\Delta t) - v(t)\,. \tag{4.3.23}$$

The x***-component of the average acceleration*** for the time interval Δt is defined to be

$$\vec{\mathbf{a}}_{ave} = a_{ave}\,\hat{\mathbf{i}} \equiv \frac{\Delta v}{\Delta t}\,\hat{\mathbf{i}} = \frac{(v(t+\Delta t) - v(t))}{\Delta t}\,\hat{\mathbf{i}}\,. \tag{4.3.24}$$

The SI units for average acceleration are meters per second squared, $[\mathrm{m}\cdot\mathrm{s}^{-2}]$.

4.4.2 Instantaneous Acceleration

Consider the graph of the x-component of velocity, $v(t)$, (Figure 4.7).

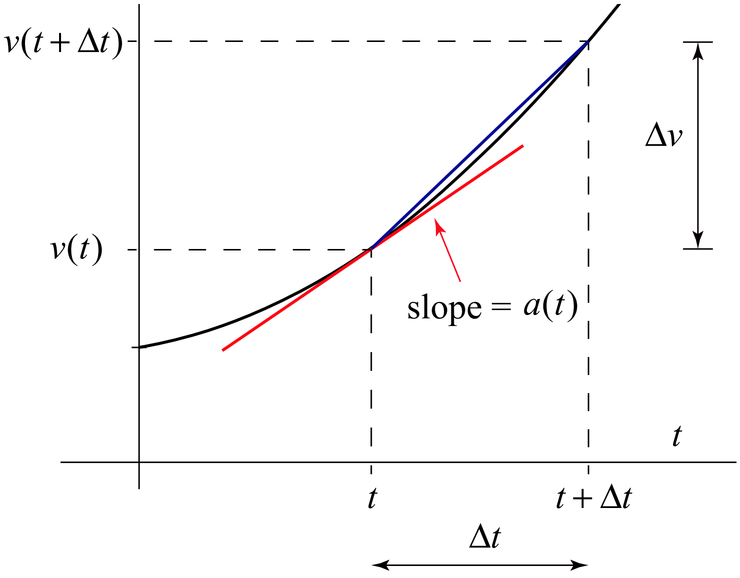

Figure 4.7 Graph of velocity *vs*. time showing the tangent line at time t.

The average acceleration for a fixed time interval Δt is the slope of the straight line connecting the two points $(t, v(t))$ and $(t+\Delta t, v(t+\Delta t))$. In order to define the x-component of the instantaneous acceleration at time t, we employ the same limiting argument as we did when we defined the instantaneous velocity in terms of the slope of the tangent line.

> The x***-component of the instantaneous acceleration*** *at time* t is the slope of the tangent line at time t of the graph of the x-component of the velocity as a function of time,

$$a(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \lim_{\Delta t \to 0} \frac{(v(t+\Delta t) - v(t))}{\Delta t} \equiv \frac{dv}{dt}. \qquad (4.3.25)$$

The instantaneous acceleration vector at time t is then

$$\vec{\mathbf{a}}(t) = a(t)\,\hat{\mathbf{i}}. \qquad (4.3.26)$$

Because the velocity is the derivative of position with respect to time, the x-component of the acceleration is the second derivative of the position function,

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}. \qquad (4.3.27)$$

Example 4.3 Determining Acceleration from Velocity

Let's continue Example 4.1, in which the position function for the body is given by $x = x_0 + (1/2)bt^2$, and the x-component of the velocity is $v = bt$. The x-component of the instantaneous acceleration is the first derivative (with respect to time) of the x-component of the velocity:

$$a = \frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{bt + b\Delta t - bt}{\Delta t} = b. \qquad (4.3.28)$$

Note that in Eq. (4.3.28), the ratio $\Delta v / \Delta t$ is independent of t, consistent with the constant slope as shown in Figure 4.5.

4.5 Constant Acceleration

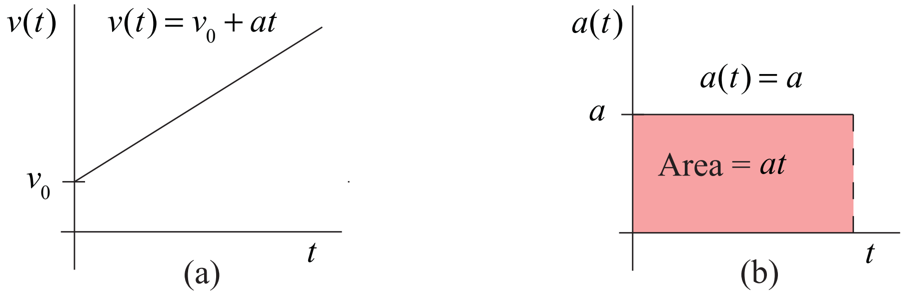

Figure 4.8 Constant acceleration: (a) velocity, (b) acceleration

When the x-component of the velocity is a linear function (Figure 4.8(a)), the average acceleration, $\Delta v / \Delta t$, is a constant and hence is equal to the instantaneous acceleration (Figure 4.8(b)). Let's consider a body undergoing constant acceleration for a time interval $[0,t]$, where $\Delta t = t$. Denote the x-component of the velocity at time $t=0$ by $v_0 \equiv v(t=0)$. Therefore the x-component of the acceleration is given by

$$a(t)=\frac{\Delta v}{\Delta t}=\frac{v(t)-v_0}{t}. \tag{4.4.1}$$

Thus the x-component of the velocity is a linear function of time given by

$$v(t)=v_0+at. \tag{4.4.2}$$

4.5.1 Velocity: Area Under the Acceleration *vs.* Time Graph

In Figure 4.8(b), the area under the acceleration *vs.* time graph, for the time interval $\Delta t = t - 0 = t$, is

$$\text{Area}(a(t),t)=at. \tag{4.4.3}$$

From Eq. (4.4.2), the area is the change in the x-component of the velocity for the interval $[0,t]$:

$$\text{Area}(a(t),t)=at=v(t)-v_0=\Delta v. \tag{4.4.4}$$

4.5.2 Displacement: Area Under the Velocity *vs.* Time Graph

In Figure 4.9 shows a graph of the x-component of the velocity *vs.* time for the case of constant acceleration (Eq. (4.4.2)).

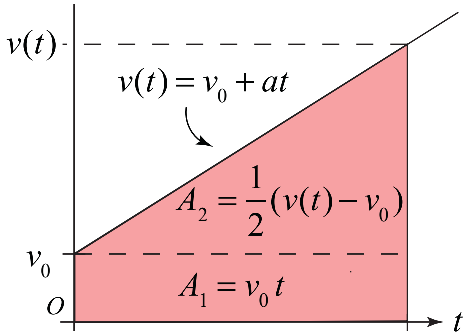

Figure 4.9 Graph of velocity as a function of time for a constant.

The region under the velocity *vs*. time curve is a trapezoid, formed from a rectangle with area $A_1 = v_0 t$, and a triangle with area $A_2 = (1/2)(v(t) - v_0)$. The total area of the trapezoid is given by

$$\text{Area}(v(t),t) = A_1 + A_2 = v_0 t + \frac{1}{2}(v(t) - v_0). \tag{4.4.5}$$

Substituting for the velocity (Eq. (4.4.2)) yields

$$\text{Area}(v(t),t) = v_0 t + \frac{1}{2} a t^2. \tag{4.4.6}$$

Recall that from Example 4.2 (setting $b = a$ and $\Delta t = t$),

$$v_{ave} = v_0 + \frac{1}{2} at = \Delta x / t, \tag{4.4.7}$$

therefore Eq. (4.4.6) can be rewritten as

$$\text{Area}(v(t),t) = (v_0 + \frac{1}{2} a t)t = v_{ave} t = \Delta x \tag{4.4.8}$$

The displacement is equal to the area under the graph of the x-component of the velocity *vs*. time. The position as a function of time can now be found by rewriting Equation (4.4.8) as

$$x(t) = x_0 + v_0 t + \frac{1}{2} a t^2. \tag{4.4.9}$$

Figure 4.10 shows a graph of this equation. Notice that at $t = 0$ the slope is non-zero, corresponding to the initial velocity component v_0.

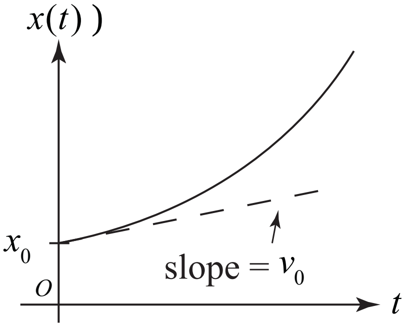

Figure 4.10 Graph of position vs. time for constant acceleration.

Example 4.4 Accelerating Car

A car, starting at rest at $t = 0$, accelerates in a straight line for 100 m with an unknown constant acceleration. It reaches a speed of $20\text{ m}\cdot\text{s}^{-1}$ and then continues at this speed for another 10 s. (a) Write down the equations for position and velocity of the car as a function of time. (b) How long was the car accelerating? (c) What was the magnitude of the acceleration? (d) Plot speed *vs.* time, acceleration *vs.* time, and position *vs.* time for the entire motion. (e) What was the average velocity for the entire trip?

Solutions: (a) For the acceleration a, the position $x(t)$ and velocity $v(t)$ as a function of time t for a car starting from rest are

$$\begin{aligned} x(t) &= (1/2)at^2 \\ v_x(t) &= at. \end{aligned} \tag{4.4.10}$$

b) Denote the time interval during which the car accelerated by t_1. We know that the position $x(t_1) = 100\text{m}$ and $v(t_1) = 20\text{ m}\cdot\text{s}^{-1}$. Note that we can eliminate the acceleration a between the Equations (4.4.10) to obtain

$$x(t) = (1/2)v(t)\,t. \tag{4.4.11}$$

We can solve this equation for time as a function of the distance and the final speed giving

$$t = 2\frac{x(t)}{v(t)}. \tag{4.4.12}$$

We can now substitute our known values for the position $x(t_1) = 100\text{m}$ and $v(t_1) = 20\text{ m}\cdot\text{s}^{-1}$ and solve for the time interval that the car has accelerated

$$t_1 = 2\frac{x(t_1)}{v(t_1)} = 2\frac{100\text{ m}}{20\text{ m}\cdot\text{s}^{-1}} = 10\,\text{s}\,. \qquad (4.4.13)$$

c) We can substitute into either of the expressions in Equation (4.4.10); the second is slightly easier to use,

$$a = \frac{v(t_1)}{t_1} = \frac{20\text{ m}\cdot\text{s}^{-1}}{10\,\text{s}} = 2.0\,\text{m}\cdot\text{s}^{-2}\,. \qquad (4.4.14)$$

d) The x -component of acceleration vs. time, x -component of the velocity vs. time, and the position vs. time are piece-wise functions given by

$$a(t) = \begin{cases} 2\text{ m}\cdot\text{s}^{-2}; & 0 < t \le 10\text{ s} \\ 0; & 10\text{ s} < t < 20\text{ s} \end{cases},$$

$$v(t) = \begin{cases} (2\text{ m}\cdot\text{s}^{-2})t; & 0 < t \le 10\text{ s} \\ 20\text{ m}\cdot\text{s}^{-1}; & 10\text{ s} \le t \le 20\text{ s} \end{cases},$$

$$x(t) = \begin{cases} (1/2)(2\text{ m}\cdot\text{s}^{-2})t^2; & 0 < t \le 10\text{ s} \\ 100\text{ m} + (20\text{ m}\cdot\text{s}^{-2})(t - 10\text{ s}); & 10\text{ s} \le t \le 20\text{ s} \end{cases}.$$

The graphs of the x -component of acceleration vs. time, x -component of the velocity vs. time, and the position vs. time are shown in Figure 4.11.

(e) After accelerating, the car travels for an additional ten seconds at constant speed and during this interval the car travels an additional distance $\Delta x = v(t_1) \times 10\,\text{s} = 200\,\text{m}$ (note that this is twice the distance traveled during the $10\,\text{s}$ of acceleration), so the total distance traveled is $300\,\text{m}$ and the total time is $20\,\text{s}$, for an average velocity of

$$v_{\text{ave}} = \frac{300\,\text{m}}{20\,\text{s}} = 15\,\text{m}\cdot\text{s}^{-1}\,. \qquad (4.4.15)$$

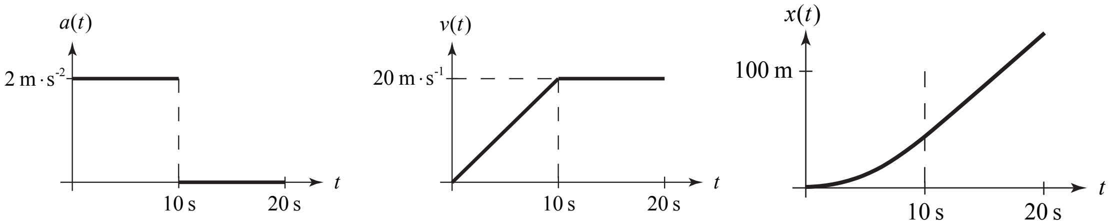

Figure 4.11 Graphs of the x-components of acceleration, velocity and position as piece-wise functions

Example 4.5 Catching a Bus

At the instant a traffic light turns green, a car starts from rest with a given constant acceleration, $3.0\ \text{m}\cdot\text{s}^{-2}$. Just as the light turns green, a bus, traveling with a given constant velocity, $1.6\times10^{1}\ \text{m}\cdot\text{s}^{-1}$, passes the car. The car speeds up and passes the bus some time later. How far down the road has the car traveled, when the car passes the bus?

Solution:

There are two moving objects, bus and the car. Each object undergoes one stage of one-dimensional motion. We are given the acceleration of the car, the velocity of the bus, and infer that the position of the car and the bus are equal when the bus just passes the car. Figure 4.12 shows a qualitative sketch of the position of the car and bus as a function of time.

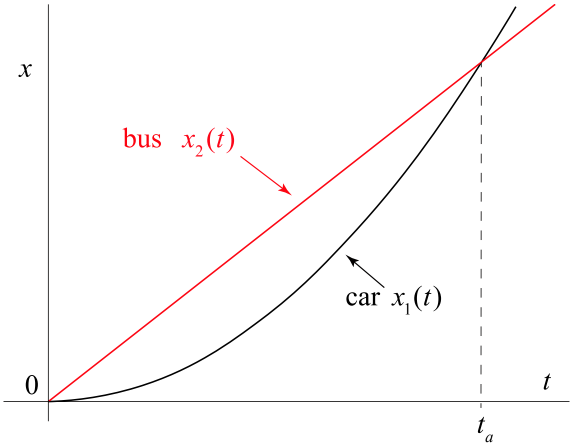

Figure 4.12 Position vs. time of the car and bus

Choose a coordinate system with the origin at the traffic light and the positive x-direction such that car and bus are travelling in the positive x-direction. Set time $t = 0$ as the instant the car and bus pass each other at the origin when the light turns green. Figure 4.13 shows the position of the car and bus at time t.

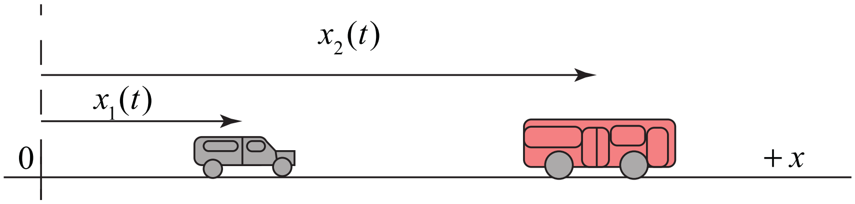

Figure 4.13 Coordinate system for car and bus

Let $x_1(t)$ denote the position function of the car, and $x_2(t)$ the position function for the bus. The initial position and initial velocity of the car are both zero, $x_{1,0} = 0$ and $v_{1,0} = 0$, and the acceleration of the car is non-zero $a_1 \neq 0$. Therefore the position and velocity functions of the car are given by

$$x_1(t) = \frac{1}{2}a_1 t^2 ,$$
$$v_1(t) = a_1 t .$$

The initial position of the bus is zero, $x_{2,0} = 0$, the initial velocity of the bus is non-zero, $v_{2,0} \neq 0$, and the acceleration of the bus is zero, $a_2 = 0$. Therefore the velocity is constant, $v_2(t) = v_{2,0}$, and the position function for the bus is given by $x_2(t) = v_{2,0}t$.

Let $t = t_a$ correspond to the time that the car passes the bus. Then at that instant, the position functions of the bus and car are equal, $x_1(t_a) = x_2(t_a)$. We can use this condition to solve for t_a:

$$(1/2)a_1 t_a{}^2 = v_{2,0}t_a \Rightarrow t_a = \frac{2v_{2,0}}{a_1} = \frac{(2)(1.6\times 10^1\,\mathrm{m\cdot s^{-1}})}{(3.0\,\mathrm{m\cdot s^{-2}})} = 1.1\times 10^1\,\mathrm{s}.$$

Therefore the position of the car at t_a is

$$x_1(t_a) = \frac{1}{2}a_1 t_a{}^2 = \frac{2v_{2,0}{}^2}{a_1} = \frac{(2)(1.6\times 10^1\,\mathrm{m\cdot s^{-1}})^2}{(3.0\ \mathrm{m\cdot s^{-2}})} = 1.7\times 10^2\,\mathrm{m}.$$

4.6 One Dimensional Kinematics and Integration

When the acceleration $a(t)$ of an object is a non-constant function of time, we would like to determine the time dependence of the position function $x(t)$ and the x-component of the velocity $v(t)$. Because the acceleration is non-constant we no longer can use Eqs. (4.4.2) and (4.4.9). Instead we shall use integration techniques to determine these functions.

4.6.1 Change of Velocity as the Indefinite Integral of Acceleration

Consider a time interval $t_1 < t < t_2$. Recall that by definition the derivative of the velocity $v(t)$ is equal to the acceleration $a(t)$,

$$\frac{dv(t)}{dt} = a(t) \ . \qquad (4.5.1)$$

Integration is defined as the inverse operation of differentiation or the 'anti-derivative'. For our example, the function $v(t)$ is called the ***indefinite integral*** of $a(t)$ with respect to t, and is unique up to an additive constant C. We denote this by writing

$$v(t)+C=\int a(t)\,dt \quad . \tag{4.5.2}$$

The symbol $\int ...dt$ means the 'integral, with respect to t, of …", and is thought of as the inverse of the symbol $\dfrac{d}{dt}$..... Equivalently we can write the differential $dv(t)=a(t)dt$, called the ***integrand***, and then Eq. (4.5.2) can be written as

$$v(t)+C=\int dv(t) \quad , \tag{4.5.3}$$

which we interpret by saying that the integral of the differential of function is equal to the function plus a constant.

Example 4.6 Non-constant acceleration

Suppose an object at time $t=0$ has initial non-zero velocity v_0 and acceleration $a(t)=bt^2$, where b is a constant. Then $dv(t)=bt^2dt=d(bt^3/3)$. The velocity is then $v(t)+C=\int d(bt^3/3)=bt^3/3$. At $t=0$, we have that $v_0+C=0$. Therefore $C=-v_0$ and the velocity as a function of time is then $v(t)=v_0+(bt^3/3)$.

4.6.2 Area as the Indefinite Integral of Acceleration

Consider the graph of a positive-valued acceleration function $a(t)$ vs. t for the interval $t_1\le t\le t_2$, shown in Figure 4.14a. Denote the area under the graph of $a(t)$ over the interval $t_1\le t\le t_2$ by $A_{t_1}^{t_2}$.

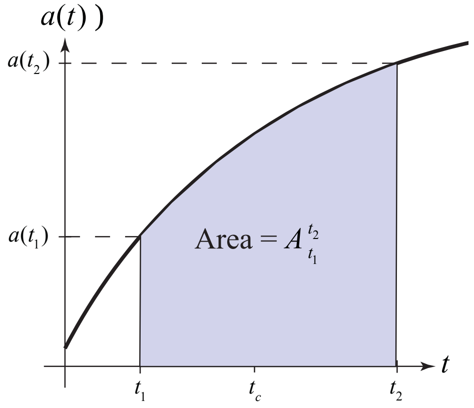

Figure 4.14a: Area under the graph of acceleration over an interval $t_1\le t\le t_2$

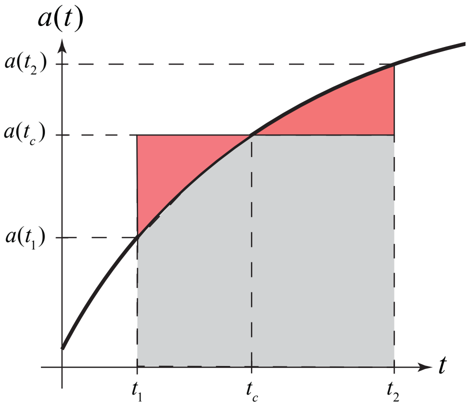

Figure 4.14b: Intermediate value Theorem. The shaded regions above and below the curve have equal areas.

The ***Intermediate Value Theorem*** states that there is at least one time t_c such that the area $A_{t_1}^{t_2}$ is equal to

$$A_{t_1}^{t_2} = a(t_c)(t_2 - t_1) \ . \tag{4.5.4}$$

In Figure 4.14b, the shaded regions above and below the curve have equal areas, and hence the area $A_{t_1}^{t_2}$ under the curve is equal to the area of the rectangle given by $a(t_c)(t_2 - t_1)$.

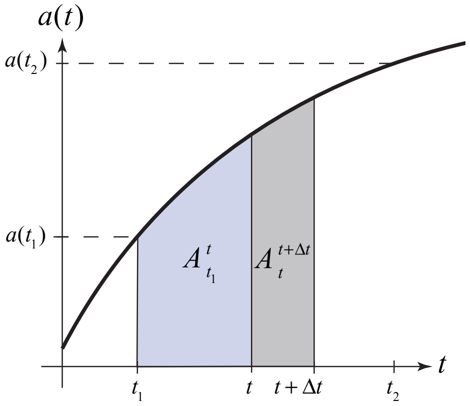

Figure 4.15 Area function is additive

We shall now show that the derivative of the area function is equal to the acceleration and thererfore we can write the area function as an indefinite integral. From Figure 4.15, the area function satisfies the condition that

$$A_{t_1}^{t} + A_{t}^{t+\Delta t} = A_{t_1}^{t+\Delta t} \ . \tag{4.5.5}$$

Let the small increment of area be denoted by $\Delta A_{t_1}^{t} = A_{t_1}^{t+\Delta t} - A_{t_1}^{t} = A_{t}^{t+\Delta t}$. By the Intermediate Value Theorem

$$\Delta A_{t_1}^{t} = a(t_c)\Delta t \ , \tag{4.5.6}$$

where $t \le t_c \le t + \Delta t$. In the limit as $\Delta t \to 0$,

$$\frac{dA_{t_1}^{t}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A_{t_1}^{t}}{\Delta t} = \lim_{t_c \to t} a(t_c) = a(t) \ , \tag{4.5.7}$$

with the initial condition that when $t = t_1$, the area $A_{t_1}^{t_1} = 0$ is zero. Because $v(t)$ is also an integral of $a(t)$, we have that

$$A_{t_1}^{t} = \int a(t)\,dt = v(t) + C \ . \tag{4.5.8}$$

When $t = t_1$, the area $A_{t_1}^{t_1} = 0$ is zero, therefore $v(t_1) + C = 0$, and so $C = -v(t_1)$. Therefore Eq. (4.5.8) becomes

$$A_{t_1}^{t} = v(t) - v(t_1) = \int a(t)\,dt \ . \tag{4.5.9}$$

When we set $t = t_2$, Eq. (4.5.9) becomes

$$A_{t_1}^{t_2} = v(t_2) - v(t_1) = \int a(t)\,dt \ . \tag{4.5.10}$$

The area under the graph of the positive-valued acceleration function for the interval $t_1 \le t \le t_2$ can be found by integrating $a(t)$.

4.6.3 Change of Velocity as the Definite Integral of Acceleration

Let $a(t)$ be the acceleration function over the interval $t_i \le t \le t_f$. Recall that the velocity $v(t)$ is an integral of $a(t)$ because $dv(t)/dt = a(t)$. Divide the time interval $[t_i, t_f]$ into n equal time subintervals $\Delta t = (t_f - t_i)/n$. For each subinterval $[t_j, t_{j+1}]$, where the index $j = 1,\ 2,\ \ldots, n$, $t_1 = t_i$ and $t_{n+1} = t_f$, let t_{c_j} be a time such that $t_j \le t_{c_j} \le t_{j+1}$. Let

$$S_n = \sum_{j=1}^{j=n} a(t_{c_j})\,\Delta t \ . \tag{4.5.11}$$

S_n is the sum of the blue rectangle shown in Figure 4.16a for the case $n = 4$. The **Fundamental Theorem of Calculus** states that in the limit as $n \to \infty$, the sum is equal to the change in the velocity during the interval $[t_i, t_f]$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{j=1}^{j=n} a(t_{c_j})\,\Delta t = v(t_f) - v(t_i) \ . \qquad (4.5.12)$$

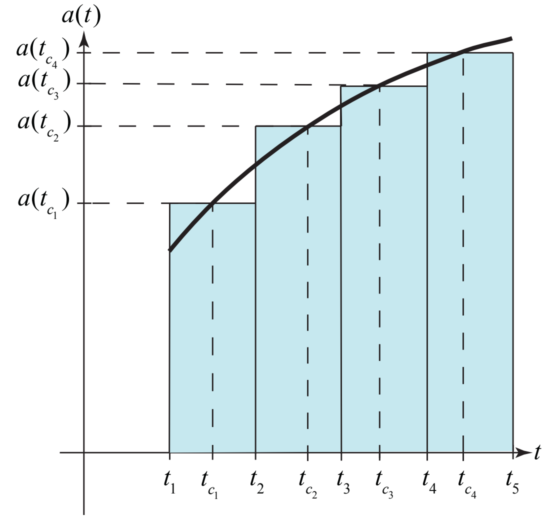

Figure 4.16a Graph of $a(t)$ vs. t

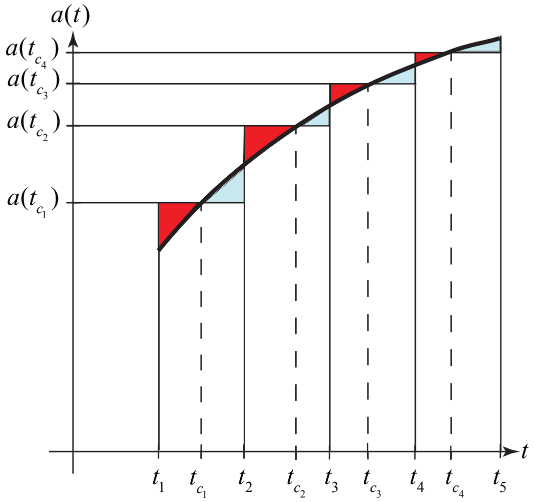

Figure 4.16b Graph of $a(t)$ vs. t

The limit of the sum in Eq. (4.5.12) is a number, which we denote by the symbol

$$\int_{t_i}^{t_f} a(t)\,dt \equiv \lim_{n\to\infty} \sum_{j=1}^{j=n} a(t_{c_j})\,\Delta t = v(t_f) - v(t_i) \ , \qquad (4.5.13)$$

and is called the ***definite integral*** of $a(t)$ from t_i to t_f. The times t_i and t_f are called the limits of integration, t_i the lower limit and t_f the upper limit. The definite integral is a linear map that takes a function $a(t)$ defined over the interval $[t_i, t_f]$ and gives a number. The map is linear because

$$\int_{t_i}^{t_f} (a_1(t) + a_2(t))\,dt = \int_{t_i}^{t_f} a_1(t)\,dt + \int_{t_i}^{t_f} a_2(t)\,dt \ , \qquad (4.5.14)$$

Suppose the times t_{c_j}, $j = 1,...,n$, are selected such that each t_{c_j} satisfies the Intermediate Value Theorem,

$$\Delta v_j \equiv v(t_{j+1}) - v(t_j) = \frac{dv(t_{c_j})}{dt}\,\Delta t = a(t_{c_j})\Delta t \ , \qquad (4.5.15)$$

where $a(t_{c_j})$ is the instantaneous acceleration at t_{c_j}, (Figure 4.16b). Then the sum of the changes in the velocity for the interval $[t_i, t_f]$ is

$$\begin{aligned}\sum_{j=1}^{j=n} \Delta v_j &= (v(t_2) - v(t_1)) + (v(t_3) - v(t_2)) + \cdots + (v(t_{n+1}) - v(t_n)) = v(t_{n+1}) - v(t_1) \\ &= v(t_f) - v(t_i).\end{aligned} \tag{4.5.16}$$

where $v(t_f) = v(t_{n+1})$ and $v(t_i) = v(t_1)$. Substituting Eq. (4.5.15) into Equation (4.5.16) yields the exact result that the change in the x-component of the velocity is give by this finite sum.

$$v(t_f) - v(t_i) = \sum_{j=1}^{j=n} \Delta v_j = \sum_{j=1}^{j=n} a(t_{c_j})\, \Delta t. \tag{4.5.17}$$

We do not specifically know the intermediate values $a(t_{c_j})$ and so Eq. (4.5.17) is not useful as a calculating tool. The statement of the Fundamental Theorem of Calculus is that the limit as $n \to \infty$ of the sum in Eq. (4.5.12) is independent of the choice of the set of t_{c_j}. Therefore the exact result in Eq. (4.5.17) is the limit of the sum.

Thus we can evaluate the definite integral if we know any indefinite integral of the integrand $a(t)dt = dv(t)$.

Additionally, provided the acceleration function has only non-negative values, the limit is also equal to the area under the graph of $a(t)$ vs. t for the time interval, $[t_i, t_f]$:

$$A_{t_i}^{t_f} = \int_{t_i}^{t_f} a(t)\, dt\ . \tag{4.5.18}$$

In Figure 4.14, the red areas are an overestimate and the blue areas are an underestimate. As $N \to \infty$, the sum of the red areas and the sum of the blue areas both approach zero. If there are intervals in which $a(t)$ has negative values, then the summation is a sum of signed areas, positive area above the t-axis and negative area below the t-axis.

We can determine both the change in velocity for the time interval $[t_i, t_f]$ and the area under the graph of $a(t)$ vs. t for $[t_i, t_f]$ by integration techniques instead of limiting arguments. We can turn the linear map into a function of time, instead of just giving a number, by setting $t_f = t$. In that case, Eq. (4.5.13) becomes

$$v(t) - v(t_i) = \int_{t'=t_i}^{t'=t} a(t')dt' . \qquad (4.5.19)$$

Because the upper limit of the integral, $t_f = t$, is now treated as a variable, we shall use the symbol t' as the integration variable instead of t.

4.6.4 Displacement as the Definite Integral of Velocity

We can repeat the same argument for the definite integral of the x-component of the velocity $v(t)$ *vs*. time t. Because $x(t)$ is an integral of $v(t)$ the definite integral of $v(t)$ for the time interval $[t_i, t_f]$ is the displacement

$$x(t_f) - x(t_i) = \int_{t'=t_i}^{t'=t_f} v(t')dt' . \qquad (4.5.20)$$

If we set $t_f = t$, then the definite integral gives us the position as a function of time

$$x(t) = x(t_i) + \int_{t'=t_i}^{t'=t} v(t')dt' . \qquad (4.5.21)$$

Summarizing the results of these last two sections, for a given acceleration $a(t)$, we can use integration techniques, to determine the change in velocity and change in position for an interval $[t_i, t]$, and given initial conditions (x_i, v_i), we can determine the position $x(t)$ and the x-component of the velocity $v(t)$ as functions of time.

Example 4.5 Non-constant Acceleration

Let's consider a case in which the acceleration, $a(t)$, is not constant in time,

$$a(t) = b_0 + b_1 t + b_2 t^2 . \qquad (4.5.22)$$

The graph of the x-component of the acceleration *vs*. time is shown in Figure 4.16

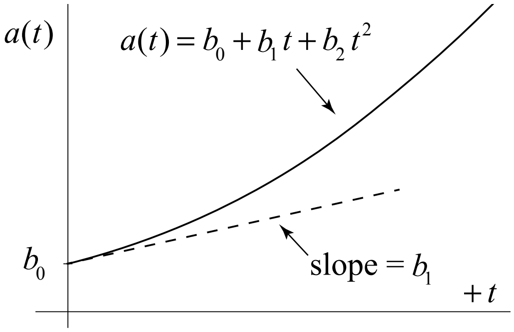

Figure 4.16 Non-constant acceleration *vs*. time graph.

Denote the initial velocity at $t = 0$ by v_0. Then, the change in the x-component of the velocity as a function of time can be found by integration:

$$v(t) - v_0 = \int_{t'=0}^{t'=t} a(t')\,dt' = \int_{t'=0}^{t'=t} (b_0 + b_1 t' + b_2 t'^2)\,dt' = b_0 t + \frac{b_1 t^2}{2} + \frac{b_2 t^3}{3}. \qquad (4.5.23)$$

The x-component of the velocity as a function in time is then

$$v(t) = v_0 + b_0 t + \frac{b_1 t^2}{2} + \frac{b_2 t^3}{3}. \qquad (4.5.24)$$

Denote the initial position at $t = 0$ by x_0. The displacement as a function of time is

$$x(t) - x_0 = \int_{t'=0}^{t'=t} v(t')\,dt'. \qquad (4.5.25)$$

Use Equation (4.5.27) for the x-component of the velocity in Equation (4.5.24) and then integrate to determine the displacement as a function of time:

$$\begin{aligned} x(t) - x_0 &= \int_{t'=0}^{t'=t} v(t')\,dt' \\ &= \int_{t'=0}^{t'=t} \left(v_0 + b_0 t' + \frac{b_1 t'^2}{2} + \frac{b_2 t'^3}{3} \right) dt' = v_0 t + \frac{b_0 t^2}{2} + \frac{b_1 t^3}{6} + \frac{b_2 t^4}{12}. \end{aligned} \qquad (4.5.26)$$

Finally the position as a function of time is then

$$x(t) = x_0 + v_{x,0} t + \frac{b_0 t^2}{2} + \frac{b_1 t^3}{6} + \frac{b_2 t^4}{12}. \qquad (4.5.27)$$

Example 4.6 Bicycle and Car

A car is driving through a green light at $t = 0$ located at $x = 0$ with an initial speed $v_{c,0} = 12\ \text{m}\cdot\text{s}^{-1}$. At time $t_1 = 1\ \text{s}$, the car starts braking until it comes to rest at time t_2. The acceleration of the car as a function of time is given by the piecewise function

$$a_c(t) = \begin{cases} 0; & 0 < t < t_1 = 1\ \text{s} \\ b(t - t_1); & 1\ \text{s} < t < t_2 \end{cases},$$

where $b = -(6\ \text{m}\cdot\text{s}^{-3})$.

(a) Find the x-component of the velocity and the position of the car as a function of time.
(b) A bicycle rider is riding at a constant speed of $v_{b,0}$ and at $t = 0$ is $17\ \text{m}$ behind the car. The bicyclist reaches the car when the car just comes to rest. Find the speed of the bicycle.

Solution: a) In order to apply Eq. (4.5.19), we shall treat each stage separately. For the time interval $0 < t < t_1$, the acceleration is zero so the x-component of the velocity is constant. For the second time interval $\text{t}_1 < t < t_2$, the definite integral becomes

$$v_c(t) - v_c(t_1) = \int_{t'=t_1}^{t'=t} b(t' - t_1)\,dt'$$

Because $v_c(t_1) = v_{c0}$, the x-component of the velocity is then

$$v_c(t) = \begin{cases} v_{c0}; & 0 < t \le t_1 \\ v_{c0} + \displaystyle\int_{t'=t_1}^{t'=t} b(t' - t_1)\,dt'; & t_1 \le t < t_2 \end{cases}.$$

Integrate and substitute the two endpoints of the definite integral, yields

$$v_c(t) = \begin{cases} v_{c0}; & 0 < t \le t_1 \\ v_{c0} + \dfrac{1}{2}b(t - t_1)^2; & t_1 \le t < t_2 \end{cases}.$$

In order to use Eq. (4.5.25), we need to separate the definite integral into two integrals corresponding to the two stages of motion, using the correct expression for the velocity for each integral. The position function is then

$$x_c(t)=\begin{cases} x_{c0}+\int\limits_{t'=0}^{t'=t_1} v_{c0}\,dt'; & 0<t\le t_1 \\ x_c(t_1)+\int\limits_{t'=t_1}^{t'=t}\left(v_{c0}+\frac{1}{2}b(t'-t_1)^2\right)dt; & t_1\le t<t_2 \end{cases}.$$

Upon integration we have

$$x_c(t)=\begin{cases} x_c(0)+v_{c0}t; & 0<t\le t_1 \\ x_c(t_1)+\left(v_{c0}(t'-t_1)+\frac{1}{6}b(t'-t_1)^3\right)\Bigg|_{t'=t_1}^{t'=t}; & t_1\le t<t_2 \end{cases}.$$

We chose our coordinate system such that the initial position of the car was at the origin, $x_{c0}=0$, therefore $x_c(t_1)=v_{c0}t_1$. So after substituting in the endpoints of the integration interval we have that

$$x_c(t)=\begin{cases} v_{c0}t; & 0<t\le t_1 \\ v_{c0}t_1+v_{c0}(t-t_1)+\frac{1}{6}b(t-t_1)^3; & t_1\le t<t_2 \end{cases}.$$

(b) We are looking for the instant t_2 that the car has come to rest. So we use our expression for the x-component of the velocity the interval $t_1\le t<t_2$, where we set $t=t_2$ and $v_c(t_2)=0$:

$$0=v_c(t_2)=v_{c0}+\frac{1}{2}b(t_2-t_1)^2.$$

Solving for t_2 yields

$$t_2=t_1+\sqrt{-\frac{2v_{c0}}{b}},$$

where we have taken the positive square root. Substitute the given values then yields

$$t_2=1\,\mathrm{s}+\sqrt{-\frac{2(12\ \mathrm{m\cdot s^{-1}})}{(-6\ \mathrm{m\cdot s^{-3}})}}=3\,\mathrm{s}.$$

The position of the car at t_2 is then given by

$$x_c(t_2) = v_{c0}\,t_1 + v_{c0}(t_2 - t_1) + \frac{1}{6}b(t_2 - t_1)^3$$

$$x_c(t_2) = v_{c0}\,t_1 + v_{c0}\sqrt{-2v_{c0}\,/\,b} + \frac{1}{6}b(-2v_{c0}\,/\,b)^{3/2}$$

$$x_c(t_2) = v_{c0}\,t_1 + \frac{2\sqrt{2}(v_{c0}{}^{3/2})}{3(-b)^{1/2}}$$

where we used the condition that $t_2 - t_1 = \sqrt{-2v_{c0}\,/\,b}$. Substitute the given values then yields

$$x_c(t_2) = v_{c0}\,t_1 + 2\frac{4\sqrt{2}(v_{c0})^{3/2}}{3(-b)^{1/2}} = (12\text{ m}\cdot\text{s}^{-1})(1\text{ s}) + \frac{4\sqrt{2}((12\text{ m}\cdot\text{s}^{-1})^{3/2}}{3((6\text{ m}\cdot\text{s}^{-3}))^{1/2}} = 28\text{ m}\,.$$

b) Because the bicycle is traveling at a constant speed with an initial position $x_{b0} = -17\text{ m}$, the position of the bicycle is given by $x_b(t) = -17\text{ m} + v_b t$. The bicycle and car intersect at time $t_2 = 3\text{ s}$, where $x_b(t_2) = x_c(t_2)$. Therefore $-17\text{ m} + v_b(3\text{ s}) = 28\text{ m}$. So the speed of the bicycle is $v_b = 15\text{ m}\cdot\text{s}^{-1}$.

Chapter 5 Two Dimensional Kinematics

Chapter 5 Two Dimensional Kinematics

> *Where was the chap I saw in the picture somewhere? Ah yes, in the dead sea floating on his back, reading a book with a parasol open. Couldn't sink if you tried: so thick with salt. Because the weight of the water, no, the weight of the body in the water is equal to the weight of the what? Or is it the volume equal to the weight? It's a law something like that. Vance in High school cracking his fingerjoints, teaching. The college curriculum. Cracking curriculum. What is weight really when you say weight? Thirtytwo feet per second per second. Law of falling bodies: per second per second. They all fall to the ground. The earth. It's the force of gravity of the earth is the weight.* [1]
>
> *James Joyce*

5.1 Introduction to the Vector Description of Motion in Two Dimensions

We have introduced the concepts of position, velocity and acceleration to describe motion in one dimension; however we live in a multidimensional universe. In order to explore and describe motion in more than one dimension, we shall study the motion of a projectile in two-dimension moving under the action of uniform gravitation.

We extend our definitions of position, velocity, and acceleration for an object that moves in two dimensions (in a plane) by treating each direction independently, which we can do with vector quantities by resolving each of these quantities into components. For example, our definition of velocity as the derivative of position holds for each component separately. In Cartesian coordinates, the position vector $\vec{\mathbf{r}}(t)$ with respect to some choice of origin for the object at time t is given by

$$\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}}. \tag{5.1.1}$$

The velocity vector $\vec{\mathbf{v}}(t)$ at time t is the derivative of the position vector,

$$\vec{\mathbf{v}}(t) = \frac{dx(t)}{dt}\,\hat{\mathbf{i}} + \frac{dy(t)}{dt}\,\hat{\mathbf{j}} \equiv v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}}, \tag{5.1.2}$$

where $v_x(t) \equiv dx(t)/dt$ and $v_y(t) \equiv dy(t)/dt$ denote the x- and y-components of the velocity respectively.

The acceleration vector $\vec{\mathbf{a}}(t)$ is defined in a similar fashion as the derivative of the velocity vector,

---

[1] James Joyce, *Ulysses*, The Corrected Text edited by Hans Walter Gabler with Wolfhard Steppe and Claus Melchior, Random House, New York.

$$\vec{\mathbf{a}}(t) = \frac{dv_x(t)}{dt}\,\hat{\mathbf{i}} + \frac{dv_y(t)}{dt}\,\hat{\mathbf{j}} \equiv a_x(t)\,\hat{\mathbf{i}} + a_y(t)\,\hat{\mathbf{j}}, \tag{5.1.3}$$

where $a_x(t) \equiv dv_x(t)/dt$ and $a_y(t) \equiv dv_y(t)/dt$ denote the x- and y-components of the acceleration.

5.2 Projectile Motion

Consider the motion of a body that is released at time $t = 0$ with an initial velocity $\vec{\mathbf{v}}_0$. Two paths are shown in Figure 5.1.

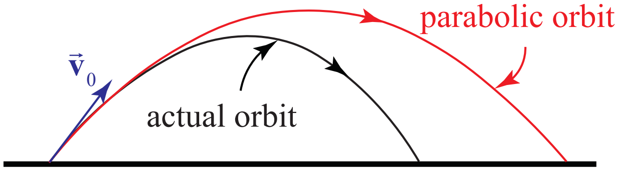

Figure 5.1 Actual orbit accounting for air resistance and parabolic orbit of a projectile

The dotted path represents a *parabolic* trajectory and the solid path represents the actual trajectory. The difference between the two paths is due to air resistance acting on the object, $\vec{\mathbf{F}}^{air} = -bv^2\hat{\mathbf{v}}$, where $\hat{\mathbf{v}}$ is a unit vector in the direction of the velocity. (For the orbits shown in Figure 5.1, $b = 0.01\,\mathrm{N \cdot s^2 \cdot m^{-2}}$, $\left|\vec{\mathbf{v}}_0\right| = 30.0\,\mathrm{m \cdot s}$, the initial launch angle with respect to the horizontal $\theta_0 = 21^\circ$, and the actual horizontal distance traveled is 71.7% of the projectile orbit.). There are other factors that can influence the path of motion; a rotating body or a special shape can alter the flow of air around the body, which may induce a curved motion or lift like the flight of a baseball or golf ball. We shall begin our analysis by neglecting all interactions except the gravitational interaction.

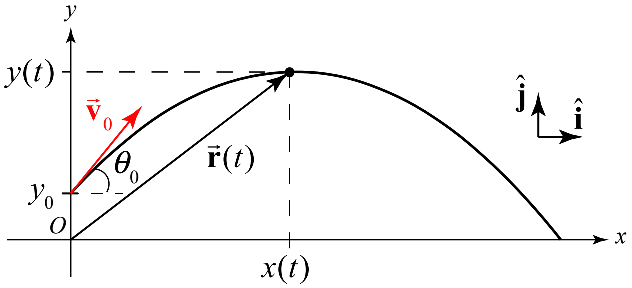

Figure 5.2 A coordinate sketch for parabolic motion.

Choose coordinates with the positive *y*-axis in the upward vertical direction and the positive *x*-axis in the horizontal direction in the direction that the object is moving horizontally. Choose the origin at the ground immediately below the point the object is released. Figure 5.2 shows our coordinate system with the position of the object $\vec{\mathbf{r}}(t)$ at time t, the initial velocity $\vec{\mathbf{v}}_0$, and the initial angle θ_0 with respect to the horizontal, and the coordinate functions $x(t)$ and $y(t)$.

Initial Conditions:

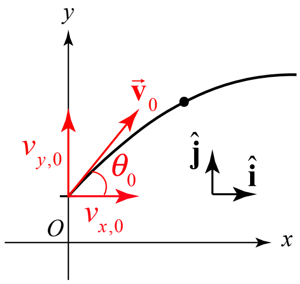

Figure 5.3 A vector decomposition of the initial velocity

Decompose the initial velocity vector into its components:

$$\vec{\mathbf{v}}_0 = v_{x,0}\,\hat{\mathbf{i}} + v_{y,0}\,\hat{\mathbf{j}}. \tag{5.1.4}$$

The vector decomposition for the initial velocity is shown in Figure 5.3. Often the description of the flight of a projectile includes the statement, "a body is projected with an initial speed v_0 at an angle θ_0 with respect to the horizontal." The components of the initial velocity can be expressed in terms of the initial speed and angle according to

$$v_{x,0} = v_0 \cos\theta_0 \,, \tag{5.1.5}$$

$$v_{y,0} = v_0 \sin\theta_0 \,. \tag{5.1.6}$$

Because the initial speed is the magnitude of the initial velocity, we have that

$$v_0 = (v_{x,0}^2 + v_{y,0}^2)^{1/2} \,. \tag{5.1.7}$$

The angle θ_0 is related to the components of the initial velocity by

$$\theta_0 = \tan^{-1}(v_{y,0} / v_{x,0}) \,. \tag{5.1.8}$$

Equation (5.1.8) will give two values for the angle θ_0, so care must be taken to choose the correct physical value. The initial position vector generally is given by

$$\vec{\mathbf{r}}_0 = x_0\,\hat{\mathbf{i}} + y_0\,\hat{\mathbf{j}}. \tag{5.1.9}$$

Note that the trajectory in Figure 5.3 has $x_0 = 0$, but this will not always be the case.

Force Diagram:

We begin by neglecting all forces other than the gravitational interaction between the object and the earth. This force acts downward with magnitude mg, where m is the mass of the object and $g = 9.8\ \mathrm{m \cdot s^{-2}}$. Figure 5.4 shows the force diagram on the object.

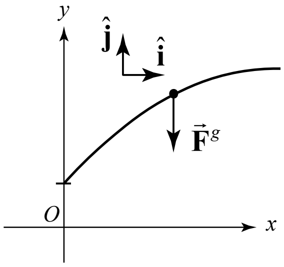

Figure 5.4 Free-body force diagram on the object with the action of gravity

The vector decomposition of the force is

$$\vec{\mathbf{F}}^g = -mg\,\hat{\mathbf{j}}. \tag{5.1.10}$$

Equations of Motions:

The force diagram reminds us that the force is acting in the y-direction. Newton's Second Law states that the sum of the force, $\vec{\mathbf{F}}^{\text{total}}$, acting on the object is equal to the product of the mass m and the acceleration vector $\vec{\mathbf{a}}$,

$$\vec{\mathbf{F}}^{\text{total}} = m\vec{\mathbf{a}}. \tag{5.1.11}$$

Because we are modeling the motion with only one force, we have that $\vec{\mathbf{F}}^{\text{total}} = \vec{\mathbf{F}}^g$. This is a vector equation; the components are equated separately:

$$-mg = ma_y, \tag{5.1.12}$$

$$0 = ma_x. \tag{5.1.13}$$

Therefore the y-component of the acceleration is

$$a_y = -g. \tag{5.1.14}$$

We see that the acceleration is a constant and is independent of the mass of the object. Notice that $a_y < 0$. This is because we chose our positive y-direction to point upwards. The sign of the y-component of acceleration is determined by how we choose our coordinate system. Because there are no horizontal forces acting on the object, we conclude that the acceleration in the horizontal direction is also zero

$$a_x = 0\,. \tag{5.1.15}$$

Therefore the x-component of the velocity remains unchanged throughout the flight of the object.

The acceleration in the vertical direction is constant for all bodies near the surface of the Earth, independent of the mass of the object, thus confirming Galileo's Law of Free Falling Bodies. Notice that the equation of motion (Equation (5.1.14)) generalizes the experimental observation that objects fall with constant acceleration. Our statement about the acceleration of objects near the surface of Earth depends on our model force law Eq. (5.1.10), and if subsequent observations show the acceleration is not constant then we either must include additional forces (for example, air resistance), or modify the force law (for objects that are no longer near the surface of Earth, or consider that Earth is a non-symmetric non-uniform body), or take into account the rotational motion of the Earth.

We can now integrate the equation of motions (Eqs. (5.1.14) and (5.1.15)) separately for the x- and y- directions to find expressions for the x- and y-components of velocity and position:

$$v_x(t) - v_{x,0} = \int_{t'=0}^{t'=t} a_x(t')\,dt' = 0 \Rightarrow v_x(t) = v_{x,0}$$

$$x(t) - x_0 = \int_{t'=0}^{t'=t} v_x(t')\,dt' = \int_{t'=0}^{t'=t} v_{x,0}\,dt' = v_{x,0}t \Rightarrow x(t) = x_0 + v_{x,0}t$$

$$v_y(t) - v_{y,0} = \int_{t'=0}^{t'=t} a_y(t')\,dt' = -\int_{t'=0}^{t'=t} g\,dt' = -gt \Rightarrow v_y(t) = v_{y,0} - gt$$

$$y(t) - y_0 = \int_{t'=0}^{t'=t} v_y(t')\,dt' = \int_{t'=0}^{t'=t} (v_{y,0} - gt)\,dt' = v_{y,0}t - (1/2)gt^2 \Rightarrow y(t) = y_0 + v_{y,0}t - (1/2)gt^2.$$

The complete set of vector equations for position and velocity for each independent direction of motion are given by

$$\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} = (x_0 + v_{x,0}t)\,\hat{\mathbf{i}} + (y_0 + v_{y,0}\,t + (1/2)a_y\,t^2)\,\hat{\mathbf{j}}\,, \tag{5.1.16}$$

$$\vec{\mathbf{v}}(t) = v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}} = v_{x,0}\;\hat{\mathbf{i}} + (v_{y,0} + a_y\,t)\,\hat{\mathbf{j}}\,, \tag{5.1.17}$$

$$\vec{\mathbf{a}}(t) = a_x(t)\,\hat{\mathbf{i}} + a_y(t)\,\hat{\mathbf{j}} = a_y\;\hat{\mathbf{j}}\,. \tag{5.1.18}$$

Example 5.1 Time of Flight and Maximum Height of a Projectile

A person throws a stone at an initial angle $\theta_0 = 45^\circ$ from the horizontal with an initial speed of $v_0 = 20\ \text{m} \cdot \text{s}^{-1}$. The point of release of the stone is at a height $d = 2\ \text{m}$ above the ground. You may neglect air resistance. a) How long does it take the stone to reach the highest point of its trajectory? b) What was the maximum vertical displacement of the stone? Ignore air resistance.

Solution: Choose the origin on the ground directly underneath the point where the stone is released. We choose the positive y-axis in the upward vertical direction and the positive x-axis in the horizontal direction in the direction that the object is moving horizontally. Set $t = 0$ the instant the stone is released. At $t = 0$ the initial conditions are then $x_0 = 0$ and $y_0 = d$. The initial x- and y-components of the velocity are given by Eqs. (5.1.5) and (5.1.6).

At time t the stone has coordinates $(x(t), y(t))$. These coordinate functions are shown in Figure 5.5.

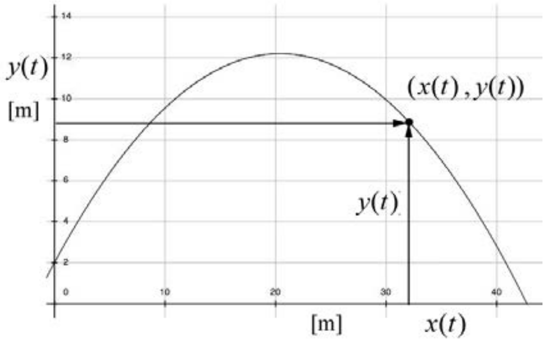

Figure 5.5: Coordinate functions for stone

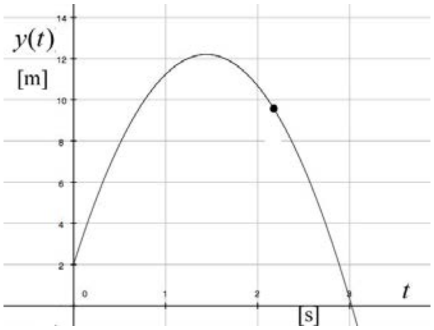

Figure 5.6 Plot of the y-component of the position as a function of time

The slope of this graph at any time t yields the instantaneous y-component of the velocity $v_y(t)$ at that time t. Figure 5.5 is a plot of $y(t)$ vs. $x(t)$ and Figure 5.6 is a plot of $y(t)$ vs. t. There are several important things to notice about Figures 5.5 and 5.6. The first point is that the abscissa axes are different in both figures. The second thing to notice is that at $t = 0$, the slope of the graph in Figure 5.5 is equal to

$$\left.\frac{dy}{dx}\right|_{t=0} = \left.\left(\frac{dy/dt}{dx/dt}\right)\right|_{t=0} = \frac{v_{y,0}}{v_{x,0}} = \tan\theta_0 \ , \tag{5.1.19}$$

while at $t = 0$ the slope of the graph in Figure 5.6 is equal to

$$\left.\frac{dy}{dt}\right|_{t=0} = v_{y,0} \ . \tag{5.1.20}$$

The slope of this graph in Figure 5.6 at any time t yields the instantaneous y-component of the velocity $v_y(t)$ at that time t. Let $t = t_1$ correspond to the instant the stone is at its maximal vertical position, the highest point in the flight. The final thing to notice about Figure 5.6 is that at $t = t_1$ the slope is zero or $v_y(t = t_1) = 0$. Therefore

$$v_y(t_1) = v_0 \sin\theta_0 - g t_1 = 0 \ . \tag{5.1.21}$$

Solving Eq. (5.1.21) for t_1 yields,

$$t_1 = \frac{v_0 \sin\theta_0}{g} = \frac{(20 \text{ m}\cdot\text{s}^{-1})\sin(45^\circ)}{9.8 \ \text{m}\cdot\text{s}^{-2}} = 1.44 \text{ s} . \tag{5.1.22}$$

The graph in Figure 5.7 shows a plot of $v_y(t)$ as a function of time. Notice that at $t = 0$ the intercept is positive indicting that $v_{y,0}$ is positive which means that the stone was thrown upwards. The y-component of the velocity changes sign at $t = t_1$ indicating that the stone is reversing its direction and starting to move downwards.

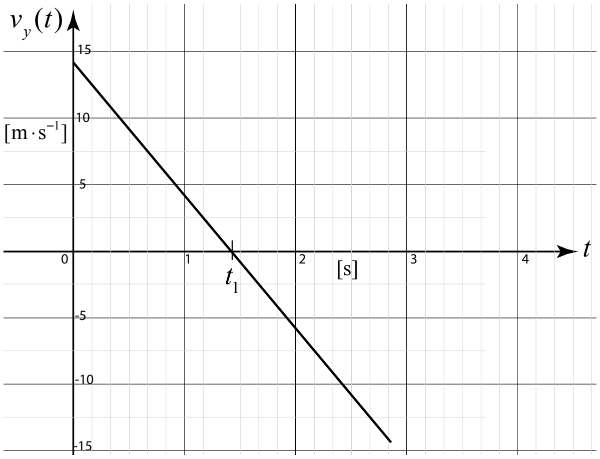

Figure 5.7 y-component of the velocity as a function of time

We now substitute the expression for $t = t_{top}$ (Eq. (5.1.22)) into the y-component of the position in Eq. (5.1.16) to find the maximal height of the stone above the ground

$$\begin{aligned} y(t = t_{top}) &= d + v_0 \sin\theta_0 \frac{v_0 \sin\theta_0}{g} - \frac{1}{2} g \left(\frac{v_0 \sin\theta_0}{g} \right)^2 \\ &= d + \frac{v_0{}^2 \sin^2\theta_0}{2g} = 2\text{ m} + \frac{(20\text{ m}\cdot\text{s}^{-1})^2 \sin^2(45^\circ)}{2(9.8\ \text{ m}\cdot\text{s}^{-2})} = 12.2\text{ m} \end{aligned} , \qquad (5.1.23)$$

5.2.1 Orbit equation

So far our description of the motion has emphasized the independence of the spatial dimensions, treating all of the kinematic quantities as functions of time. We shall now eliminate time from our equation and find the *orbit equation* of the body undergoing projectile motion. We begin with the x-component of the position in Eq. (5.1.16),

$$x(t) = x_0 + v_{x,0} t \qquad (5.1.24)$$

and solve Equation (5.1.24) for time t as a function of $x(t)$,

$$t = \frac{x(t) - x_0}{v_{x,0}}. \qquad (5.1.25)$$

The y-component of the position in Eq. (5.1.16) is given by

$$y(t) = y_0 + v_{y,0} t - \frac{1}{2} g t^2. \qquad (5.1.26)$$

We then substitute Eq. (5.1.25) into Eq. (5.1.26) yielding

$$y(t) = y_0 + v_{y,0} \left(\frac{x(t) - x_0}{v_{x,0}} \right) - \frac{1}{2} g \left(\frac{x(t) - x_0}{v_{x,0}} \right)^2. \qquad (5.1.27)$$

A little algebraic simplification yields the equation for a parabola:

$$y(t) = -\frac{1}{2} \frac{g}{v_{x,0}^2} x(t)^2 + \left(\frac{g\, x_0}{v_{x,0}^2} + \frac{v_{y,0}}{v_{x,0}} \right) x(t) - \frac{v_{y,0}}{v_{x,0}} x_0 - \frac{1}{2} \frac{g}{v_{x,0}^2} x_0^2 + y_0. \qquad (5.1.28)$$

The graph of $y(t)$ as a function of $x(t)$ is shown in Figure 5.8.

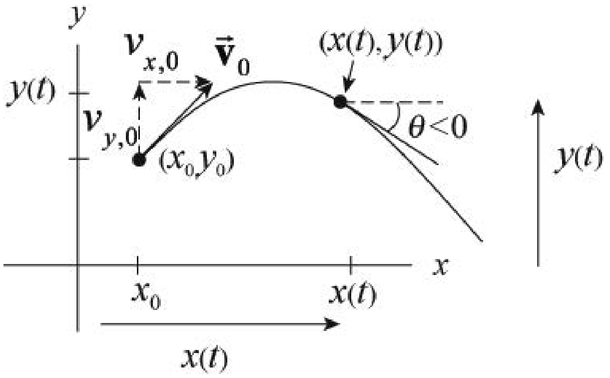

Figure 5.8 The parabolic orbit

The velocity vector is given by

$$\vec{\mathbf{v}}(t) = \frac{dx(t)}{dt}\,\hat{\mathbf{i}} + \frac{dy(t)}{dt}\,\hat{\mathbf{j}} \equiv v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}}. \qquad (5.1.29)$$

The direction of the velocity vector at a point $(x(t), y(t))$ can be determined from the components. Let θ be the angle that the velocity vector forms with respect to the positive x-axis. Then

$$\theta = \tan^{-1}\left(\frac{v_y(t)}{v_x(t)}\right) = \tan^{-1}\left(\frac{dy/dt}{dx/dt}\right) = \tan^{-1}\left(\frac{dy}{dx}\right). \qquad (5.1.30)$$

Differentiating Eq. (5.1.28) with respect to x yields

$$\frac{dy}{dx} = -\frac{g}{v_{x,0}^2}x + \left(\frac{g\,x_0}{v_{x,0}^2} + \frac{v_{y,0}}{v_{x,0}}\right). \qquad (5.1.31)$$

The direction of the velocity vector at a point $(x(t), y(t))$ is therefore

$$\theta = \tan^{-1}\left(-\frac{g}{v_{x,0}^2}x + \left(\frac{g\,x_0}{v_{x,0}^2} + \frac{v_{y,0}}{v_{x,0}}\right)\right). \qquad (5.1.32)$$

Although we can determine the angle of the velocity, we cannot determine how fast the body moves along the parabolic orbit from our graph of $y(x)$; the magnitude of the velocity cannot be determined from information about the tangent line.

If we choose our origin at the initial position of the body at $t = 0$, then $x_0 = 0$ and $y_0 = 0$. Our orbit equation, Equation (5.1.28) can now be simplified to

$$y(t) = -\frac{1}{2}\frac{g}{v_{x,0}^2}x(t)^2 + \frac{v_{y,0}}{v_{x,0}}x(t)\,. \tag{5.1.33}$$

Example 5.2 Hitting the Bucket

A person is holding a pail while standing on a ladder. The person releases the pail from rest at a height h_1 above the ground. A second person, standing a horizontal distance s from the pail, aims and throws a ball the instant the pail is released in order to hit the pail. The person releases the ball at a height h_2 above the ground, with an initial speed v_0, and at an angle θ_0 with respect to the horizontal. Assume that v_0 is large enough so that the stone will at least travel a horizontal distance s before it hits the ground. You may ignore air resistance.

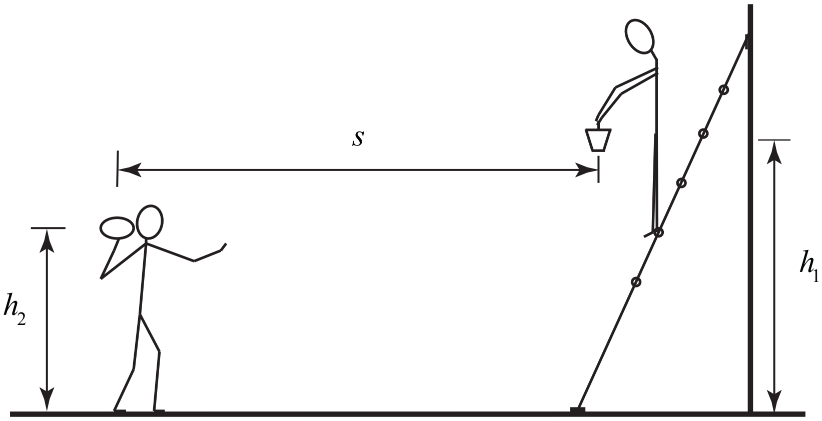

Figure 5.9: Example 5.2

a) Find an expression for the angle θ_0 that the person aims the ball in order to hit the pail. Does the answer depend on the initial velocity?

b) Find an expression for the time of collision as a function of the initial speed of the ball v_0, and the quantities h_1, h_2, and s.

c) Find an expression for the height above the ground where the collision occurred as a function of the initial speed of the ball v_0, and the quantities h_1, h_2, and s.

Solution:

There are two objects involved in this problem. Each object is undergoing free fall, so there is only one stage of motion for each object. The pail is undergoing one-dimensional motion. The ball is undergoing two-dimensional motion. The parameters h_1, h_2, v_0, and s are unspecified, so our answers will be functions of those quantities. Figure 5.9 shows a sketch of the motion of all the bodies in this problem.

Choose an origin on the ground directly underneath the point where the ball is released, upwards for the positive y-direction and towards the pail for the positive x-direction. Choose position coordinates for the pail as follows. The horizontal coordinate is constant and given by $x_1 = s$. The vertical coordinate represents the height above the ground and is denoted by $y_1(t)$. The ball has coordinates $(x_2(t), y_2(t))$. We show these coordinates in the Figure 5.10.

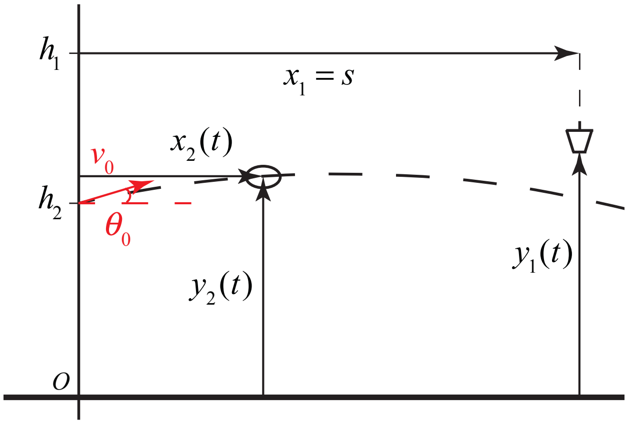

Figure 5.10: Coordinate System

The pail undergoes constant acceleration $a_{1,y} = -g$ in the vertical direction and the ball undergoes uniform motion in the horizontal direction and constant acceleration in the vertical direction, with $a_{2,x} = 0$ and $a_{2,y} = -g$.

The initial conditions for the pail are $(v_{1,0})_y = 0$, $x_{1,0} = s$, $y_{1,0} = h_1$. The equations for position and velocity of the pail simplify to

$$y_1(t) = h_1 - \frac{1}{2}gt^2 \tag{5.1.34}$$

$$v_{y,1}(t) = -gt\,. \tag{5.1.35}$$

The initial position is given by $x_{2,0} = 0, y_{2,0} = h_2$. The components of the initial velocity are given by $(v_{2,0})_y = v_0 \sin(\theta_0)$ and $(v_{2,0})_x = v_0 \cos(\theta_0)$, where v_0 is the magnitude of the initial velocity and θ_0 is the initial angle with respect to the horizontal. The equations for the position and velocity of the ball simplify to

$$x_2(t) = v_0 \cos(\theta_0)t \tag{5.1.36}$$

$$v_{2,x}(t) = v_0 \cos(\theta_0) \tag{5.1.37}$$

$$y_2(t) = h_2 + v_0 \sin(\theta_0)t - \frac{1}{2}gt^2 \tag{5.1.38}$$

$$v_{2,y}(t) = v_0 \sin(\theta_0) - gt\,. \tag{5.1.39}$$

Note that the quantities h_1, h_2, v_0, and s should be treated as known quantities although no numerical values were given. There are six independent equations with 8 as yet unspecified quantities $y_1(t)$, t, $y_2(t)$, $x_2(t)$, $v_{1,y}(t)$, $v_{2,y}(t)$, $v_{2,x}(t)$, and θ_0.

So we need two more conditions, in order to find expressions for the initial angle, θ_0, the time of collision, t_a, and the spatial location of the collision point specified by $y_1(t_a)$ or $y_2(t_a)$. At the collision time $t = t_a$, the collision occurs when the two balls are located at the same position. Therefore

$$y_1(t_a) = y_2(t_a) \tag{5.1.40}$$

$$x_2(t_a) = x_1 = s\,. \tag{5.1.41}$$

We shall now apply these conditions that must be satisfied in order for the ball to hit the pail.

$$h_1 - \frac{1}{2}g{t_a}^2 = h_2 + v_0 \sin\left(\theta_0\right)t_a - \frac{1}{2}g{t_a}^2 \tag{5.1.42}$$

$$s = v_0 \cos(\theta_0)t_a\,. \tag{5.1.43}$$

Eq. (5.1.42) simplifies to

$$v_0 \sin(\theta_0)t_a = h_1 - h_2 \tag{5.1.44}$$

Dividing Eq. (5.1.44) by Eq. (5.1.43) yields

$$\frac{v_0 \sin(\theta_0)t_a}{v_0 \cos(\theta_0)t_a} = \tan(\theta_0) = \frac{h_1 - h_2}{s_2}\,. \tag{5.1.45}$$

So the initial angle θ_0 is independent of v_0, and is given by

$$\theta_0 = \tan^{-1}((h_1 - h_2)/s)\,. \tag{5.1.46}$$

From the Figure 5.11 we can see that $\tan(\theta_0) = (h_1 - h_2)/s$ implies that the second person aims the ball at the initial position of the pail.

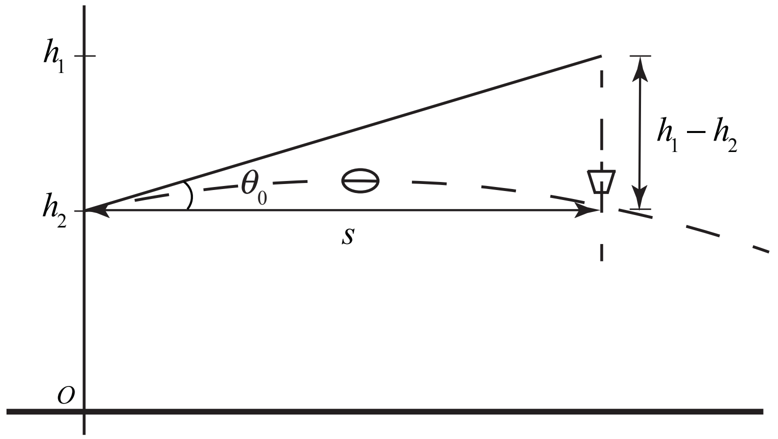

Figure 5.11: Geometry of collision

In order to find the time that the ball collides with the pail, we begin by squaring both Eqs. (5.1.44) and (5.1.43), then utilize the trigonometric identity $\sin^2(\theta_0)+\cos^2(\theta_0)=1$. Our squared equations become

$$v_0{}^2\sin^2(\theta_0)t_a{}^2 = (h_1 - h_2)^2 \tag{5.1.47}$$

$$v_0{}^2\cos^2(\theta_0)t_a{}^2 = s^2 . \tag{5.1.48}$$

Adding these equations together and using the identity $\sin^2(\theta_0)+\cos^2(\theta_0)=1$ and taking square roots yields

$$v_0 t_a = (s^2 + (h_1 - h_2)^2)^{1/2} . \tag{5.1.49}$$

We can solve Eq. (5.1.49) for the time of collision

$$t_a = \frac{1}{v_0}(s^2 + (h_1 - h_2)^2)^{1/2} . \tag{5.1.50}$$

We can now use the y-coordinate function of either the ball or the pail at $t = t_a$ to find the height that the ball collides with the pail. Because the pail had no initial y-component of the velocity, it's easier to use the condition for the pail,

$$y_1(t_a) = h_1 - \frac{g(s^2 + (h_1 - h_2)^2)}{2v_0{}^2} . \tag{5.1.51}$$

Comments:

(1) Eqs. (5.1.49) and (5.1.50) can be arrived at in a very direct way. Suppose we analyze the motion in a reference frame that is accelerating downward with $\vec{\mathbf{A}} = -g\,\hat{\mathbf{j}}$. In that

reference frame both the pail and the stone are not accelerating; the pail is at rest and the stone is travelling with speed v_0, at an angle θ_0. Therefore in order to hit the stationary pail, the stone must be thrown at the angle given by Eq. (5.1.46) and the time that it takes to hit the stone is just given by distance traveled divided by speed, Eq. (5.1.50).

Chapter 6 Circular Motion

Chapter 6 Central Motion

> *And the seasons they go round and round*
> *And the painted ponies go up and down*
> *We're captive on the carousel of time*
> *We can't return we can only look*
> *Behind from where we came*
> *And go round and round and round*
> *In the circle game* [1]
>
> *Joni Mitchell*

6.1 Introduction

We shall now investigate a special class of motions, motion in a plane about a central point, a motion we shall refer to as central motion, the most outstanding case of which is circular motion. Special cases often dominate our study of physics, and circular motion about a central point is certainly no exception. There are many instances of central motion about a point; a bicycle rider on a circular track, a ball spun around by a string, and the rotation of a spinning wheel are just a few examples. Various planetary models described the motion of planets in circles before any understanding of gravitation. The motion of the moon around the earth is nearly circular. The motions of the planets around the sun are nearly circular. Our sun moves in nearly a circular orbit about the center of our galaxy, 50,000 light years from a massive black hole at the center of the galaxy. When Newton solved the two-body under a gravitational central force, he discovered that the orbits can be circular, elliptical, parabolic or hyperbolic. All of these orbits still display central force motion about the center of mass of the two-body system. Another example of central force motion is the scattering of particles by a Coulombic central force, for example Rutherford scattering of an alpha particle (two protons and two neutrons bound together into a particle identical to a helium nucleus) against an atomic nucleus such as a gold nucleus.

We shall begin by describing the kinematics of circular motion, the position, velocity, and acceleration, as a special case of two-dimensional motion. We will see that unlike linear motion, where velocity and acceleration are directed along the line of motion, in circular motion the direction of velocity is always tangent to the circle. This means that as the object moves in a circle, the direction of the velocity is always changing. When we examine this motion, we shall see that the direction of the change of the velocity is towards the center of the circle. This means that there is a non-zero component of the acceleration directed radially inward, which is called the *centripetal acceleration*. If our object is increasing its speed or slowing down, there is also a non-zero *tangential acceleration* in the direction of motion. But when the object is moving at a constant speed in a circle then only the centripetal acceleration is non-zero.

---

[1] Joni Mitchell, *The Circle Game,* Siquomb Publishing Company.

In 1666, twenty years before Newton published his *Principia*, he realized that the moon is always "falling" towards the center of the earth; otherwise, by the First Law, it would continue in some linear trajectory rather than follow a circular orbit. Therefore there must be a *centripetal force*, a radial force pointing inward, producing this centripetal acceleration.

In all of these instances, when an object is constrained to move in a circle, there must exist a force $\vec{\mathbf{F}}$ acting on the object directed towards the center. Because Newton's Second Law is a vector equality, the radial component of the Second Law is

$$F_r = m a_r . \tag{6.1.1}$$

6.2 Circular Motion: Velocity and Angular Velocity

We begin our description of circular motion by choosing polar coordinates. In Figure 6.1 we sketch the position vector $\vec{\mathbf{r}}(t)$ of the object moving in a circular orbit of radius r.

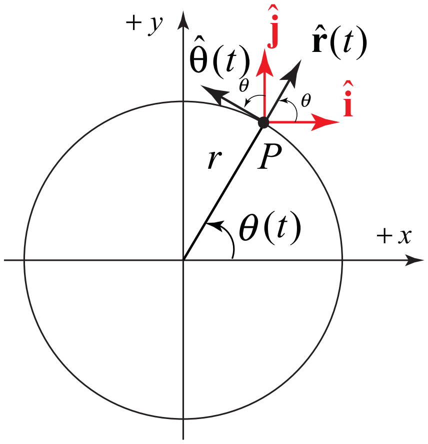

Figure 6.1 A circular orbit with unit vectors.

At time t, the particle is located at the point P with coordinates $(r, \theta(t))$ and position vector given by

$$\vec{\mathbf{r}}(t) = r\,\hat{\mathbf{r}}(t) . \tag{6.2.1}$$

At the point P, consider two sets of unit vectors ($\hat{\mathbf{r}}(t)$, $\hat{\boldsymbol{\theta}}(t)$) and ($\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$), as shown in Figure 6.1. The vector decomposition expression for $\hat{\mathbf{r}}(t)$ and $\hat{\boldsymbol{\theta}}(t)$ in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ is given by

$$\hat{\mathbf{r}}(t) = \cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}} , \tag{6.2.2}$$

$$\hat{\boldsymbol{\theta}}(t) = -\sin\theta(t)\,\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}} . \tag{6.2.3}$$

Before we calculate the velocity, we shall calculate the time derivatives of Eqs. (6.2.2) and (6.2.3). Let's first begin with $d\hat{\mathbf{r}}(t)/dt$:

$$\begin{aligned}\frac{d\hat{\mathbf{r}}(t)}{dt}&=\frac{d}{dt}(\cos\theta(t)\,\hat{\mathbf{i}}+\sin\theta(t)\,\hat{\mathbf{j}})=(-\sin\theta(t)\frac{d\theta(t)}{dt}\,\hat{\mathbf{i}}+\cos\theta(t)\frac{d\theta(t)}{dt}\,\hat{\mathbf{j}})\\&=\frac{d\theta(t)}{dt}(-\sin\theta(t)\,\hat{\mathbf{i}}+\cos\theta(t)\,\hat{\mathbf{j}})=\frac{d\theta(t)}{dt}\hat{\boldsymbol{\theta}}(t)\end{aligned}\;; \tag{6.2.4}$$

where we used the chain rule to calculate that

$$\frac{d}{dt}\cos\theta(t)=-\sin\theta(t)\frac{d\theta(t)}{dt}\;, \tag{6.2.5}$$

$$\frac{d}{dt}\sin\theta(t)=\cos\theta(t)\frac{d\theta(t)}{dt}\;. \tag{6.2.6}$$

The calculation for $d\hat{\boldsymbol{\theta}}(t)/dt$ is similar:

$$\begin{aligned}\frac{d\hat{\boldsymbol{\theta}}(t)}{dt}&=\frac{d}{dt}(-\sin\theta(t)\,\hat{\mathbf{i}}+\cos\theta(t\,\hat{\mathbf{j}})=(-\cos\theta(t)\frac{d\theta(t)}{dt}\,\hat{\mathbf{i}}-\sin(t)\frac{d\theta(t)}{dt}\,\hat{\mathbf{j}})\\&=\frac{d\theta(t)}{dt}(-\cos\theta(t)\,\hat{\mathbf{i}}-\sin\theta(t)\,\hat{\mathbf{j}})=-\frac{d\theta(t)}{dt}\hat{\mathbf{r}}(t)\end{aligned}\;. \tag{6.2.7}$$

The velocity vector is then

$$\vec{\mathbf{v}}(t)=\frac{d\vec{\mathbf{r}}(t)}{dt}=r\frac{d\hat{\mathbf{r}}}{dt}=r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}(t)=v_\theta\,\hat{\boldsymbol{\theta}}(t), \tag{6.2.8}$$

where the $\hat{\boldsymbol{\theta}}$-component of the velocity is given by

$$v_\theta=r\frac{d\theta}{dt}, \tag{6.2.9}$$

a quantity we shall refer to as the *tangential component of the velocity*. Denote the magnitude of the velocity by $v\equiv\left|\vec{\mathbf{v}}\right|$, The angular speed is the magnitude of the rate of change of angle with respect to time, which we denote by the Greek letter ω,

$$\omega\equiv\left|\frac{d\theta}{dt}\right|. \tag{6.2.10}$$

6.2.1 Geometric Derivation of the Velocity for Circular Motion

Consider a particle undergoing circular motion. At time t, the position of the particle is $\vec{\mathbf{r}}(t)$. During the time interval Δt, the particle moves to the position $\vec{\mathbf{r}}(t+\Delta t)$ with a displacement $\Delta\vec{\mathbf{r}}$.

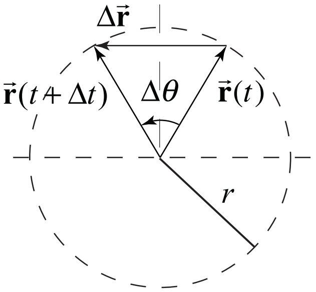

Figure 6.2 Displacement vector for circular motion

The magnitude of the displacement, $\left|\Delta\vec{\mathbf{r}}\right|$, is represented by the length of the horizontal vector $\Delta\vec{\mathbf{r}}$ joining the heads of the displacement vectors in Figure 6.2 and is given by

$$\left|\Delta\vec{\mathbf{r}}\right| = 2r\sin(\Delta\theta/2). \tag{6.2.11}$$

When the angle $\Delta\theta$ is small, we can approximate

$$\sin(\Delta\theta/2) \cong \Delta\theta/2. \tag{6.2.12}$$

This is called the *small angle approximation*, where the angle $\Delta\theta$ (and hence $\Delta\theta/2$) is measured in radians. This fact follows from an infinite power series expansion for the sine function given by

$$\sin\left(\frac{\Delta\theta}{2}\right) = \frac{\Delta\theta}{2} - \frac{1}{3!}\left(\frac{\Delta\theta}{2}\right)^3 + \frac{1}{5!}\left(\frac{\Delta\theta}{2}\right)^5 - \cdots. \tag{6.2.13}$$

When the angle $\Delta\theta/2$ is small, only the first term in the infinite series contributes, as successive terms in the expansion become much smaller. For example, when $\Delta\theta/2 = \pi/30 \cong 0.1$, corresponding to 6°, $(\Delta\theta/2)^3/3! \cong 1.9\times10^{-4}$; this term in the power series is three orders of magnitude smaller than the first and can be safely ignored for small angles.

Using the small angle approximation, the magnitude of the displacement is

$$\left|\Delta \vec{\mathbf{r}}\right| \cong r\ \Delta\theta\ . \tag{6.2.14}$$

This result should not be too surprising since in the limit as $\Delta\theta$ approaches zero, the length of the chord approaches the arc length $r\,\Delta\theta$.

The magnitude of the velocity, v, is proportional to the rate of change of the magnitude of the angle with respect to time,

$$v \equiv \left|\vec{\mathbf{v}}(t)\right| = \lim_{\Delta t\to 0}\frac{\left|\Delta\vec{\mathbf{r}}\right|}{\Delta t} = \lim_{\Delta t\to 0}\frac{r\left|\Delta\theta\right|}{\Delta t} = r\lim_{\Delta t\to 0}\frac{\left|\Delta\theta\right|}{\Delta t} = r\left|\frac{d\theta}{dt}\right| = r\omega\ . \tag{6.2.15}$$

The direction of the velocity can be determined by considering that in the limit as $\Delta t \to 0$ (note that $\Delta\theta \to 0$), the direction of the displacement $\Delta\vec{\mathbf{r}}$ approaches the direction of the tangent to the circle at the position of the particle at time t (Figure 6.3).

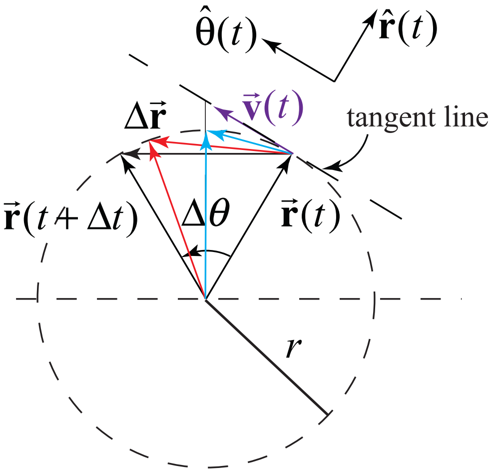

Figure 6.3 Direction of the displacement approaches the direction of the tangent line

Thus, in the limit $\Delta t \to 0$, $\Delta\vec{\mathbf{r}} \perp \vec{\mathbf{r}}$, and so the direction of the velocity $\vec{\mathbf{v}}(t)$ at time t is perpendicular to the position vector $\vec{\mathbf{r}}(t)$ and tangent to the circular orbit in the $+\hat{\boldsymbol{\theta}}$-direction for the case shown in Figure 6.3.

6.3 Circular Motion: Tangential and Radial Acceleration

When the motion of an object is described in polar coordinates, the acceleration has two components, the tangential component a_θ, and the radial component, a_r. We can write the acceleration vector as

$$\vec{\mathbf{a}} = a_r\ \hat{\mathbf{r}}(t) + a_\theta\ \hat{\boldsymbol{\theta}}(t)\ . \tag{6.3.1}$$

Keep in mind that as the object moves in a circle, the unit vectors $\hat{\mathbf{r}}(t)$ and $\hat{\boldsymbol{\theta}}(t)$ change direction and hence are not constant in time.

We will begin by calculating the tangential component of the acceleration for circular motion. Suppose that the tangential velocity $v_\theta = r\,d\theta / dt$ is changing in magnitude due to the presence of some tangential force; we shall now consider that $d\theta / dt$ is changing in time, (the magnitude of the velocity is changing in time). Recall that in polar coordinates the velocity vector Eq. (6.2.8) can be written as

$$\vec{\mathbf{v}}(t) = r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}(t) . \tag{6.3.2}$$

We now use the product rule to determine the acceleration.

$$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}(t)}{dt} = r\frac{d^2\theta(t)}{dt^2}\hat{\boldsymbol{\theta}}(t) + r\frac{d\theta(t)}{dt}\frac{d\hat{\boldsymbol{\theta}}(t)}{dt} . \tag{6.3.3}$$

Recall from Eq. (6.2.3) that $\hat{\boldsymbol{\theta}}(t) = -\sin\theta(t)\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}$. So we can rewrite Eq. (6.3.3) as

$$\vec{\mathbf{a}}(t) = r\frac{d^2\theta(t)}{dt^2}\hat{\boldsymbol{\theta}}(t) + r\frac{d\theta(t)}{dt}\frac{d}{dt}(-\sin\theta(t)\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}) . \tag{6.3.4}$$

We again use the chain rule (Eqs. (6.2.5) and (6.2.6)) and find that

$$\vec{\mathbf{a}}(t) = r\frac{d^2\theta(t)}{dt^2}\hat{\boldsymbol{\theta}}(t) + r\frac{d\theta(t)}{dt}\left(-\cos\theta(t)\frac{d\theta(t)}{dt}\hat{\mathbf{i}} - \sin\theta(t)\frac{d\theta(t)}{dt}\,\hat{\mathbf{j}}\right). \tag{6.3.5}$$

Recall that $\omega \equiv d\theta / dt$, and from Eq. (6.2.2), $\hat{\mathbf{r}}(t) = \cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}}$, therefore the acceleration becomes

$$\vec{\mathbf{a}}(t) = r\frac{d^2\theta(t)}{dt^2}\hat{\boldsymbol{\theta}}(t) - r\left(\frac{d\theta(t)}{dt}\right)^2\hat{\mathbf{r}}(t) . \tag{6.3.6}$$

The *tangential component of the acceleration* is then

$$a_\theta = r\frac{d^2\theta(t)}{dt^2} . \tag{6.3.7}$$

The *radial component of the acceleration* is given by

$$a_r = -r\left(\frac{d\theta(t)}{dt}\right)^2 = -r\omega^2 < 0 \ . \qquad (6.3.8)$$

Because $a_r < 0$, that radial vector component $\vec{\mathbf{a}}_r(t) = -r\omega^2\,\hat{\mathbf{r}}(t)$ is always directed towards the center of the circular orbit.

Example 6.1 Circular Motion Kinematics

A particle is moving in a circle of radius R. At $t = 0$, it is located on the x-axis. The angle the particle makes with the positive x-axis is given by $\theta(t) = At^3 - Bt$, where A and B are positive constants. Determine (a) the velocity vector, and (b) the acceleration vector. Express your answer in polar coordinates. At what time is the centripetal acceleration zero?

Solution:

The derivatives of the angle function $\theta(t) = At^3 - Bt$ are $d\theta / dt = 3At^2 - B$ and $d^2\theta / dt^2 = 6At$. Therefore the velocity vector is given by

$$\vec{\mathbf{v}}(t) = R\frac{d\theta(t)}{dt}\,\hat{\boldsymbol{\theta}}(t) = R(3At^2 - Bt)\hat{\boldsymbol{\theta}}(t) \,.$$

The acceleration is given by

$$\begin{aligned}\vec{\mathbf{a}}(t) &= R\frac{d^2\theta(t)}{dt^2}\,\hat{\boldsymbol{\theta}}(t) - R\left(\frac{d\theta(t)}{dt}\right)^2\hat{\mathbf{r}}(t) \\ &= R(6At)\,\hat{\boldsymbol{\theta}}(t) - R\left(3At^2 - B\right)^2\,\hat{\mathbf{r}}(t)\end{aligned} \,.$$

The centripetal acceleration is zero at time $t = t_1$ when

$$3At_1^2 - B = 0 \Rightarrow t_1 = \sqrt{B / 3A} \,.$$

6.4 Period and Frequency for Uniform Circular Motion

If the object is constrained to move in a circle and the total tangential force acting on the object is zero, $F_\theta^{\text{total}} = 0$, then (Newton's Second Law), the tangential acceleration is zero,

$$a_\theta = 0 \,. \qquad (6.4.1)$$

This means that the magnitude of the velocity (the speed) remains constant. This motion is known as *uniform circular motion*. The acceleration is then given by only the acceleration radial component vector

$$\vec{\mathbf{a}}_r(t) = -r\omega^2(t)\,\hat{\mathbf{r}}(t) \quad \text{uniform circular motion}\,. \tag{6.4.2}$$

Because the speed $v = r|\omega|$ is constant, the amount of time that the object takes to complete one circular orbit of radius r is also constant. This time interval, T, is called the *period***.** In one period the object travels a distance $s = vT$ equal to the circumference, $s = 2\pi r$; thus

$$s = 2\pi r = vT\,. \tag{6.4.3}$$

The period T is then given by

$$T = \frac{2\pi r}{v} = \frac{2\pi r}{r\omega} = \frac{2\pi}{\omega}\,. \tag{6.4.4}$$

The *frequency* f is defined to be the reciprocal of the period,

$$f = \frac{1}{T} = \frac{\omega}{2\pi}\,. \tag{6.4.5}$$

The SI unit of frequency is the inverse second, which is defined as the hertz, $\left[\mathrm{s}^{-1}\right] \equiv [\mathrm{Hz}]$.

The magnitude of the radial component of the acceleration can be expressed in several equivalent forms since both the magnitudes of the velocity and angular velocity are related by $v = r\omega$. Thus we have several alternative forms for the magnitude of the centripetal acceleration. The first is that in Equation (6.5.3). The second is in terms of the radius and the angular velocity,

$$|a_r| = r\,\omega^2\,. \tag{6.4.6}$$

The third form expresses the magnitude of the centripetal acceleration in terms of the speed and radius,

$$|a_r| = \frac{v^2}{r}. \tag{6.4.7}$$

Recall that the magnitude of the angular velocity is related to the frequency by $\omega = 2\pi f$, so we have a fourth alternate expression for the magnitude of the centripetal acceleration in terms of the radius and frequency,

$$|a_r| = 4\pi^2 r f^2\,. \tag{6.4.8}$$

A fifth form commonly encountered uses the fact that the frequency and period are related by $f = 1/T = \omega / 2\pi$. Thus we have the fourth expression for the centripetal acceleration in terms of radius and period,

$$|a_r| = \frac{4\pi^2 r}{T^2}. \tag{6.4.9}$$

Other forms, such as $4\pi^2 r^2 f / T$ or $2\pi r \omega f$, while valid, are uncommon.

Often we decide which expression to use based on information that describes the orbit. A convenient measure might be the orbit's radius. We may also independently know the period, or the frequency, or the angular velocity, or the speed. If we know one, we can calculate the other three but it is important to understand the meaning of each quantity.

6.4.1 Geometric Interpretation for Radial Acceleration for Uniform Circular Motion

An object traveling in a circular orbit is always accelerating towards the center. Any radial inward acceleration is called *centripetal acceleration*. Recall that the direction of the velocity is always tangent to the circle. Therefore the direction of the velocity is constantly changing because the object is moving in a circle, as can be seen in Figure 6.4. Because the velocity changes direction, the object has a nonzero acceleration.

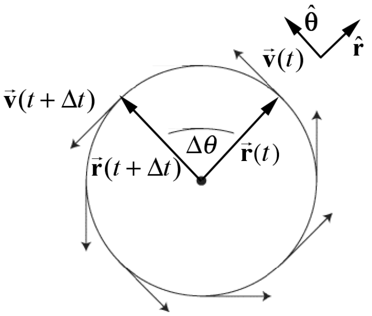

Figure 6.4 Direction of the velocity for circular motion.

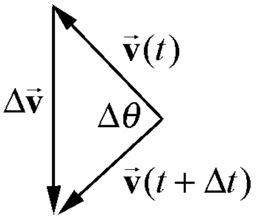

Figure 6.5 Change in velocity vector.

The calculation of the magnitude and direction of the acceleration is very similar to the calculation for the magnitude and direction of the velocity for circular motion, but the change in velocity vector, $\Delta\vec{\mathbf{v}}$, is more complicated to visualize. The change in velocity $\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}(t+\Delta t) - \vec{\mathbf{v}}(t)$ is depicted in Figure 6.5. The velocity vectors have been given a common point for the tails, so that the change in velocity, $\Delta\vec{\mathbf{v}}$, can be visualized. The length $\left|\Delta\vec{\mathbf{v}}\right|$ of the vertical vector can be calculated in exactly the same way as the displacement $\left|\Delta\vec{\mathbf{r}}\right|$. The magnitude of the change in velocity is

$$\left|\Delta\vec{\mathbf{v}}\right| = 2v\sin(\Delta\theta/2). \qquad (6.5.1)$$

We can use the small angle approximation $\sin\left(\Delta\theta/2\right) \cong \Delta\theta/2$ to approximate the magnitude of the change of velocity,

$$\left|\Delta\vec{\mathbf{v}}\right| \cong v\left|\Delta\theta\right|. \qquad (6.5.2)$$

The magnitude of the radial acceleration is given by

$$\left|a_r\right| = \lim_{\Delta t\to 0}\frac{\left|\Delta\vec{\mathbf{v}}\right|}{\Delta t} = \lim_{\Delta t\to 0}\frac{v\left|\Delta\theta\right|}{\Delta t} = v\lim_{\Delta t\to 0}\frac{\left|\Delta\theta\right|}{\Delta t} = v\left|\frac{d\theta}{dt}\right| = v\left|\omega\right|. \qquad (6.5.3)$$

The direction of the radial acceleration is determined by the same method as the direction of the velocity; in the limit $\Delta\theta \to 0$, $\Delta\vec{\mathbf{v}} \perp \vec{\mathbf{v}}$, and so the direction of the acceleration radial component vector $\vec{\mathbf{a}}_r(t)$ at time t is perpendicular to position vector $\vec{\mathbf{v}}(t)$ and directed inward, in the $-\hat{\mathbf{r}}$-direction.

6.5 Angular Velocity and Angular Acceleration

6.5.1. Angular Velocity

We shall always choose a right-handed cylindrical coordinate system. If the positive z-axis points up, then we choose θ to be increasing in the counterclockwise direction as shown in Figures 6.6.

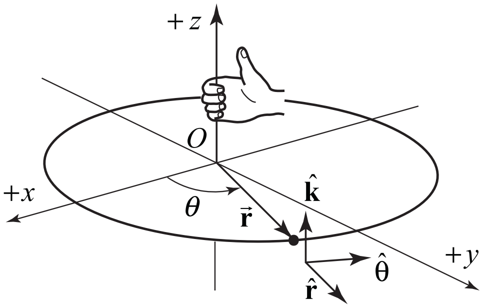

Figure 6.6 Right handed coordinate system

For a point object undergoing circular motion about the z-axis, the angular velocity vector $\vec{\boldsymbol{\omega}}$ is directed along the z-axis with z-component equal to the time derivative of the angle θ,

$$\vec{\boldsymbol{\omega}} = \frac{d\theta}{dt}\hat{\mathbf{k}} = \omega_z \hat{\mathbf{k}} . \tag{6.5.4}$$

The SI units of angular velocity are $[\text{rad} \cdot \text{s}^{-1}]$. Note that the angular speed is just the magnitude of the z-component of the angular velocity,

$$\omega \equiv \left|\omega_z\right| = \left|\frac{d\theta}{dt}\right| . \tag{6.5.5}$$

If the velocity of the object is in the $+\hat{\boldsymbol{\theta}}$-direction, (rotating in the counterclockwise direction in Figure 6.7(a)), then the z-component of the angular velocity is positive, $\omega_z = d\theta / dt > 0$. The angular velocity vector then points in the $+\hat{\mathbf{k}}$-direction as shown in Figure 6.7(a). If the velocity of the object is in the $-\hat{\boldsymbol{\theta}}$-direction, (rotating in the clockwise direction in Figure 6.7(b)), then the z-component of the angular velocity angular velocity is negative, $\omega_z = d\theta / dt < 0$. The angular velocity vector then points in the $-\hat{\mathbf{k}}$-direction as shown in Figure 6.7(b).

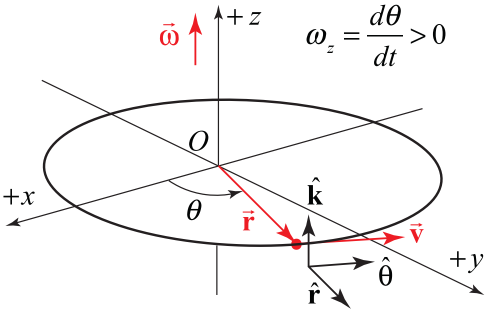

Figure 6.7(a) Angular velocity vector vector for motion with $d\theta / dt > 0$.

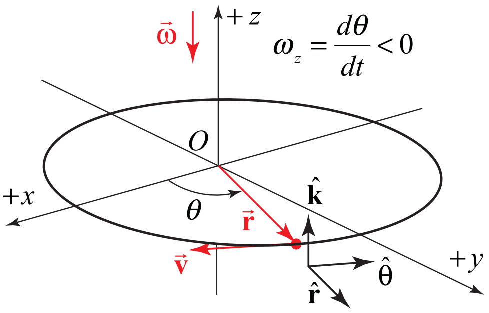

Figure 6.7(b) Angular velocity for motion with $d\theta / dt < 0$.

The velocity and angular velocity are related by

$$\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}} = \frac{d\theta}{dt}\hat{\mathbf{k}} \times r\hat{\mathbf{r}} = r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} . \tag{6.5.6}$$

Example 6.2 Angular Velocity

A particle is moving in a circle of radius R. At $t = 0$, it is located on the x-axis. The angle the particle makes with the positive x-axis is given by $\theta(t) = At - Bt^3$, where A and B are positive constants. Determine (a) the angular velocity vector, and (b) the velocity vector. Express your answer in polar coordinates. (c) At what time, $t = t_1$, is the angular velocity zero? (d) What is the direction of the angular velocity for (i) $t < t_1$, and (ii) $t > t_1$?

Solution: The derivative of $\theta(t) = At - Bt^3$ is

$$\frac{d\theta(t)}{dt} = A - 3Bt^2 .$$

Therefore the angular velocity vector is given by

$$\vec{\boldsymbol{\omega}}(t) = \frac{d\theta(t)}{dt}\hat{\mathbf{k}} = (A - 3Bt^2)\hat{\mathbf{k}} .$$

The velocity is given by

$$\vec{\mathbf{v}}(t) = R\frac{d\theta(t)}{dt}\hat{\boldsymbol{\theta}}(t) = R(A - 3Bt^2)\hat{\boldsymbol{\theta}}(t) .$$

The angular velocity is zero at time $t = t_1$ when

$$A - 3Bt_1^2 = 0 \Rightarrow t_1 = \sqrt{A / 3B} .$$

For $t < t_1$, $\dfrac{d\theta(t)}{dt} = A - 3Bt_1^2 > 0$ hence $\vec{\boldsymbol{\omega}}(t)$ points in the positive $\hat{\mathbf{k}}$-direction.

For $t > t_1$, $\dfrac{d\theta(t)}{dt} = A - 3Bt_1^2 < 0$ hence $\vec{\boldsymbol{\omega}}(t)$ points in the negative $\hat{\mathbf{k}}$-direction.

6.5.2 Angular Acceleration

In a similar fashion, for a point object undergoing circular motion about the fixed z-axis, the angular acceleration is defined as

$$\vec{\boldsymbol{\alpha}} = \frac{d^2\theta}{dt^2}\hat{\mathbf{k}} = \alpha_z\hat{\mathbf{k}} . \tag{6.5.7}$$

The SI units of angular acceleration are $[\text{rad}\cdot\text{s}^{-2}]$. The magnitude of the angular acceleration is denoted by the Greek symbol alpha,

$$\alpha \equiv \left|\vec{\boldsymbol{\alpha}}\right| = \left|\frac{d^2\theta}{dt^2}\right| . \tag{6.5.8}$$

There are four special cases to consider for the direction of the angular velocity. Let's first consider the two types of motion with $\vec{\boldsymbol{\alpha}}$ pointing in the $+\hat{\mathbf{k}}$-direction: (i) if the object is rotating counterclockwise and speeding up then both $d\theta / dt > 0$ and $d^2\theta / dt^2 > 0$ (Figure 6.8(a), (ii) if the object is rotating clockwise and slowing down then $d\theta / dt < 0$ but $d^2\theta / dt^2 > 0$ (Figure 6.8(b). There are two corresponding cases in which $\vec{\boldsymbol{\alpha}}$ pointing in the $-\hat{\mathbf{k}}$-direction: (iii) if the object is rotating counterclockwise and slowing down then $d\theta / dt > 0$ but $d^2\theta / dt^2 < 0$ (Figure 6.9(a), (iv) if the object is rotating clockwise and speeding up then both $d\theta / dt < 0$ and $d^2\theta / dt^2 < 0$ (Figure 6.9(b).

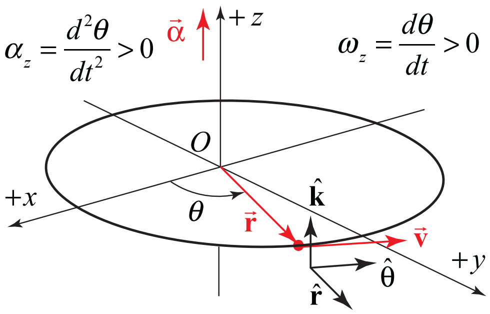

Figure 6.8(a) Angular acceleration vector vector for motion with $d\theta / dt > 0$, and $d^2\theta / dt^2 > 0$.

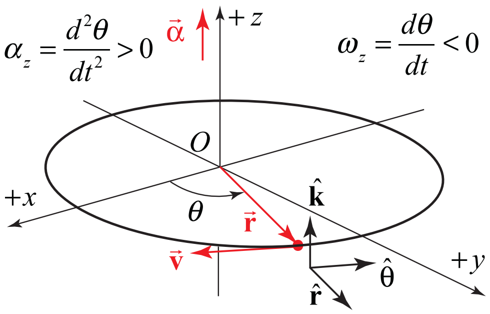

Figure 6.8(b) Angular velocity for motion with $d\theta / dt < 0$, and $d^2\theta / dt^2 > 0$.

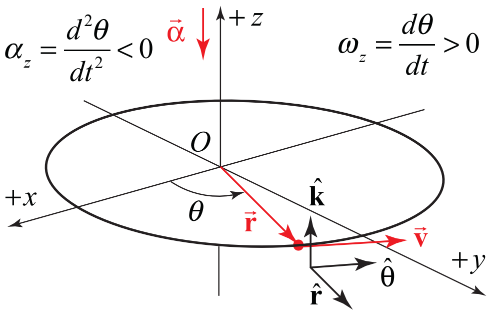

Figure 6.9(a) Angular acceleration vector vector for motion with $d\theta / dt > 0$, and $d^2\theta / dt^2 < 0$.

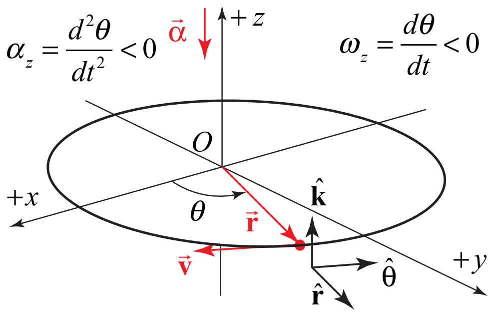

Figure 6.9(b) Angular velocity for motion with $d\theta / dt < 0$, and $d^2\theta / dt^2 < 0$.

Example 6.3 Integration and Circular Motion Kinematics

A point-like object is constrained to travel in a circle. The z-component of the angular acceleration of the object for the time interval $[0, t_1]$ is given by the function

$$\alpha_z(t) = \begin{cases} b\left(1 - \dfrac{t}{t_1}\right); \ 0 \le t \le t_1 \\ 0; \, t > t_1 \end{cases},$$

where b is a positive constant with units $\text{rad} \cdot \text{s}^{-2}$.

a) Determine an expression for the angular velocity of the object at $t = t_1$.

b) Through what angle has the object rotated at time $t = t_1$?

Solution:

a) The angular velocity at time $t = t_1$ is given by

$$\omega_z(t_1) - \omega_z(t = 0) = \int_{t'=0}^{t'=t_1} \alpha_z(t')\,dt' = \int_{t'=0}^{t'=t_1} b\left(1 - \frac{t'}{t_1}\right) dt' = b\left(t_1 - \frac{t_1^{\,2}}{2t_1}\right) = \frac{bt_1}{2}$$

b) In order to find the angle $\theta(t_1) - \theta(t = 0)$ that the object has rotated through at time $t = t_1$, you first need to find $\omega_z(t)$ by integrating the z-component of the angular acceleration

$$\omega_z(t) - \omega_z(t = 0) = \int_{t'=0}^{t'=t} \alpha_z(t')\,dt' = \int_{t'=0}^{t'=t} b\left(1 - \frac{t'}{t_1}\right) dt' = b\left(t - \frac{t^2}{2t_1}\right).$$

Because it started from rest, $\omega_z(t = 0) = 0$, hence $\omega_z(t) = b\left(t - \dfrac{t^2}{2t_1}\right); \ 0 \le t \le t_1$.

Then integrate $\omega_z(t)$ between $t = 0$ and $t = t_1$ to find that

$$\theta(t_1) - \theta(t = 0) = \int_{t'=0}^{t'=t_1} \omega_z(t')\,dt' = \int_{t'=0}^{t'=t_1} b\left(t' - \frac{t'^2}{2t_1}\right) dt' = b\left(\frac{t_1^{\,2}}{2} - \frac{t_1^{\,3}}{6t_1}\right) = \frac{bt_1^{\,2}}{3}.$$

6.5 Non-circular Central Planar Motion

Let's now consider central motion in a plane that is non-circular. In Figure 6.10, we show the spiral motion of a moving particle. In polar coordinates, the key point is that the time derivative dr / dt of the position function r is no longer zero. The second derivative d^2r / dt^2 also may or may not be zero. In the following calculation we will drop all explicit references to the time dependence of the various quantities. The position vector is still given by Eq. (6.2.1), which we shall repeat below

$$\vec{\mathbf{r}} = r\,\hat{\mathbf{r}}\,. \tag{6.5.9}$$

Because $dr / dt \neq 0$, when we differentiate Eq. (6.5.9), we need to use the product rule

$$\vec{\mathbf{v}} = \frac{d\,\vec{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\,\hat{\mathbf{r}}}{dt}\,. \tag{6.5.10}$$

Substituting Eq. (6.2.4) into Eq. (6.5.10)

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{dr}{dt}\,\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} = v_r\,\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}\,. \tag{6.5.11}$$

The velocity is no longer tangential but now has a radial component as well

$$v_r = \frac{dr}{dt}\,. \tag{6.5.12}$$

In order to determine the acceleration, we now differentiate Eq. (6.5.11), again using the product rule, which is now a little more involved:

$$\vec{\mathbf{a}} = \frac{d\,\vec{\mathbf{v}}}{dt} = \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\,\hat{\mathbf{r}}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\,\hat{\boldsymbol{\theta}}}{dt}\,. \tag{6.5.13}$$

Now substitute Eqs. (6.2.4) and (6.2.7) for the time derivatives of the unit vectors in Eq. (6.5.13), and after collecting terms yields

$$\begin{aligned}\vec{\mathbf{a}} &= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{\mathbf{r}} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\hat{\boldsymbol{\theta}} \\ &= a_r\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}}\end{aligned}\,. \tag{6.5.14}$$

The radial and tangential components of the acceleration are now more complicated than then in the case of circular motion due to the non-zero derivatives of dr / dt and d^2r / dt^2. The radial component is

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 . \qquad (6.5.15)$$

and the tangential component is

$$a_\theta = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} . \qquad (6.5.16)$$

The firs term in the tangential component of the acceleration, $2(dr/dt)(d\theta/dt)$ has a special name, the *coriolis acceleration*,

$$a_{cor} = 2\frac{dr}{dt}\frac{d\theta}{dt} . \qquad (6.5.17)$$

Example 6.4 Spiral Motion

A particle moves outward along a spiral starting from the origin at $t = 0$. Its trajectory is given by $r = b\theta$, where b is a positive constant with units $[\text{m}\cdot\text{rad}^{-1}]$. θ increases in time according to $\theta = ct^2$, where $c > 0$ is a positive constant (with units $[\text{rad}\cdot\text{s}^{-2}]$).

a) Determine the acceleration as a function of time.
b) Determine the time at which the radial acceleration is zero.
c) What is the angle when the radial acceleration is zero?
d) Determine the time at which the radial and tangential accelerations have equal magnitude.

Solution:

a) The position coordinate as a function of time is given by $r = b\theta = bct^2$. The acceleration is given by Eq. (6.5.14). In order to calculate the acceleration, we need to calculate the four derivatives $dr/dt = 2bct$, $d^2r/dt^2 = 2bc$, $d\theta/dt = 2ct$, and $d^2\theta/dt^2 = 2c$. The acceleration is then

$$\vec{\mathbf{a}} = \left(2bc - 4bc^3t^4\right)\hat{\mathbf{r}} + \left(8bc^2t^2 + 2bc^2t^2\right)\hat{\boldsymbol{\theta}} = \left(2bc - 4bc^3t^4\right)\hat{\mathbf{r}} + 10bc^2t^2\,\hat{\boldsymbol{\theta}} .$$

b) The radial acceleration is zero when

$$t_1 = \left(\frac{1}{2c^2}\right)^{1/4} .$$

c) The angle when the radial acceleration is zero is

$$\theta_1 = ct_1^{\,2} = \sqrt{2}\,/\,2\,.$$

d) The radial and tangential accelerations have equal magnitude when after some algebra

$$(2bc - 4bc^3t^4) = 10bc^2t^2 \Rightarrow 0 = t^4 + (5/2c)t^2 - (1/2c^2)\,.$$

This equation has as only positive solution for t^2:

$$t_2^{\,2} = \frac{-(5/2c) \pm ((5/2c)^2 + 2c^2)^{1/2}}{2} = \frac{\sqrt{33} - 5}{4c}\,.$$

Therefore the magnitudes of the two components are equal when

$$t_2 = \sqrt{\frac{\sqrt{33} - 5}{4c}}\,.$$

Chapter 7 Newton's Laws of Motion

Chapter 7 Newton's Laws of Motion

I have not as yet been able to discover the reason for these properties of gravity from phenomena, and I do not feign hypotheses. For whatever is not deduced from the phenomena must be called a hypothesis; and hypotheses, whether metaphysical or physical, or based on occult qualities, or mechanical, have no place in experimental philosophy. In this philosophy particular propositions are inferred from the phenomena, and afterwards rendered general by induction. [1]

Isaac Newton

7.1 Force and Quantity of Matter

In our daily experience, we can cause a body to move by either pushing or pulling that body. Ordinary language use describes this action as the effect of a person's strength or *force.* However, bodies placed on inclined planes, or when released at rest and undergo free fall, will move without any push or pull. Galileo referred to a force acting on these bodies, a description of which he published in *Mechanics* in 1623. In 1687, Isaac Newton published his three laws of motion in the *Philosophiae Naturalis Principia Mathematica* ("Mathematical Principles of Natural Philosophy"), which extended Galileo's observations. The First Law expresses the idea that when no force acts on a body, it will remain at rest or maintain uniform motion; when a force is applied to a body, it will change its state of motion.

Many scientists, especially Galileo, recognized the idea that force produces motion before Newton but Newton extended the concept of force to any circumstance that produces acceleration. When a body is initially at rest, the direction of our push or pull corresponds to the direction of motion of the body. If the body is moving, the direction of the applied force may change both the direction of motion of the body and how fast it is moving. Newton defined the force acting on an object as proportional to the acceleration of the object.

An impressed force is an action exerted upon a body, in order to change its state, either of rest, or of uniform motion in a right line.[2]

In order to define the magnitude of the force, he introduced a constant of proportionality, the *inertial mass*, which Newton called "quantity of matter".

---

[1] Isaac Newton (1726). *Philosophiae Naturalis Principia Mathematica*, General Scholium. Third edition, page 943 of I. Bernard Cohen and Anne Whitman's 1999 translation, University of California Press.

[2] Isaac Newton. *Mathematical Principles of Natural Philosophy*. Translated by Andrew Motte (1729). Revised by Florian Cajori. Berkeley: University of California Press, 1934. p. 2.

> *The quantity of matter is the measure of the same, arising from its density and bulk conjointly.*
>
> *Thus air of double density, in a double space, is quadruple in quantity; in a triple space, sextuple in quantity. The same thing is to be understood of snow, and fine dust or powders, that are condensed by compression or liquefaction, and of all bodies that are by any causes whatever differently condensed. I have no regard in this place to a medium, if any such there is, that freely pervades the interstices between the parts of bodies. It is this quantity that I mean hereafter everywhere under the name of body or mass. And the same is known by the weight of each body, for it is proportional to the weight, as I have found by experiment on pendulums, very accurately made, which shall be shown hereafter.*[3]

Suppose we apply a force to a body that is an identical copy of the standard mass, (we shall refer to this body as a standard body). The force will induce the standard body to accelerate with magnitude $\left|\vec{\mathbf{a}}\right|$ that can be measured by an accelerometer (any device that measures acceleration). The magnitude of the force $\left|\vec{\mathbf{F}}\right|$ acting on the standard body is **defined** to be the product of the standard mass m_s with the magnitude of the acceleration $\left|\vec{\mathbf{a}}\right|$. Force is a vector quantity. The direction of the force on the standard body is **defined** to be the direction of the acceleration of the body. Thus

$$\vec{\mathbf{F}} \equiv m_s \vec{\mathbf{a}} \tag{7.1.1}$$

In order to justify the statement that force is a vector quantity, we need to apply two forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ simultaneously to our standard body and show that the resultant force $\vec{\mathbf{F}}^T$ is the vector sum of the two forces when the forces are applied one at a time.

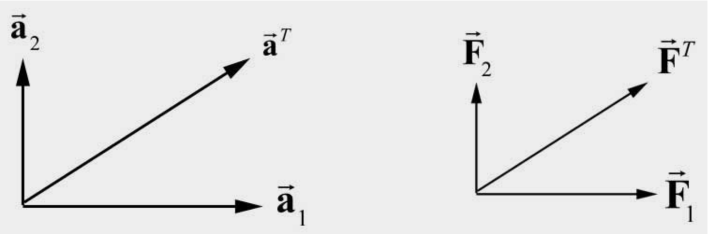

Figure 7.1 Acceleration add as vectors **Figure 7.2** Force adds as vectors.

We apply each force separately and measure the accelerations $\vec{\mathbf{a}}_1$ and $\vec{\mathbf{a}}_{2.}$, noting that

$$\vec{\mathbf{F}}_1 = m_s \vec{\mathbf{a}}_1 \tag{7.1.2}$$

---

[3] Ibid. p. 1.

$$\vec{\mathbf{F}}_2 = m_s \vec{\mathbf{a}}_2 . \tag{7.1.3}$$

When we apply the two forces simultaneously, we measure the acceleration $\vec{\mathbf{a}}$. The force by definition is now

$$\vec{\mathbf{F}}^T \equiv m_s \vec{\mathbf{a}} . \tag{7.1.4}$$

We then compare the accelerations. The results of these three measurements, and for that matter any similar experiment, confirms that the accelerations add as vectors (Figure 7.1)

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}_1 + \vec{\mathbf{a}}_2 . \tag{7.1.5}$$

Therefore the forces add as vectors as well (Figure 7.2),

$$\vec{\mathbf{F}}^T = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 . \tag{7.1.6}$$

This last statement is not a definition but a consequence of the experimental result described by Equation (7.1.5) and our definition of force.

Example 7.1 Vector Decomposition Solution

Two horizontal ropes are attached to a post that is stuck in the ground. The ropes pull the post producing the vector forces $\vec{\mathbf{F}}_1 = 70\text{ N}\,\hat{\mathbf{i}} + 20\text{ N}\,\hat{\mathbf{j}}$ and $\vec{\mathbf{F}}_2 = -30\text{ N}\,\hat{\mathbf{i}} + 40\text{ N}\,\hat{\mathbf{j}}$ as shown in Figure 7.3. Find the direction and magnitude of the horizontal component of a contact force of the ground on the post.

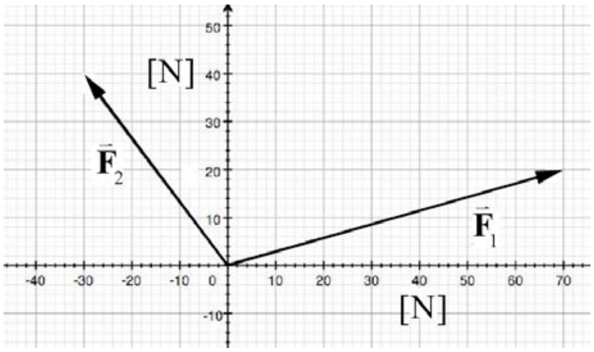

Figure 7.3 Example 7.1

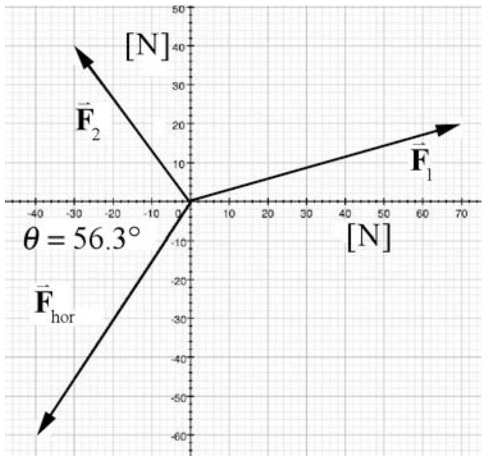

Figure 7.4 Vector sum of the horizontal forces

Solution: Because the ropes are pulling the post horizontally, the contact force must have a horizontal component that is equal to the negative of the sum of the two horizontal forces exerted by the rope on the post (Figure 7.4). There is an additional vertical component of the contact force that balances the gravitational force exerted on the post by the earth. We restrict our attention to the horizontal component of the contact force. Let $\vec{\mathbf{F}}_3$ denote the sum of the forces due to the ropes. Then we can write the vector $\vec{\mathbf{F}}_3$ as

$$\begin{aligned}\vec{\mathbf{F}}_3 &= (F_{1x} + F_{2x})\,\hat{\mathbf{i}} + (F_{1y} + F_{2y})\,\hat{\mathbf{j}} = (70\text{ N} + -30\text{ N})\,\hat{\mathbf{i}} + (20\text{ N} + 40\text{ N})\,\hat{\mathbf{j}} \\ &= (40\text{ N})\,\hat{\mathbf{i}} + (60\text{ N})\,\hat{\mathbf{j}}\end{aligned}$$

Therefore the horizontal component of the contact force satisfies the condition that

$$\vec{\mathbf{F}}_{hor} = -\vec{\mathbf{F}}_3 = -(\vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2) = (-40\text{ N})\,\hat{\mathbf{i}} + (-60\text{ N})\,\hat{\mathbf{j}}.$$

The magnitude is $\left|\vec{\mathbf{F}}_{hor}\right| = \sqrt{(-40\text{ N})^2 + (-60\text{ N})^2} = 72\text{ N}$. The horizontal component of the contact force makes an angle

$$\theta = \tan^{-1}\left[\frac{60\text{ N}}{40\text{ N}}\right] = 56.3^\circ$$

as shown in the figure above.

7.1.1 Mass Calibration

So far, we have only used the standard body to measure force. Instead of performing experiments on the standard body, we can calibrate the masses of all other bodies in terms of the standard mass by the following experimental procedure. We shall refer to the mass measured in this way as the **inertial mass** and denote it by m_{in}.

We apply a force of magnitude F to the standard body and measure the magnitude of the acceleration a_{s}. Then we apply the same force to a second body of unknown mass m_{in} and measure the magnitude of the acceleration a_{in}. Because the same force is applied to both bodies,

$$F = m_{in}\, a_{in} = m_{\text{s}}\, a_{\text{s}}, \tag{1.7}$$

the ratio of the inertial mass to the standard mass is equal to the inverse ratio of the magnitudes of the accelerations,

$$\frac{m_{in}}{m_{\text{s}}} = \frac{a_{\text{s}}}{a_{in}}. \tag{1.8}$$

Therefore the second body has inertial mass equal to

$$m_{in} = m_s \frac{a_s}{a_{in}}. \tag{1.9}$$

This method is justified by the fact that we can repeat the experiment using a different force and still find that the ratios of the acceleration are the same. For simplicity we shall denote the inertial mass by m.

7.2 Newton's First Law

The First Law of Motion, commonly called the "Principle of Inertia," was first realized by Galileo. (Newton did not acknowledge Galileo's contribution.) Newton was particularly concerned with how to phrase the First Law in Latin, but after many rewrites Newton choose the following expression for the First Law (in English translation):

> *Law 1: Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.*
>
> *Projectiles continue in their motions, so far as they are not retarded by the resistance of air, or impelled downwards by the force of gravity. A top, whose parts by their cohesion are continually drawn aside from rectilinear motions, does not cease its rotation, otherwise than as it is retarded by air. The greater bodies of planets and comets, meeting with less resistance in freer spaces, preserve their motions both progressive and circular for a much longer time.*[4]

The first law is an experimental statement about the motions of bodies. When a body moves with constant velocity, there are either no forces present or the sum of all the forces acting on the body is zero. If the body changes its velocity, it has non-zero acceleration, and hence the sum of all the forces acting on the body must be non-zero as well. If the velocity of a body changes in time, then either the direction or magnitude changes, or both can change.

After a bus or train starts, the acceleration is often so small we can barely perceive it. We are often startled because it seems as if the station is moving in the opposite direction while we seem to be at rest. Newton's First Law states that there is no physical way to distinguish between whether we are moving or the station is moving, because there is nearly zero total force acting on the body. Once we reach a constant velocity, our minds dismiss the idea that the ground is moving backwards because we think it is impossible, but there is no actual way for us to distinguish whether the train is moving or the ground is moving.

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[4] Ibid. p. 13.

7.3 Momentum, Newton's Second Law and Third Law

Newton began his analysis of the cause of motion by introducing the quantity of motion:

> ***Definition: Quantity of Motion***
>
> *The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjointly.*
>
> *The motion of the whole is the sum of the motion of all its parts; and therefore in a body double in quantity, with equal velocity, the motion is double, with twice the velocity, it is quadruple.*[5]

Our modern term for quantity of motion is *momentum* and it is a vector quantity

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}}\,, \tag{7.3.1}$$

where m is the inertial mass and $\vec{\mathbf{v}}$ is the velocity of the body. Newton's Second Law states that

> *Law II: The change of motion is proportional to the motive force impressed, and is made in the direction of the right line in which that force is impressed.*
>
> *If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force is impressed altogether and at once or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added or subtracted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both.*[6]

Suppose that a force is applied to a body for a time interval Δt. The impressed force or *impulse* (a vector quantity $\vec{\mathbf{I}}$) produces a change in the momentum of the body,

$$\vec{\mathbf{I}} = \vec{\mathbf{F}}\Delta t = \Delta\vec{\mathbf{p}}\,. \tag{7.3.2}$$

From the commentary to the second law, Newton also considered forces that were applied continually to a body instead of impulsively. The instantaneous action of the total

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[5] Ibid. p. 1.

[6] Ibid. p. 13.

force acting on a body at a time t is defined by taking the mathematical limit as the time interval Δt becomes smaller and smaller,

$$\vec{\mathbf{F}} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\mathbf{p}}}{\Delta t} \equiv \frac{d\vec{\mathbf{p}}}{dt}. \tag{7.3.3}$$

When the mass remains constant in time, the Second Law can be recast in its more familiar form,

$$\vec{\mathbf{F}} = m\frac{d\vec{\mathbf{v}}}{dt}. \tag{7.3.4}$$

Because the derivative of velocity is the acceleration, the force is the product of mass and acceleration,

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}}. \tag{7.3.5}$$

Because we defined force in terms of change in motion, the Second Law appears to be a restatement of this definition, and devoid of predictive power since force is only determined by measuring acceleration. What transforms the Second Law from just a definition is the additional input that comes from *force laws* that are based on experimental observations on the interactions between bodies. Throughout this book, we shall investigate these force laws and learn to use them in order to determine the forces and accelerations acting on a body (left-hand-side of Newton's Second Law). When a physical body is constrained to move along a surface, or inside a container (for example gas molecules in a container), there are *constraint forces* that are not determined beforehand by any force law but are only determined by their effect on the motion of the body. For any given constrained motion, these constraint forces are unknown and must be determined by the particular motion of the body that we are studying, for example the contact force of the surface on the body, or the force of the wall on the gas particles.

The right-hand-side of Newton's Second Law is the product of mass with acceleration. Acceleration is a mathematical description of how the velocity of a body changes. Knowledge of all the forces acting on the body enables us to predict the acceleration. Eq. (7.3.5) is known as the *equation of motion.* Once we know this equation we may be able to determine the velocity and position of that body at all future times by integration techniques, or computational techniques. For constrained motion, if we know the acceleration of the body, we can also determine the constraint forces acting on the body.

7.4 Newton's Third Law: Action-Reaction Pairs

Newton realized that when two bodies interact via a force, then the force on one body is equal in magnitude and opposite in direction to the force acting on the other body.

> *Law III: To every action there is always opposed an equal reaction: or, the mutual action of two bodies upon each other are always equal, and directed to contrary parts.*
>
> *Whatever draws or presses another is as much drawn or pressed by that other. If you press on a stone with your finger, the finger is also pressed by the stone.*[7]

The Third Law, commonly known as the "action-reaction" law, is the most surprising of the three laws. Newton's great discovery was that when two objects interact, they each exert the same magnitude of force on each other but in opposite directions. We shall refer to the pair of forces between two interacting bodies as an *interaction pair of force*, or more briefly as an *interaction pair*.

Consider two bodies engaged in a mutual interaction. Label the bodies 1 and 2 respectively. Let $\vec{\mathbf{F}}_{1,2}$ be the force on body 2 due to the interaction with body 1, and $\vec{\mathbf{F}}_{2,1}$ be the force on body 1 due to the interaction with body 2. These forces are depicted in Figure 7.5.

Figure 7.5 Interaction pair of forces

These two vector forces are equal in magnitude and opposite in direction,

$$\vec{\mathbf{F}}_{1,2} = -\vec{\mathbf{F}}_{2,1}. \qquad (7.4.1)$$

We shall employ these definitions, Newton's three laws, and force laws to describe the motion of bodies, a subject known as *classical mechanics* or *Newtonian Mechanics*, and hence explain a vast range of phenomena. Newtonian mechanics has important limits. It does not satisfactorily explain systems of objects moving at speeds comparable to the speed of light ($v > 0.1\,c$) where we need the theory of special relativity, nor does it adequately explain the motion of electrons in atoms, where we need *quantum mechanics*. We also need *general relativity* and *cosmology* to explain the large-scale structure of the universe.

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[7] Ibid p. 13.

Chapter 8 Applications of Newton's Second Law

Chapter 8 Applications of Newton's Second Law

> *Those who are in love with practice without knowledge are like the sailor who gets into a ship without rudder or compass and who never can be certain whether he is going. Practice must always be founded on sound theory...*[1]
>
> *Leonardo da Vinci*

8.1 Force Laws

There are forces that don't change appreciably from one instant to another, which we refer to as constant in time, and forces that don't change appreciably from one point to another, which we refer to as constant in space. The gravitational force on an object near the surface of the earth is an example of a force that is constant in space.

There are forces that depend on the configuration of a system. When a mass is attached to one end of a spring, the spring force acting on the object increases in strength whether the spring is extended or compressed.

There are forces that spread out in space such that their influence becomes less with distance. Common examples are the gravitational and electrical forces. The gravitational force between two objects falls off as the inverse square of the distance separating the objects provided the objects are of a small dimension compared to the distance between them. More complicated arrangements of attracting and repelling interactions give rise to forces that fall off with other powers of r: constant, $1/r$, $1/r^2$, $1/r^3$, …,.

A force may remain constant in magnitude but change direction; for example the gravitational force acting on a planet undergoing circular motion about a star is directed towards the center of the circle. This type of attractive central force is called a ***centripetal force***.

A ***force law*** describes the relationship between the force and some measurable property of the objects involved. We shall see that some interactions are describable by force laws and other interactions cannot be so simply described.

8.1.1 Hooke's Law

In order to stretch or compress a spring from its equilibrium length, a force must be exerted on the spring. Consider an object of mass m that is lying on a horizontal surface. Attach one end of a spring to the object and fix the other end of the spring to a wall. Let l_0 denote the equilibrium length of the spring (neither stretched or compressed). Assume

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[1] Notebooks of Leonardo da Vinci Complete, tr. Jean Paul Richter, 1888, Vol.1.

that the contact surface is smooth and hence frictionless in order to consider only the effect of the spring force. If the object is pulled to stretch the spring or pushed to compress the spring, then by Newton's Third Law the force of the spring on the object is equal and opposite to the force that the object exerts on the spring. We shall refer to the force of the spring on the object as the *spring force* and experimentally determine a relationship between that force and the amount of stretch or compress of the spring.

Choose a coordinate system with the origin located at the point of contact of the spring and the object when the spring-object system is in the equilibrium configuration. Choose the $\hat{\mathbf{i}}$ unit vector to point in the direction the object moves when the spring is being stretched. Choose the coordinate function x to denote the position of the object with respect to the origin (Figure 8.1).

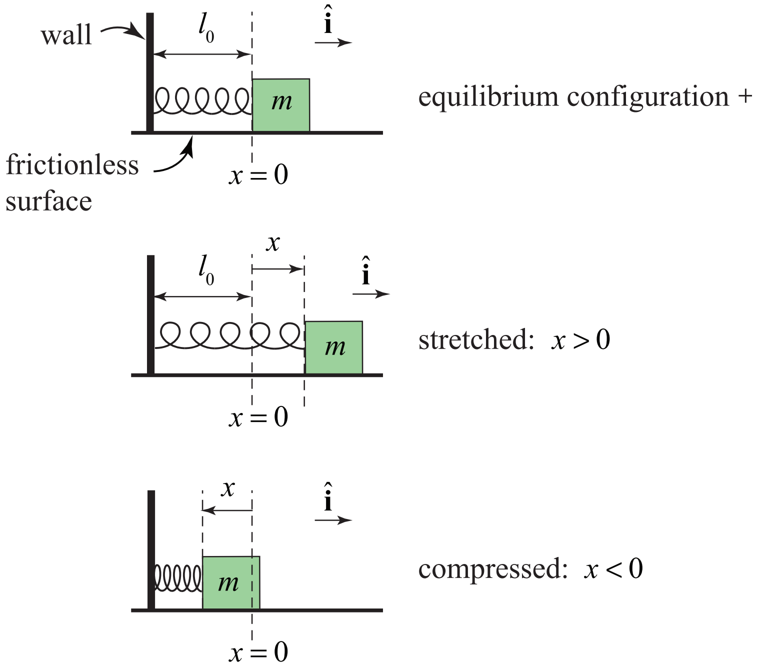


Figure 8.1 Spring attached to a wall and an object

Initially stretch the spring until the object is at position x. Then release the object and measure the acceleration of the object the instant the object is released. The magnitude of the spring force acting on the object is $|\vec{\mathbf{F}}| = m|\vec{\mathbf{a}}|$. Now repeat the experiment for a range of stretches (or compressions). Experiments show that for each spring, there is a range of maximum values $x_{\text{max}} > 0$ for stretching and minimum values $x_{\text{min}} < 0$ for compressing such that the magnitude of the measured force is proportional to the stretched or compressed length and is given by the formula

$$|\vec{\mathbf{F}}| = k|x|, \tag{8.1.1}$$

where the ***spring constant*** k has units $\mathrm{N \cdot m^{-1}}$. The free-body force diagram is shown in Figure 8.2.

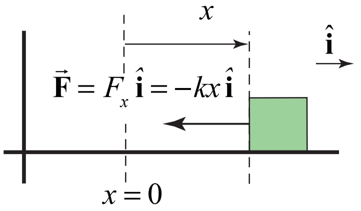


Figure 8.2 Spring force acting on object

The constant k is equal to the negative of the slope of the graph of the force *vs.* the compression or stretch (Figure 8.3).

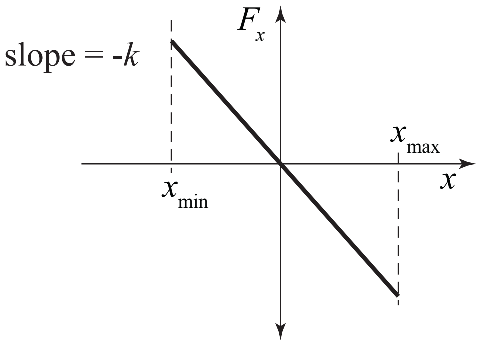


Figure 8.3 Plot of x-component of the spring force F_x vs. x

The direction of the acceleration is always towards the equilibrium position whether the spring is stretched or compressed. This type of force is called a ***restoring force***. Let F_x denote the x-component of the spring force. Then

$$F_x = -kx\,. \tag{8.1.2}$$

Now perform similar experiments on other springs. For a range of stretched lengths, each spring exhibits the same proportionality between force and stretched length, although the spring constant may differ for each spring.

It would be extremely impractical to experimentally determine whether this proportionality holds for all springs, and because a modest sampling of springs has confirmed the relation, we shall *infer* that all *ideal springs* will produce a restoring force, which is linearly proportional to the stretched (or compressed) length. This experimental relation regarding force and stretched (or compressed) lengths for a finite set of springs has now been *inductively* generalized into the above mathematical model for ideal springs, a force law known as a **Hooke's Law**.

This inductive step, referred to as ***Newtonian induction,*** is the critical step that makes physics a predictive science. Suppose a spring, attached to an object of mass m, is stretched by an amount Δx. Use the force law to predict the magnitude of the force between the rubber band and the object, $|\vec{\mathbf{F}}| = k|\Delta x|$, without having to experimentally measure the acceleration. Now use Newton's Second Law to predict the magnitude of the acceleration of the object

$$|\vec{\mathbf{a}}| = \frac{|\vec{\mathbf{F}}|}{m} = \frac{k|\Delta x|}{m}. \tag{8.1.3}$$

Carry out the experiment, and measure the acceleration within some error bounds. If the magnitude of the predicted acceleration disagrees with the measured result, then the model for the force law needs modification. The ability to adjust, correct or even reject models based on new experimental results enables a description of forces between objects to cover larger and larger experimental domains.

Many real springs have been wound such that a force of magnitude F_0 must be applied before the spring begins to stretch. The value of F_0 is referred to as the *pre-tension* of the spring. Under these circumstances, Hooke's law must be modified to account for this pretension,

$$\begin{cases} F_x = -F_0 - kx, & x > 0 \\ F_x = +F_1 - kx, & x < 0 \end{cases}. \tag{8.1.4}$$

Note the value of the pre-tension F_0 and F_1 may differ for compressing or stretching a spring.

8.2 Fundamental Laws of Nature

Force laws are mathematical models of physical processes. They arise from observation and experimentation, and they have limited ranges of applicability. Does the linear force law for the spring hold for all springs? Each spring will most likely have a different range of linear behavior. So the model for stretching springs still lacks a universal character. As such, there should be some hesitation to generalize this observation to all springs unless some property of the spring, universal to all springs, is responsible for the force law.

Perhaps springs are made up of very small components, which when pulled apart tend to contract back together. This would suggest that there is some type of force that contracts spring molecules when they are pulled apart. What holds molecules together? Can we find some fundamental property of the interaction between atoms that will suffice to explain the macroscopic force law? This search for ***fundamental forces*** is a central task of physics.

In the case of springs, this could lead into an investigation of the composition and structural properties of the atoms that compose the steel in the spring. We would investigate the geometric properties of the lattice of atoms and determine whether there is some fundamental property of the atoms that create this lattice. Then we ask how stable is this lattice under deformations. This may lead to an investigation into the electron configurations associated with each atom and how they overlap to form bonds between atoms. These particles carry charges, which obey Coulomb's Law, but also the Laws of Quantum Mechanics. So in order to arrive at a satisfactory explanation of the elastic restoring properties of the spring, we need models that describe the fundamental physics that underline Hooke's Law.

8.2.1 Universal Law of Gravitation

At points significantly far away from the surface of Earth, the gravitational force is no longer constant with respect to the distance to the center of Earth. ***Newton's Universal Law of Gravitation*** describes the gravitational force between two objects with masses, m_1 and m_2. This force points along the line connecting the objects, is attractive, and its magnitude is proportional to the inverse square of the distance, $r_{1,2}$, between the two point-like objects (Figure 8.4a). The force on object 2 due to the gravitational interaction between the two objects is given by

$$\vec{\mathbf{F}}^{G}_{1,2} = -G\frac{m_1 m_2}{r^2_{1,2}}\hat{\mathbf{r}}_{1,2}, \tag{8.2.1}$$

where $\vec{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1$ is a vector directed from object 1 to object 2, $r_{1,2} = \left|\vec{\mathbf{r}}_{1,2}\right|$, and $\hat{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_{1,2} / \left|\vec{\mathbf{r}}_{1,2}\right|$ is a unit vector directed from object 1 to object 2 (Figure 8.4b). The constant of proportionality in SI units is $G = 6.67 \times 10^{-11}\,\mathrm{N} \cdot \mathrm{m}^2 \cdot \mathrm{kg}^{-2}$.

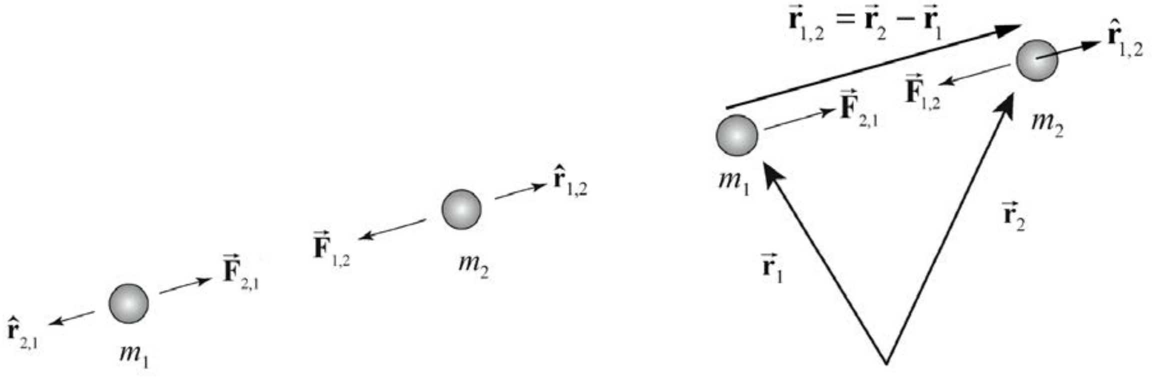


Figure 8.4 (a) Gravitational force between two point-like objects. **Figure 8.4 (b)** Coordinate system for the two-body problem.

8.2.2 Principle of Equivalence:

The Principle of Equivalence states that the mass that appears in the Universal Law of Gravity is identical to the inertial mass that is determined with respect to the standard kilogram. From this point on, the equivalence of inertial and gravitational mass will be assumed and the mass will be denoted by the symbol m.

8.2.3 Gravitational Force near the Surface of the Earth

Near the surface of Earth, the gravitational interaction between an object and Earth is mutually attractive and has a magnitude of

$$\left|\vec{\mathbf{F}}^{G}_{earth,object}\right| = mg \tag{8.2.2}$$

where g is a positive constant.

The International Committee on Weights and Measures has adopted as a standard value for the acceleration of an object freely falling in a vacuum $g = 9.80665\ \mathrm{m\cdot s^{-2}}$. The actual value of g varies as a function of elevation and latitude. If ϕ is the latitude and h the elevation in meters then the acceleration of gravity in SI units is

$$g = (9.80616 - 0.025928\cos(2\phi) + 0.000069\cos^2(2\phi) - 3.086\times 10^{-4} h)\ \mathrm{m\cdot s^{-2}}. \tag{8.2.3}$$

This is known as Helmert's equation. The strength of the gravitational force on the standard kilogram at 42° latitude is $9.80345\ \mathrm{N\cdot kg^{-1}}$, and the acceleration due to gravity at sea level is therefore $g = 9.80345\ \mathrm{m\cdot s^{-2}}$ for all objects. At the equator, $g = 9.78\ \mathrm{m\cdot s^{-2}}$ and at the poles $g = 9.83\ \mathrm{m\cdot s^{-2}}$. This difference is primarily due to the earth's rotation, which introduces an apparent (fictitious) repulsive force that affects the determination of g as given in Equation (8.2.2) and also flattens the spherical shape of Earth (the distance from the center of Earth is larger at the equator than it is at the poles by about $26.5\ \mathrm{km}$). Both the magnitude and the direction of the gravitational force also show variations that depend on local features to an extent that's useful in prospecting for oil, investigating the water table, navigating submerged submarines, and as well as many other practical uses. Such variations in g can be measured with a sensitive spring balance. Local variations have been much studied over the past two decades in attempts to discover a proposed "fifth force" which would fall off faster than the gravitational force that falls off as the inverse square of the distance between the objects.

8.2.4 Electric Charge and Coulomb's Law

Matter has properties other than mass. Matter can also carry one of two types of observed ***electric charge***, positive and negative. Like charges repel, and opposite charges attract each other. The unit of charge in the SI system of units is called the *coulomb* [C].

The smallest unit of "free" charge known in nature is the charge of an electron or proton, which has a magnitude of

$$e = 1.602\times 10^{-19}\ \mathrm{C}. \tag{8.2.4}$$

It has been shown experimentally that charge carried by ordinary objects is quantized in integral multiples of the magnitude of this free charge. The electron carries one unit of negative charge ($q_e = -e$) and the proton carries one unit of positive charge ($q_p = +e$). In an isolated system, the charge stays constant; in a closed system, an amount of unbalanced charge can neither be created nor destroyed. Charge can only be transferred from one object to another.

Consider two point-like objects with charges q_1 and q_2, separated by a distance $r_{1,2}$ in vacuum. By experimental observation, the two objects repel each other if they are both positively or negatively charged (Figure 8.4a). They attract each other if they are oppositely charged (Figure 8.5b). The force exerted on object 2 due to the interaction between objects 1 and 2 is given by Coulomb's Law,

$$\vec{\mathbf{F}}^{E}_{1,2} = k_e \frac{q_1 q_2}{r^2_{1,2}} \hat{\mathbf{r}}_{1,2} \tag{8.2.5}$$

where $\hat{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_{1,2} / \left|\vec{\mathbf{r}}_{1,2}\right|$ is a unit vector directed from object 1 to object 2, and in SI units, $k_e = 8.9875 \times 10^9 \mathrm{N} \cdot \mathrm{m}^2 \cdot \mathrm{C}^{-2}$, as illustrated in the Figure 8.5a. This law was derived empirically by Charles Augustin de Coulomb in the late 18th century.

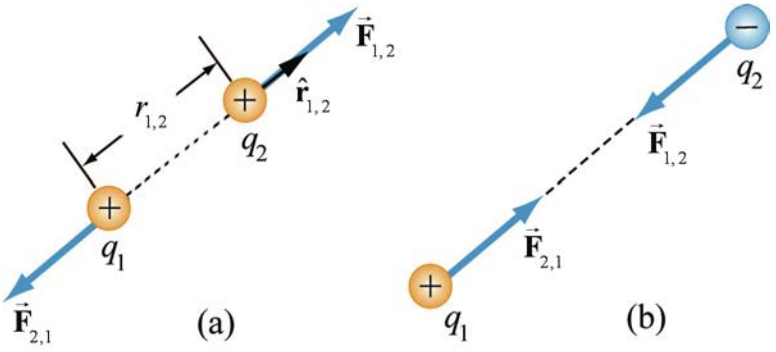


Figure 8.5 (a) and 8.5 (b) Coulomb interaction between two charges

Example 8.1 Coulomb's Law and the Universal Law of Gravitation

Show that both Coulomb's Law and the Universal Law of Gravitation satisfy Newton's Third Law.

Solution: To see this, interchange 1 and 2 in the Universal Law of Gravitation to find the force on object 1 due to the interaction between the objects. The only quantity to change sign is the unit vector

$$\hat{\mathbf{r}}_{2,1} = -\hat{\mathbf{r}}_{1,2} \,. \tag{8.2.6}$$

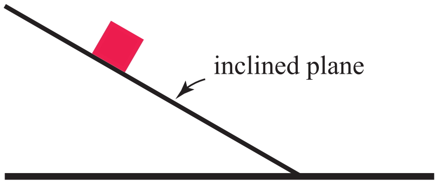
Then

$$\vec{\mathbf{F}}^{G}_{2,1} = -G\frac{m_2\, m_1}{r^2_{2,1}}\hat{\mathbf{r}}_{2,1} = G\frac{m_1\, m_2}{r^2_{1,2}}\hat{\mathbf{r}}_{1,2} = -\vec{\mathbf{F}}^{G}_{1,2} \,. \tag{8.2.7}$$

Coulomb's Law also satisfies Newton's Third Law since the only quantity to change sign is the unit vector, just as in the case of the Universal Law of Gravitation.

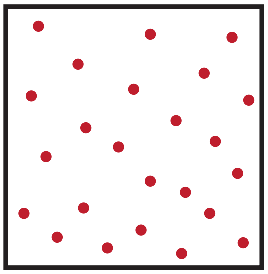
8.3 Constraint Forces

Knowledge of all the external and internal forces acting on each of the objects in a system and applying Newton's Second Law to each of the objects determine a set of equations of motion. These equations of motion are not necessarily independent due to the fact that the motion of the objects may be limited by equations of constraint. In addition there are forces of constraint that are determined by their effect on the motion of the objects and are not known beforehand or describable by some force law. For example: an object sliding down an inclined plane is constrained to move along the surface of the inclined plane (Figure 8.6a) and the surface exerts a contact force on the object; an object that slides down the surface of a sphere until it falls off experiences a contact force until it loses contact with the surface (Figure 8.6b); gas particles in a sealed vessel are constrained to remain inside the vessel and therefore the wall must exert force on the gas molecules to keep them inside the vessel (8.6c); and a bead constrained to slide outward along a rotating rod is acting on by time dependent forces of the rod on the bead (Figure 8.6d). We shall develop methods to determine these constraint forces although there are many examples in which the constraint forces cannot be determined.

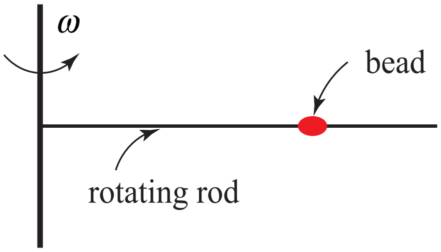


(a)

(b) '



(c)



(d) '

Figure 8.6 Constrained motions: **(a)** particle sliding down inclined plane, **(b)** particles sliding down surface of sphere, **(c)** gas molecules in a sealed vessel, and **(d)** bead sliding on a rotating rod

8.3.1 Contact Forces

Pushing, lifting and pulling are ***contact forces*** that we experience in the everyday world. Rest your hand on a table; the atoms that form the molecules that make up the table and your hand are in contact with each other. If you press harder, the atoms are also pressed closer together. The electrons in the atoms begin to repel each other and your hand is pushed in the opposite direction by the table.

According to Newton's Third Law, the force of your hand on the table is equal in magnitude and opposite in direction to the force of the table on your hand. Clearly, if you push harder the force increases. Try it! If you push your hand straight down on the table, the table pushes back in a direction perpendicular (normal) to the surface. Slide your hand gently forward along the surface of the table. You barely feel the table pushing upward, but you do feel the friction acting as a resistive force to the motion of your hand. This force acts tangential to the surface and opposite to the motion of your hand. Push downward and forward. Try to estimate the magnitude of the force acting on your hand.

The force of the table acting on your hand, $\vec{\mathbf{F}}^C \equiv \vec{\mathbf{C}}$, is called the ***contact force.*** This force has both a normal component to the surface, $\vec{\mathbf{C}}_\perp \equiv \vec{\mathbf{N}}$, called the ***normal force,*** and a tangential component to the surface, $\vec{\mathbf{C}}_{||} \equiv \vec{\mathbf{f}}$, called the ***friction force*** (Figure 8.6).

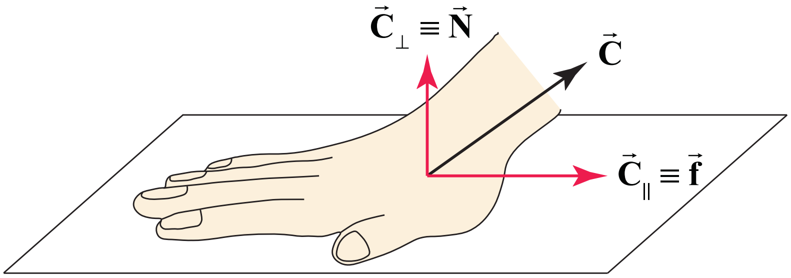


Figure 8.6 Normal and tangential components of the contact force

The contact force, written in terms of its component forces, is therefore

$$\vec{\mathbf{C}} = \vec{\mathbf{C}}_\perp + \vec{\mathbf{C}}_{||} \equiv \vec{\mathbf{N}} + \vec{\mathbf{f}} . \tag{8.3.1}$$

Any force can be decomposed into component vectors so the normal component, $\vec{\mathbf{N}}$, and the tangential component, $\vec{\mathbf{f}}$, are not independent forces but the vector components of the contact force, perpendicular and parallel to the surface of contact. The contact force is a *distributed force* acting over all the points of contact between your hand and the surface.

For most applications we shall treat the contact force as acting at single point but precaution must be taken when the distributed nature of the contact force plays a key role in constraining the motion of a rigid body.

In Figure 8.7, the forces acting on your hand are shown. These forces include the contact force, $\vec{\mathbf{C}}$, of the table acting on your hand, the force of your forearm, $\vec{\mathbf{F}}_{\text{forearm}}$, acting on your hand (which is drawn at an angle indicating that you are pushing down on your hand as well as forward), and the gravitational interaction, $\vec{\mathbf{F}}^g$, between the earth and your hand.

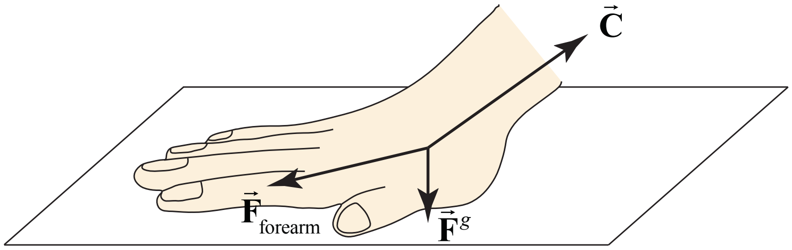


Figure 8.7 Forces on hand when moving towards the left

One point to keep in mind is that the magnitudes of the two components of the contact force depend on how hard you push or pull your hand and in what direction, a characteristic of constraint forces, in which the components are not specified by a force law but dependent on the particular motion of the hand.

Example 8.2 Normal Component of the Contact Force and Weight

Hold a block in your hand such that your hand is at rest (Figure 8.8). You can feel the "weight" of the block against your palm. But what exactly do we mean by "weight"?

Figure 8.8 Block resting in hand

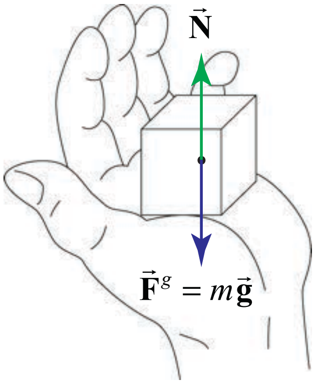


Figure 8.9 Forces on block

There are two forces acting on the block as shown in Figure 8.9. One force is the gravitational force between the earth and the block, and is denoted by $\vec{\mathbf{F}}^{g} = m\vec{\mathbf{g}}$. The other force acting on the block is the contact force between your hand and the block. Because our hand is at rest, this contact force on the block points perpendicular to the surface, and hence has only a normal component, $\vec{\mathbf{N}}$. Let N denote the magnitude of the normal force. Because the object is at rest in your hand, the vertical acceleration is zero. Therefore Newton's Second Law states that

$$\vec{\mathbf{N}} + \vec{\mathbf{F}}^{g} = \vec{\mathbf{0}}\,. \qquad (8.3.2)$$

Choose the positive direction to be upwards and then in terms of vertical components we have that

$$N - mg = 0\,. \qquad (8.3.3)$$

which can be solved for the magnitude of the normal force

$$N = mg\,. \qquad (8.3.4)$$

When we talk about the "weight" of the block, we often are referring to the effect the block has on a scale or on the feeling we have when we hold the block. These effects are actually effects of the normal force. We say that a block "feels lighter" if there is an additional force holding the block up. For example, you can rest the block in your hand, but use your other hand to apply a force upwards on the block to make it feel lighter in your supporting hand.

The word "weight," is often used to describe the gravitational force that Earth exerts on an object. We shall always refer to this force as the *gravitational force* instead of "weight." When you jump in the air, you feel "weightless" because there is no normal force acting on you, even though Earth is still exerting a gravitational force on you; clearly, when you jump, you do not turn gravity off!

This example may also give rise to a misconception that the normal force is always equal to the mass of the object times the magnitude of the gravitational acceleration at the surface of the earth. The normal force and the gravitational force are two completely different forces. In this particular example, the normal force is equal in magnitude to the gravitational force and directed in the opposite direction because the object is at rest. The normal force and the gravitational force do not form a Third Law interaction pair of forces. In this example, our system is just the block and the normal force and gravitational force are external forces acting on the block.

Let's redefine our system as the block, your hand, and Earth. Then the normal force and gravitational force are now internal forces in the system and we can now identify the various interaction pairs of forces. We explicitly introduce our interaction pair notation to enable us to identify these interaction pairs: for example, let $\vec{\mathbf{F}}^{g}_{E,B}$ denote the gravitational force on the block due to the interaction with Earth. The gravitational force on Earth due to the interaction with the block is denoted by $\vec{\mathbf{F}}^{g}_{B,E}$, and these two forces form an interaction pair. By Newton's Third Law, $\vec{\mathbf{F}}^{g}_{E,B} = -\vec{\mathbf{F}}^{g}_{B,E}$. Note that these two forces are acting on different objects, the block and Earth. The contact force on the block due to the interaction between the hand and the block is then denoted by $\vec{\mathbf{N}}_{H,B}$. The force of the block on the hand, which we denote by $\vec{\mathbf{N}}_{B,H}$, satisfies $\vec{\mathbf{N}}_{B,H} = -\vec{\mathbf{N}}_{H,B}$. Because we are including your hand as part of the system, there are two additional forces acting on the hand. There is the gravitational force on your hand $\vec{\mathbf{F}}^{g}_{E,H}$, satisfying $\vec{\mathbf{F}}^{g}_{E,H} = -\vec{\mathbf{F}}^{g}_{H,E}$, where $\vec{\mathbf{F}}^{g}_{H,E}$ is the gravitational force on Earth due to your hand. Finally there is the force of your forearm holding your hand up, which we denote $\vec{\mathbf{F}}_{F,H}$. Because we are not including the forearm in our system, this force is an external force to the system. The forces acting on your hand are shown in diagram on your hand is shown in Figure 8.10, and the just the interaction pairing of forces acting on Earth is shown in Figure 8.11 (we are not representing all other external forces acting on the Earth).

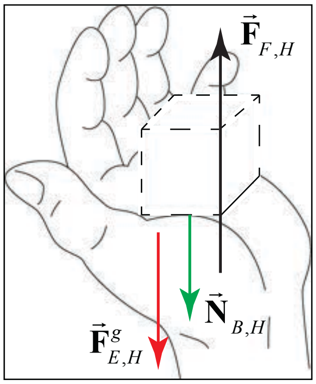


Figure 8.10 Free-body force diagram on hand

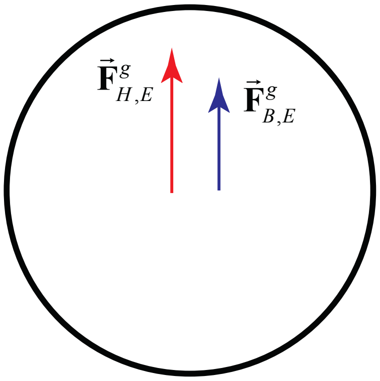


Figure 8.11 Gravitational forces on earth due to object and hand

8.3.2 Kinetic and Static Friction

When a block is pulled along a horizontal surface or sliding down an inclined plane there is a lateral force resisting the motion. If the block is at rest on the inclined plane, there is still a lateral force resisting the motion. This resistive force is known as dry friction, and there are two distinguishing types when surfaces are in contact with each other. The first type is when the two objects are moving relative to each other; the friction in that case is called ***kinetic friction*** or ***sliding friction***. When the two surfaces are non-moving but there is still a lateral force as in the example of the block at rest on an inclined plane, the force is called, ***static friction***.

Leonardo da Vinci was the first to record the results of measurements on kinetic friction over a twenty-year period between 1493–4 and about 1515. Based on his measurements, the force of kinetic friction, $\vec{\mathbf{f}}^{k}$, between two surfaces, he identified two key properties of kinetic friction. The magnitude of kinetic friction is proportional to the normal force between the two surfaces,

$$f_{\mathrm{k}} = \mu_{\mathrm{k}} N, \qquad (8.3.5)$$

where μ_{k} is called the ***coefficient of kinetic friction***. The second result is rather surprising in that the magnitude of the force is independent of the contact surface. Consider two blocks of the same mass, but different surface areas. The force necessary to move the blocks at a constant speed is the same. The block in Figure 8.12a has twice the contact area as the block shown in Figure 8.12b, but when the same external force is applied to either block, the blocks move at constant speed. These results of da Vinci were rediscovered by Guillaume Amontons and published in 1699. The third property that kinetic friction is independent of the speed of moving objects (for ordinary sliding speeds) was discovered by Charles Augustin Coulomb.

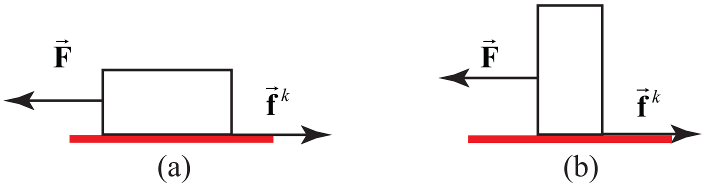


Figure 8.12 (a) and (b): kinetic friction is independent of the contact area

The kinetic friction on surface 2 moving relative to surface 1 is denoted by, $\vec{\mathbf{f}}^{k}_{1,2}$. The direction of the force is always opposed to the relative direction of motion of surface 2 relative to the surface 1. When one surface is at rest relative to our choice of reference frame we will denote the friction force on the moving object by $\vec{\mathbf{f}}^{k}$.

The second type of dry friction, static friction occurs when two surfaces are static relative to each other. Because the static friction force between two surfaces forms a third law interaction pair, will use the notation $\vec{\mathbf{f}}^{s}_{1,2}$ to denote the static friction force on surface 2 due to the interaction between surfaces 1 and 2. Push your hand forward along a surface; as you increase your pushing force, the frictional force feels stronger and stronger. Try this! Your hand will at first stick until you push hard enough, then your hand slides forward. The magnitude of the static frictional force, f_{s}, depends on how hard you push.

If you rest your hand on a table without pushing horizontally, the static friction is zero. As you increase your push, the static friction increases until you push hard enough that your hand slips and starts to slide along the surface. Thus the magnitude of static friction can vary from zero to some maximum value, $(f_s)_{\max}$, when the pushed object begins to slip,

$$0 \le f_s \le (f_s)_{\max}\,. \tag{8.3.6}$$

Is there a mathematical model for the magnitude of the maximum value of static friction between two surfaces? Through experimentation, we find that this magnitude is, like kinetic friction, proportional to the magnitude of the normal force

$$(f_s)_{\max} = \mu_s N\,. \tag{8.3.7}$$

Here the constant of proportionality is μ_s, the ***coefficient of static friction.*** This constant is slightly greater than the constant μ_k associated with kinetic friction, $\mu_s > \mu_k$. This small difference accounts for the slipping and catching of chalk on a blackboard, fingernails on glass, or a violin bow on a string.

The direction of static friction on an object is always opposed to the direction of the applied force (as long as the two surfaces are not accelerating). In Figure 8.13a, an external force, $\vec{\mathbf{F}}$, is applied the left and the static friction, $\vec{\mathbf{f}}^s$, is shown pointing to the right opposing the external force. In Figure 8.13b, the external force, $\vec{\mathbf{F}}$, is directed to the right and the static friction, $\vec{\mathbf{f}}^s$, is now pointing to the left.

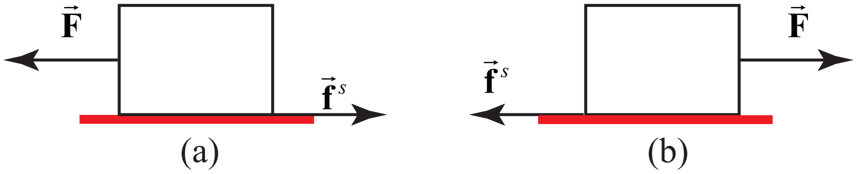


Figure 8.13 (a) and (b): External forces and the direction of static friction.

Although the force law for the maximum magnitude of static friction resembles the force law for sliding friction, there are important differences:

1. The direction and magnitude of static friction on an object always depends on the direction and magnitude of the applied forces acting on the object, where the magnitude of kinetic friction for a sliding object is fixed.

2. The magnitude of static friction has a maximum possible value. If the magnitude of the applied force along the direction of the contact surface exceeds the magnitude of the maximum value of static friction, then the object will start to slip (and be subject to kinetic friction.) We call this the ***just slipping*** condition.

8.4 Free-body Force Diagram

8.4.1 System

When we try to describe forces acting on a collection of objects we must first take care to specifically define the collection of objects that we are interested in, which define our ***system.*** Often the system is a single isolated object but it can consist of multiple objects.

Because force is a vector, the force acting on the system is a vector sum of the individual forces acting on the system

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots \tag{8.4.1}$$

A ***free-body force diagram*** is a representation of the sum of all the forces that act on a single system. We denote the system by a large circular dot, a "point". (Later on in the course we shall see that the "point" represents the center of mass of the system.) We represent each force that acts on the system by an arrow (indicating the direction of that force). We draw the arrow at the "point" representing the system. For example, the forces that regularly appear in free-body diagram are contact forces, tension, gravitation, friction, pressure forces, spring forces, electric and magnetic forces, which we shall introduce below. Sometimes we will draw the arrow representing the actual point in the system where the force is acting. When we do that, we will not represent the system by a "point" in the free-body diagram.

Suppose we choose a Cartesian coordinate system, then we can resolve the force into its component vectors

$$\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}} + F_z\,\hat{\mathbf{k}} \tag{8.4.2}$$

Each one of the component vectors is itself a vector sum of the individual component vectors from each contributing force. We can use the free-body force diagram to make these vector decompositions of the individual forces. For example, the x-component of the force is

$$F_x = F_{1,x} + F_{2,x} + \cdots . \tag{8.4.3}$$

8.4.5 Modeling

One of the most central and yet most difficult tasks in analyzing a physical interaction is developing a physical model. A physical model for the interaction consists of a description of the forces acting on all the objects. The difficulty arises in deciding which forces to include. For example in describing almost all planetary motions, the Universal Law of Gravitation was the only force law that was needed. There were anomalies, for example the small shift in Mercury's orbit. These anomalies are interesting because they may lead to new physics. Einstein corrected Newton's Law of Gravitation by introducing General Relativity and one of the first successful predictions of the new theory was the perihelion precession of Mercury's orbit. On the other hand, the anomalies may simply be due to the complications introduced by forces that are well understood but complicated to model. When objects are in motion there is always some type of friction present. Air friction is often neglected because the mathematical models for air resistance are fairly complicated even though the force of air resistance substantially changes the motion. Static or kinetic friction between surfaces is sometimes ignored but not always. The mathematical description of the friction between surfaces has a simple expression so it can be included without making the description mathematically intractable. A good way to start thinking about the problem is to make a simple model, excluding complications that are small order effects. Then we can check the predictions of the model. Once we are satisfied that we are on the right track, we can include more complicated effects.

8.5 Tension in a Rope

8.5.1 Definition of Tension in a Rope

Let's return to our example of the very light rope (object 2 with $m_2 \simeq 0$) that is attached to a block (object 1) at the point B, and pulled by an applied force at point A, $\vec{\mathbf{F}}_{\text{A},2}$ (Figure 8.18a).

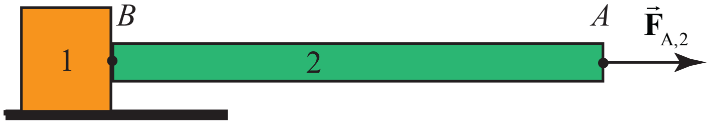


Figure 8.18a Massless rope pulling a block

Choose a coordinate system with the $\hat{\mathbf{j}}$-unit vector pointing upward in the normal direction to the surface, and the $\hat{\mathbf{i}}$-unit vector pointing in the positive x-direction, (Figure 8.18b). The force diagrams for the system consisting of the rope and block is shown in Figure 8.19, and for the rope and block separately in Figure 8.20, where $\vec{\mathbf{F}}_{2,1}$ is the force on the block (object 1) due to the rope (object 2), and $\vec{\mathbf{F}}_{1,2}$ is the force on the rope due to the block.

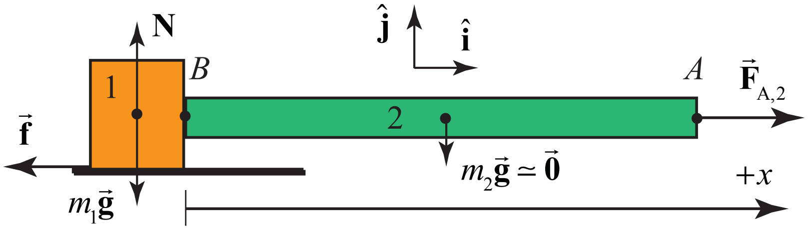


Figure 8.18b Forces acting on system consisting of block and rope

The forces on the rope and the block must each sum to zero. Because the rope is not accelerating, Newton's Second Law applied to the rope requires that $F_{A,2} - F_{1,2} = m_2 a$ (where we are using magnitudes for all the forces).

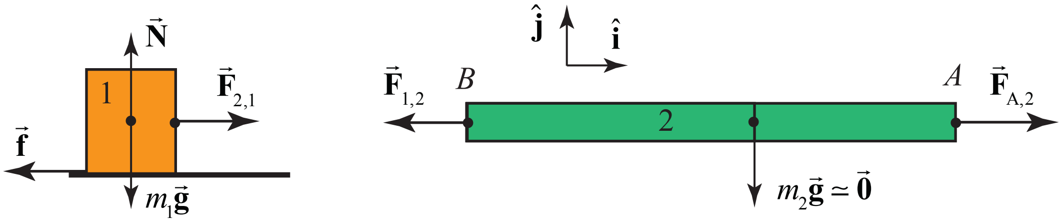


Figure 8.19 Separate force diagrams for rope and block

Because we are assuming the mass of the rope is negligible therefore

$$F_{A,2} - F_{1,2} = 0; \quad \text{(massless rope)} \tag{8.5.1}$$

.

If we consider the case that the rope is very light, then the forces acting at the ends of the rope are nearly horizontal. Then if the rope-block system is moving at constant speed or at rest, Newton's Second Law is now

$$F_{A,2} - F_{1,2} = 0; \quad \text{(constant speed or at rest)} \; . \tag{8.5.2}$$

Newton's Second Law applied to the block in the $+\hat{\mathbf{i}}$-direction requires that $F_{2,1} - f = 0$. Newton's Third Law, applied to the block-rope interaction pair requires that $F_{1,2} = F_{2,1}$. Therefore

$$F_{A,2} = F_{1,2} = F_{2,1} = f \; . \tag{8.5.3}$$

Thus the applied pulling force is transmitted through the rope to the block since it has the same magnitude as the force of the rope on the block. In addition, the applied pulling force is also equal to the friction force on the block.

How do we define "tension" at some point in a rope? Suppose make an imaginary slice of the rope at a point P, a distance x_P from point B, where the rope is attached to the block. The imaginary slice divides the rope into two sections, labeled L (left) and R (right), as shown in Figure 8.20.

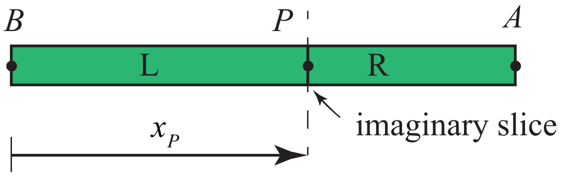


Figure 8.20 Imaginary slice through the rope

There is now a Third Law pair of forces acting between the left and right sections of the rope. Denote the force acting on the left section by $\vec{\mathbf{F}}_{\mathrm{R,L}}(x_P)$, and the force acting on the right section by $\vec{\mathbf{F}}_{\mathrm{L,R}}(x_P)$. Newton's Third Law requires that the forces in this interaction pair are equal in magnitude and opposite in direction.

$$\vec{\mathbf{F}}_{\mathrm{R,L}}(x_P) = -\vec{\mathbf{F}}_{\mathrm{L,R}}(x_P) \tag{8.5.4}$$

The force diagram for the left and right sections are shown in Figure 8.21 where $\vec{\mathbf{F}}_{1,\mathrm{L}}$ is the force on the left section of the rope due to the block-rope interaction. (We had previously denoted that force by $\vec{\mathbf{F}}_{1,2}$). Now denote the force on the right section of the rope side due to the pulling force at the point A by $\vec{\mathbf{F}}_{\mathrm{A,R}}$, (which we had previously denoted by $\vec{\mathbf{F}}_{\mathrm{A},2}$).

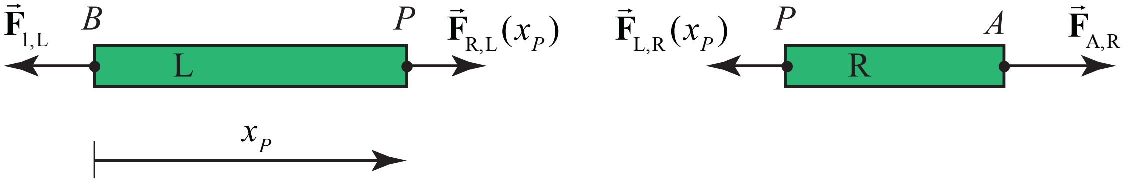


Figure 8.21 Force diagram for the left and right sections of rope

The **tension** $T(x_P)$ at a point P in rope lying a distance x from one the left end of the rope, is the magnitude of the action -reaction pair of forces acting at the point P,

$$T(x_P) = \left|\vec{\mathbf{F}}_{\mathrm{R,L}}(x_P)\right| = \left|\vec{\mathbf{F}}_{\mathrm{L,R}}(x_P)\right| . \tag{8.5.5}$$

For a rope of negligible mass, under tension, as in the above case, (even if the rope is accelerating) the sum of the horizontal forces applied to the left section and the right section of the rope are zero, and therefore the tension is uniform and is equal to the applied pulling force,

$$T = F_{\mathrm{A,R}} . \tag{8.5.6}$$

Example 8.3 Tension in a Massive Rope

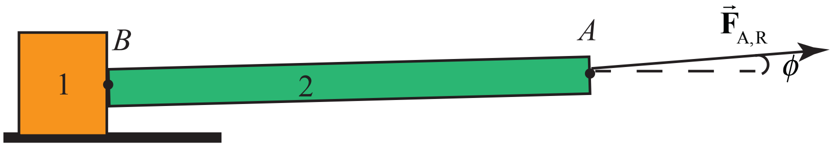


Figure 8.22a Massive rope pulling a block

Consider a block of mass m_1 that is lying on a horizontal surface. The coefficient of kinetic friction between the block and the surface is μ_k. A uniform rope of mass m_2 and length d is attached to the block. The rope is pulled from the side opposite the block with an applied force of magnitude $|\vec{\mathbf{F}}_{\mathrm{A},2}| = F_{\mathrm{A},2}$. Because the rope is now massive, the pulling force makes an angle ϕ with respect to the horizontal in order to balance the gravitational force on the rope, (Figure 8.22a). Determine the tension in the rope as a function of distance x from the block.

Solution: In the following analysis, we shall assume that the angle ϕ is very small and depict the pulling and tension forces as essentially acting in the horizontal direction even though there must be some small vertical component to balance the gravitational forces.

The key point to realize is that the rope is now massive and we must take in to account the inertia of the rope when applying Newton's Second Law. Consider an imaginary slice through the rope at a distance x from the block (Figure 8.22b), dividing the rope into two sections. The right section has length $d-x$ and mass $m_{\mathrm{R}} = (m_2 / d)(d-x)$. The left section has length x and mass $m_{\mathrm{L}} = (m_2 / d)(x)$.

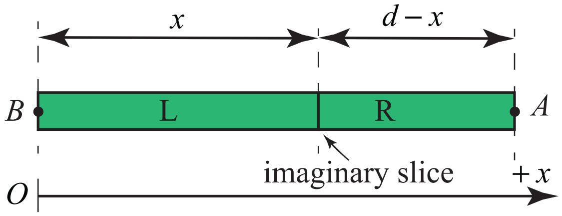


Figure 8.22b Imaginary slice through the rope

The free body force diagrams for the two sections of the rope are shown in Figure 8.22c, where $T(x)$ is the tension in the rope at a distance x from the block, and $F_{1,\mathrm{L}} = |\vec{\mathbf{F}}_{1,\mathrm{L}}| \equiv |\vec{\mathbf{F}}_{1,2}|$ is the magnitude of the force on the left-section of the rope due to the rope-block interaction.

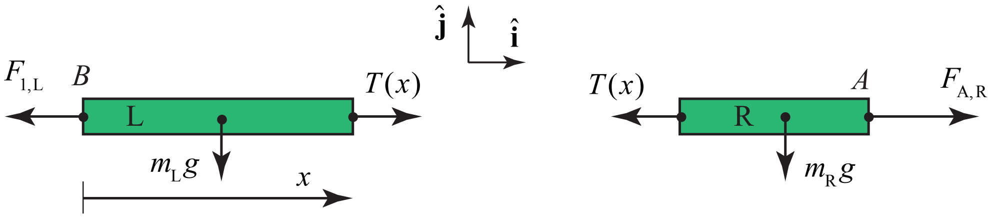


Figure 8.22c Force diagram for the left and right sections of rope

Apply Newton's Second Law to the right section of the rope yielding

$$F_{\mathrm{A,R}} - T(x) = m_{\mathrm{R}} a_{\mathrm{R}} = \frac{m_2}{d}(d - x) a_{\mathrm{R}}\,, \tag{8.5.7}$$

where a_{R} is the x-component of the acceleration of the right section of the rope. Apply Newton's Second Law to the left slice of the rope yielding

$$T(x) - F_{\mathrm{1,L}} = m_{\mathrm{L}} a_{\mathrm{L}} = (m_2 / d) x \, a_{\mathrm{L}}\,, \tag{8.5.8}$$

where a_{L} is the x-component of the acceleration of the left piece of the rope.

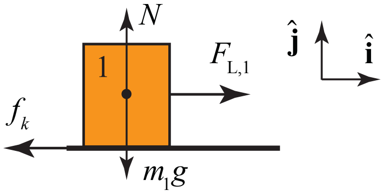


Figure 8.23 Force diagram on sliding block

The force diagram on the block is shown in Figure 8.23. Newton's Second Law on the block in the $+\hat{\mathbf{i}}$-direction is $F_{\mathrm{L,1}} - f_k = m_1 a_1$ and in the $+\hat{\mathbf{j}}$ -direction is $N - m_1 g = 0$. The kinetic friction force acting on the block is $f_k = \mu_k N = \mu_k m_1 g$. Newton's Second Law on the block in the $+\hat{\mathbf{i}}$-direction becomes

$$F_{\mathrm{L,1}} - \mu_k m_1 g = m_1 a_1\,, \tag{8.5.9}$$

Newton's Third Law for the block-rope interaction is given by $F_{\mathrm{L,1}} = F_{\mathrm{1,L}}$. Eq. (8.5.8) then becomes

$$T(x) - (\mu_k m_1 g + m_1 a_1) = (m_2 / d) x a_{\mathrm{L}}\,. \tag{8.5.10}$$

Because the rope and block move together, the accelerations are equal which we denote by the symbol $a \equiv a_1 = a_{\mathrm{L}}$. Then Eq. (8.5.10) becomes

$$T(x) = \mu_k m_1 g + (m_1 + (m_2 / d) x) a\,. \tag{8.5.11}$$

This result is not unexpected because the tension is accelerating both the block and the left section and is opposed by the frictional force.

Alternatively, the force diagram on the system consisting of the rope and block is shown in Figure 8.24.

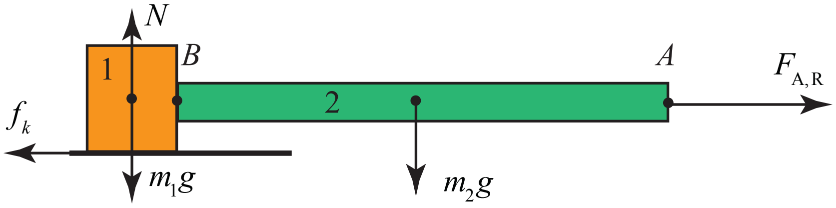


Figure 8.24 Force diagram on block-rope system

Newton's Second Law becomes

$$F_{\mathrm{A,R}} - \mu_k m_1 g = (m_2 + m_1)a \tag{8.5.12}$$

Solve Eq. (8.5.12) for $F_{\mathrm{A,R}}$ and substitute into Eq. (8.5.7), and solve for the tension yielding Eq. (8.5.11).

Example 8.4 Tension in a Suspended Rope

A uniform rope of mass M and length L is suspended from a ceiling (Figure 8.25). The magnitude of the acceleration due to gravity is g. (a) Find the tension in the rope at the upper end where the rope is fixed to the ceiling. (b) Find the tension in the rope as a function of the distance from the ceiling. (c) Find an equation for the rate of change of the tension with respect to distance from the ceiling in terms of M, L, and g.

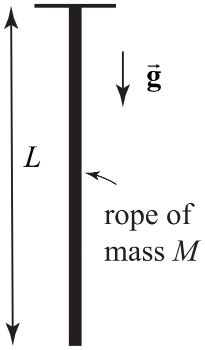


Figure 8.25 Rope suspended from ceiling

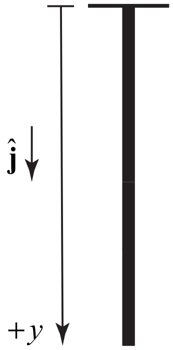


Figure 8.26 Coordinate system for suspended rope

Solution: (a) Begin by choosing a coordinate system with the origin at the ceiling and the positive y-direction pointing downward (Figure 8.26). In order to find the tension at the upper end of the rope, choose as a system the entire rope. The forces acting on the rope are the force at $y = 0$ holding the rope up, $T(y = 0)$, and the gravitational force on the entire rope. The free-body force diagram is shown in Figure 8.27.

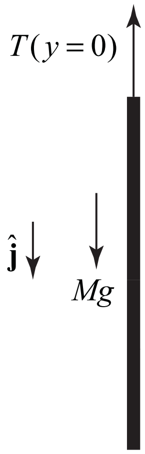


Figure 8.27 Force diagram on rope

Because the acceleration is zero, Newton's Second Law on the rope is $Mg - T(y=0) = 0$. Therefore the tension at the upper end is $T(y=0) = Mg$.

(b) Recall that the tension at a point is the magnitude of the action-reaction pair of forces acting at that point. Make an imaginary slice in the rope a distance y from the ceiling separating the rope into an upper segment 1, and lower segment 2 (Figure 8.28a). Choose the upper segment as a system with mass $m_1 = (M / L)y$. The forces acting on the upper segment are the gravitational force, the force $T(y=0)$ holding the rope up, and the tension $T(y)$ at the point y, that is pulling the upper segment down. The free-body force diagram is shown in Figure 8.28b.

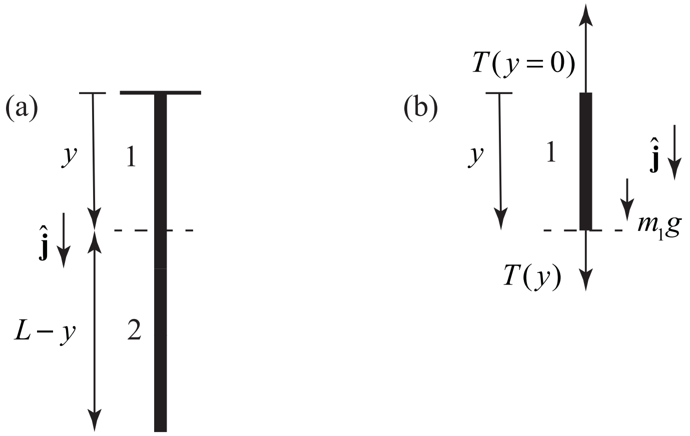


Figure 8.28 (a) Imaginary slice separates rope into two pieces**. (b)** Free-body force diagram on upper piece of rope

Apply Newton's Second Law to the upper segment: $m_1 g + T(y) - T(y=0) = 0$. Therefore the tension at a distance y from the ceiling is $T(y) = T(y=0) - m_1 g$. Because $m_1 = (M / L)y$ is the mass of the segment piece and Mg is the tension at the upper end, Newton's Second Law becomes

$$T(y) = Mg(1 - y / L) \tag{8.5.13}$$

As a check, we note that when $y = L$, the tension $T(y = L) = 0$, which is what we expect because there is no force acting at the lower end of the rope.

(c) Differentiate Eq. (8.5.13) with respect to y yielding

$$\frac{dT}{dy} = -(M / L)g\,. \tag{8.5.14}$$

The rate that the tension is changing at a constant rate with respect to distance from the top of the rope.

8.5.2 Continuous Systems and Newton's Second Law as a Differential Equations

We can determine the tension at a distance y from the ceiling in Example 8.4, by an alternative method, a technique that will generalize to many types of "continuous systems". Choose a coordinate system with the origin at the ceiling and the positive y-direction pointing downward as in Figure 8.25. Consider as the system a small element of the rope between the points y and $y + \Delta y$. This small element has length Δy, The small element has mass $\Delta m = (M / L)\Delta y$ and is shown in Figure 8.29.

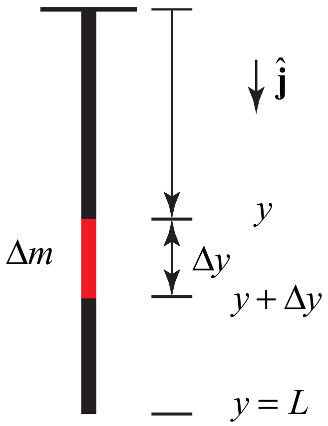


Figure 8.29 Small mass element of the rope

The forces acting on the small element are the tension, $T(y)$ at y directed upward, the tension $T(y + \Delta y)$ at $y + \Delta y$ directed downward, and the gravitational force Δmg directed downward. The tension $T(y + \Delta y)$ is equal to the tension $T(y)$ plus a small difference ΔT,

$$T(y + \Delta y) = T(y) + \Delta T\,. \tag{8.5.15}$$

The small difference in general can be positive, zero, or negative. The free body force diagram is shown in Figure 8.30.

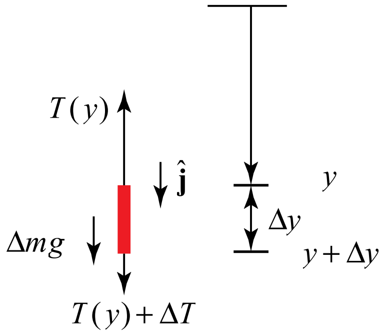


Figure 8.30 Free body force diagram on small mass element

Now apply Newton's Second Law to the small element

$$\Delta mg + T(y) - (T(y) + \Delta T) = 0 \tag{8.5.16}$$

The difference in the tension is then $\Delta T = -\Delta mg$. We now substitute our result for the mass of the element $\Delta m = (M / L)\Delta y$, and find that that

$$\Delta T = -(M / L)\Delta y g \ . \tag{8.5.17}$$

Divide through by Δy, yielding $\Delta T / \Delta y = -(M / L)g$. Now take the limit in which the length of the small element goes to zero, $\Delta y \to 0$,

$$\lim_{\Delta y \to 0} \frac{\Delta T}{\Delta y} = -(M / L)g \ . \tag{8.5.18}$$

Recall that the left hand side of Eq. (8.5.18) is the definition of the derivative of the tension with respect to y, and so we arrive at Eq. (8.5.14),

$$\frac{dT}{dy} = -(M / L)g.$$

We can solve the differential equation, Eq. (8.5.14), by a technique called **separation of variables**. We rewrite the equation as $dT = -(M / L)gdy$ and integrate both sides. Our integral will be a definite integral in which we integrate a 'dummy' integration variable y' from $y' = 0$ to $y' = y$ and the corresponding T' from $T' = T(y = 0)$ to $T' = T(y)$:

$$\int_{T'=T(y=0)}^{T'=T(y)} dT' = -(M / L)g\int_{y'=0}^{y'=y} dy' \ . \tag{8.5.19}$$

After integration and substitution of the limits, we have that

$$T(y) - T(y=0) = -(M/L)gy \ . \tag{8.5.20}$$

Us the fact that tension at the top of the rope is $T(y=0) = Mg$ and find that

$$T(y) = Mg(1 - y/L)$$

in agreement with our earlier result, Eq. (8.5.13).

8.6 Drag Forces in Fluids

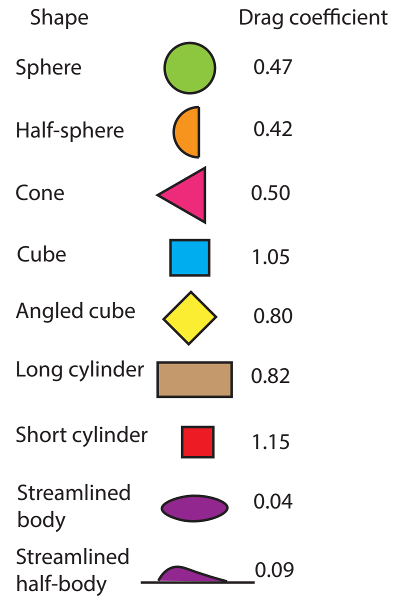
When a solid object moves through a fluid it will experience a resistive force, called the ***drag force***, opposing its motion. The fluid may be a liquid or a gas. This force is a very complicated force that depends on both the properties of the object and the properties of the fluid. The force depends on the speed, size, and shape of the object. It also depends on the density, viscosity and compressibility of the fluid.

For objects moving in air, the air drag is still quite complicated but for rapidly moving objects the resistive force is roughly proportional to the square of the speed v, the cross-sectional area A of the object in a plane perpendicular to the motion, the density ρ of the air, and independent of the viscosity of the air. Traditional the magnitude of the air drag for rapidly moving objects is written as

$$F_{\text{drag}} = \frac{1}{2} C_D A \rho v^2 \ . \tag{8.6.1}$$

The coefficient C_D is called the *drag coefficient*, a dimensionless number that is a property of the object. Table 8.1 lists the drag coefficient for some simple shapes, (each of these objects has a Reynolds number of order 10^4).

Table 8.1 Drag Coefficients



Shape	Drag coefficient
Sphere	0.47
Half-sphere	0.42
Cone	0.50
Cube	1.05
Angled cube	0.80
Long cylinder	0.82
Short cylinder	1.15
Streamlined body	0.04
Streamlined half-body	0.09

The above model for air drag does not extend to all fluids. An object dropped in oil, molasses, honey, or water will fall at different rates due to the different viscosities of the fluid. For very low speeds, the drag force depends linearly on the speed and is also proportional to the viscosity η of the fluid. For the special case of a sphere of radius R, the drag force law can be exactly deduced from the principles of fluid mechanics and is given by

$$\vec{\mathbf{F}}_{\text{drag}} = -6\pi\eta R\vec{\mathbf{v}} \qquad \text{(sphere)} \ . \tag{8.6.2}$$

This force law is known as *Stokes' Law*. The coefficient of viscosity η has SI units of $[\mathrm{N}\cdot\mathrm{m}^{-2}\cdot\mathrm{s}]=[\mathrm{Pa}\cdot\mathrm{s}]=[\mathrm{kg}\cdot\mathrm{m}^{-1}\cdot\mathrm{s}^{-1}]$; a cgs unit called the **poise** is often encountered . Some typical coefficients of viscosity are listed in Table 8.2.

Table 8.2: Coefficients of viscosity

fluid	Temperature, ^{0}C	Coefficient of viscosity η; $[\mathrm{kg}\cdot\mathrm{m}^{-1}\cdot\mathrm{s}^{-1}]$
Acetone	25	3.06×10^{-4}
Air	15	1.81×10^{-5}
Benzene	25	6.04×10^{-4}
Blood	37	$(3-4)\times10^{-3}$
Castor oil	25	0.985
Corn Syrup	25	1.3806
Ethanol	25	1.074×10^{-3}
Glycerol	20	1.2
Methanol	25	5.44×10^{-4}
Motor oil (SAE 10W)	20	6.5×10^{-2}
Olive Oil	25	8.1×10^{-2}
Water	10	1.308×10^{-3}
Water	20	1.002×10^{-3}
Water	60	0.467×10^{-3}
Water	100	0.28×10^{-3}

This law can be applied to the motion of slow moving objects in a fluid, for example: very small water droplets falling in a gravitational field, grains of sand settling in water, or the sedimentation rate of molecules in a fluid. In the later case, If we model a molecule as a sphere of radius R, the mass of the molecule is proportional to R^3 and the drag force is proportion to R, therefore different sized molecules will have different rates of acceleration. This is the basis for the design of measuring devices that separate molecules of different molecular weights.

In many physical situations the force on an object will be modeled as depending on the object's velocity. We have already seen static and kinetic friction between surfaces modeled as being independent of the surfaces' relative velocity. Common experience (swimming, throwing a Frisbee) tells us that the frictional force between an object and a fluid can be a complicated function of velocity. Indeed, these complicated relations are an important part of such topics as aircraft design.

Example 8.5 Drag Force at Low Speeds

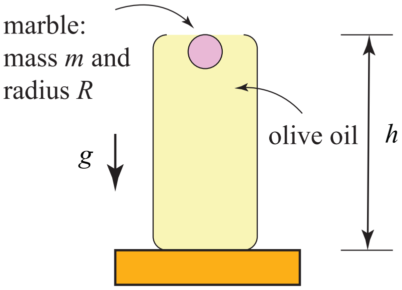


Figure 8.31 Example 8.5

A spherical marble of radius R and mass m is released from rest and falls under the influence of gravity through a jar of olive oil of viscosity η. The marble is released from rest just below the surface of the olive oil, a height h from the bottom of the jar. The gravitational acceleration is g (Figure 8.31). Neglect any force due to the buoyancy of the olive oil. (i) Determine the velocity of the marble as a function of time, (ii) what is the maximum possible velocity $\vec{\mathbf{v}}_{\infty} = \vec{\mathbf{v}}(t = \infty)$ (terminal velocity), that the marble can obtain, (iii) determine an expression for the viscosity of olive oil η in terms of g, m, R, and $v_{\infty} = \left|\vec{\mathbf{v}}_{\infty}\right|$, (iv) determine an expression for the position of the marble from just below the surface of the olive oil as a function of time.

Solution: Choose positive y-direction downwards with the origin at the initial position of the marble as shown in Figure 8.32(a).

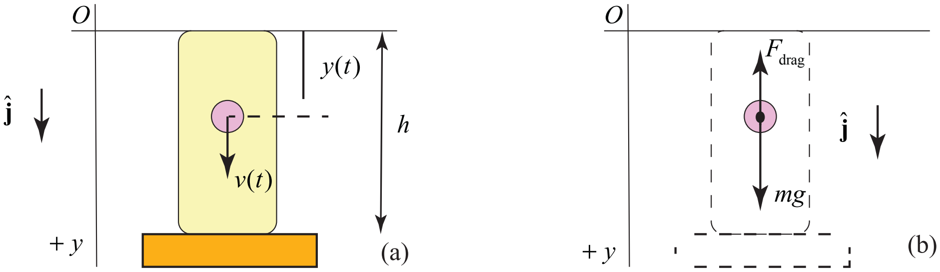


Figure 8.32 (a) Coordinate system for marble; (b) free body force diagram on marble

There are two forces acting on the marble: the gravitational force, and the drag force which is given by Eq. (8.6.2). The free body diagram is shown in the Figure 8.32(b). Newton's Second Law is then

$$mg - 6\pi\eta R v = m\frac{dv}{dt}, \qquad (8.6.3)$$

where v is the y-component of the velocity of the marble. Let $\gamma = 6\pi\eta R / m$; the SI units γ are $[\text{s}^{-1}]$. Then Eq. (8.6.3) becomes

$$g - \gamma v = \frac{dv}{dt}, \tag{8.6.4}$$

Suppose the object has an initial y-component of velocity $v(t=0)=0$. We shall solve Eq. (8.6.3) using the method of separation of variables. The differential equation may be rewritten as

$$\frac{dv}{(v - g/\gamma)} = -\gamma dt\,. \tag{8.6.5}$$

The integral version of Eq. (8.6.5) is then

$$\int_{v'=0}^{v'=v(t)} \frac{dv'}{v' - g/\gamma} = -\gamma \int_{t'=0}^{t'=t} dt'\;. \tag{8.6.6}$$

Integrating both sides of Eq. (8.6.6) yields

$$\ln\left(\frac{v(t) - g/\gamma}{-g/\gamma}\right) = -\gamma\, t\;. \tag{8.6.7}$$

Recall that $e^{\ln x} = x$, therefore upon exponentiation of Eq. (8.6.7) yields

$$\frac{v(t) - g/\gamma}{-g/\gamma} = e^{-\gamma\, t}\,. \tag{8.6.8}$$

Thus the y-component of the velocity as a function of time is given by

$$v(t) = \frac{g}{\gamma}(1 - e^{-\gamma\, t}) = \frac{mg}{6\pi\eta R}(1 - e^{-(6\pi\eta R/m)t})\;. \tag{8.6.9}$$

A plot of $v(t)$ vs. t is shown in Figure 8.31 with parameters $R = 5.00 \times 10^{-3}\text{m}$, $\eta = 8.10 \times 10^{-2}\ \text{kg}\cdot\text{m}^{-1}\cdot\text{s}^{-1}$, $m = 4.08 \times 10^{-3}\ \text{kg}$, and $g/\gamma = 1.87\ \text{m}\cdot\text{s}^{-1}$.

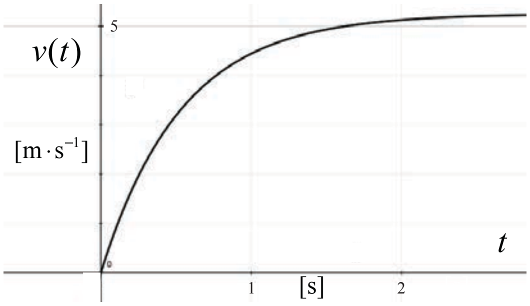


Figure 8.33 Plot of y -component of the velocity $v(t)$ vs. t for marble falling through oil with $g/\gamma = 1.87\ \mathrm{m}\cdot\mathrm{s}^{-1}$.

For large values of t, the term $e^{-(6\pi\eta R/m)t}$ approaches zero, and the marble reaches a terminal velocity

$$v_\infty = v(t=\infty) = \frac{mg}{6\pi\eta R}\cdot \tag{8.6.10}$$

The coefficient of viscosity can then be determined from the terminal velocity by the condition that

$$\eta = \frac{mg}{6\pi R v_{ter}}\cdot \tag{8.6.11}$$

Let ρ_m denote the density of the marble. The mass of the spherical marble is $m = (4/3)\rho_m R^3$. The terminal velocity is then

$$v_\infty = \frac{2\rho_m R^2 g}{9\eta}\ . \tag{8.6.12}$$

The terminal velocity depends on the square of the radius of the marble, indicating that larger marbles will reach faster terminal speeds.

The position of the marble as a function of time is given by the integral expression

$$y(t) - y(t=0) = \int_{t'=0}^{t'=t} v(t')\,dt' , \tag{8.6.13}$$

which after substitution of Eq. (8.6.9) and integration using the initial condition that $y(t=0)=0$, becomes

$$y(t) = \frac{g}{\gamma}t + \frac{g}{\gamma^2}\left(e^{-\gamma t} - 1\right)\ . \tag{8.6.14}$$

Example 8.6 Drag Forces at High Speeds

An object of mass m at time $t = 0$ is moving rapidly with velocity $\vec{\mathbf{v}}_0$ through a fluid of density ρ. Let A denote the cross-sectional area of the object in a plane perpendicular to the motion. The object experiences a retarding drag force whose magnitude is given by Eq. (8.6.1). Determine an expression for the velocity of the object as a function of time.

Solution: Choose a coordinate system such that the object is moving in the positive x-direction, $\vec{\mathbf{v}} = v\hat{\mathbf{i}}$. Set $\beta = (1/2)C_D A\rho$. Newton's Second Law can then be written as

$$-\beta v^2 = \frac{dv}{dt}. \qquad (8.6.15)$$

An integral version of Eq. (8.6.15) is then

$$\int_{v'=v_0}^{v'=v(t)} \frac{dv'}{v'^2} = -\beta \int_{t'=0}^{t'=t} dt'. \qquad (8.6.16)$$

Integration yields

$$-\left(\frac{1}{v(t)} - \frac{1}{v_0}\right) = -\beta t. \qquad (8.6.17)$$

After some algebraic rearrangement the x-component of the velocity as a function of time is given by

$$v(t) = \frac{v_0}{1 + v_0\beta t} = \frac{1}{1 + t/\tau} v_0, \qquad (8.6.18)$$

where $\tau = 1/v_0\beta$. A plot of $v(t)$ vs. t is shown in Figure 8.34 with initial conditions $v_0 = 20\ \text{m}\cdot\text{s}^{-1}$ and $\beta = 0.5\ \text{s}^{-1}$.

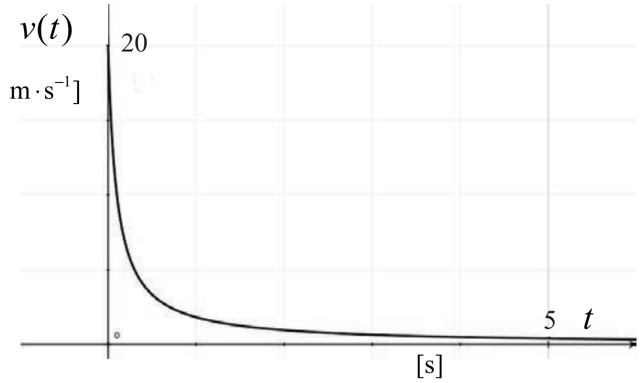


Figure 8.34 Plot of $v(t)$ vs. t for damping force $F_{\text{drag}} = \frac{1}{2} C_D A\rho v^2$

8.7 Worked Examples

Example 8.7 Staircase

An object of mass m at time $t = 0$ has speed v_0. It slides a distance s along a horizontal floor and then off the top of a staircase (Figure 8.35). The coefficient of kinetic friction between the object and the floor is μ_k. The object strikes at the far end of the third stair. Each stair has a rise of h and a run of d. Neglect air resistance and use g for the gravitational constant. (a) What is the distance s that the object slides along the floor?

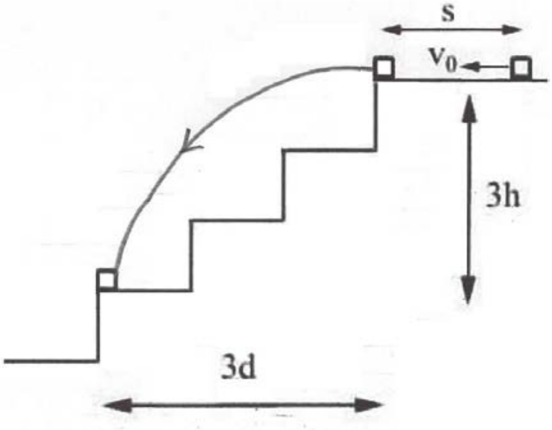


Figure 8.35 Object falling down a staircase

Solution: There are two distinct stages to the object's motion, the initial horizontal motion and then free fall. The given final position of the object, at the far end of the third stair, will determine the horizontal component of the velocity at the instant the object left the top of the stairs. This in turn can be used to determine the time the object decelerated along the floor, and hence the distance traveled on the floor. The given quantities are m, v_0, μ_k, g, h and d.

For the horizontal motion, choose coordinates with the origin at the initial position of the block. Choose the positive $\hat{\mathbf{i}}$-direction to be horizontal, directed to the left in Figure 8.35, and the positive $\hat{\mathbf{j}}$-direction to be vertical (up). The forces on the object are gravity $m\vec{\mathbf{g}} = -mg\,\hat{\mathbf{j}}$, the normal force $\vec{\mathbf{N}} = N\,\hat{\mathbf{j}}$ and the kinetic frictional force $\vec{\mathbf{f}}_k = -f_k\,\hat{\mathbf{i}}$. The components of the vectors in Newton's Second Law, $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, are

$$\begin{aligned} -f_k &= m\,a_x \\ N - m\,g &= m\,a_y. \end{aligned} \tag{8.6.19}$$

The object does not move in the y-direction; $a_y = 0$ and thus from the second expression in (8.6.19), $N = m\,g$. The magnitude of the frictional force is then $f_\text{k} = \mu_\text{k} N = \mu_\text{k} mg$, and the first expression in (8.6.19) gives the x-component of acceleration as $a_x = -\mu_\text{k} g$. Becasue the acceleration is constant the x-component of the velocity is given by

$$v_x(t) = v_0 + a_x\,t\,, \tag{8.6.20}$$

where v_0 is the x-component of the velocity of the object when it just started sliding. The displacement is given by

$$x(t) - x_0 = v_0\,t + \frac{1}{2}a_x\,t^2. \tag{8.6.21}$$

Denote the time the block just leaves the landing by t_1, where $x(t_1) = s$, and the speed just when it reaches the landing $v_x(t_1) = v_{x,1}$. The initial speed is v_0 and $x_0 = 0$. Using the initial and final conditions, and the value of the acceleration, Eq. (8.6.21) becomes

$$s = v_0\,t_1 - \frac{1}{2}\mu_k g\,t_1^{\,2}. \tag{8.6.22}$$

Solve Eq. (8.6.20) for the time the block reaches the edge of the landing,

$$t_1 = \frac{v_{x,1} - v_0}{-\mu_\text{k} g} = \frac{v_0 - v_{x,1}}{\mu_\text{k} g}. \tag{8.6.23}$$

Substituting Eq. (8.6.23) into Eq. (8.6.22) yields

$$s = v_0\left(\frac{v_0 - v_{x,1}}{\mu_\text{k} g}\right) - \frac{1}{2}\mu_k g\left(\frac{v_0 - v_{x,1}}{\mu_\text{k} g}\right)^2 \tag{8.6.24}$$

and after some algebra, we can rewrite Eq. (8.6.24) as

$$s = \frac{v_0^{\,2} - v_{x,1}^{\,2}}{2\mu_\text{k} g}. \tag{8.6.25}$$

From the top of the stair to the far end of the third stair, the object is in free fall. Choose the positive $\hat{\mathbf{i}}$-direction to be horizontal, directed to the left in Figure 8.35, and the positive $\hat{\mathbf{j}}$-direction to be vertical (up) and now choose the origin at the top of the stairs, where the object first goes into free fall. The components of acceleration are $a_x = 0$,

$a_y = -g$, the initial x-component of velocity is $v_{x,1}$, the initial y-component of velocity is $v_{y,0} = 0$, the initial x-position is $x_0 = 0$ and the initial y-position is $y_0 = 0$. Reset $t = 0$ when the object just leaves the landing. Let t_2 denote the instant the object hits the stair, where $y(t_2) = -3h$ and $x(t_2) = 3d$. The equations describing the object's position and speed at time $t = t_2$ are

$$x(t_2) = 3d = v_{x,1} t_2 \tag{8.6.26}$$

$$y(t_2) = -3h = -\frac{1}{2} g t_2^{\ 2}. \tag{8.6.27}$$

Solve Eq. (8.6.26) for t_2 to yield

$$t_2 = \frac{3d}{v_{x,1}}. \tag{8.6.28}$$

Substitute Eq. (8.6.28) into Eq. (8.6.27) and eliminate the variable t_2,

$$3h = \frac{1}{2} g \frac{9d^2}{v_{x,1}^{\ 2}}. \tag{8.6.29}$$

Eq. (8.6.29) can now be solved for the square of the horizontal component of the velocity,

$$v_{x,1}^{\ 2} = \frac{3gd^2}{2h}. \tag{8.6.30}$$

Now substitute Eq. (8.6.30) into Eq. (8.6.25) to determine the distance the object traveled on the landing,

$$s = \frac{v_0^2 - (3gd^2 / 2h)}{2\mu_k g}. \tag{8.6.31}$$

Example 8.8 Cart Moving on a Track

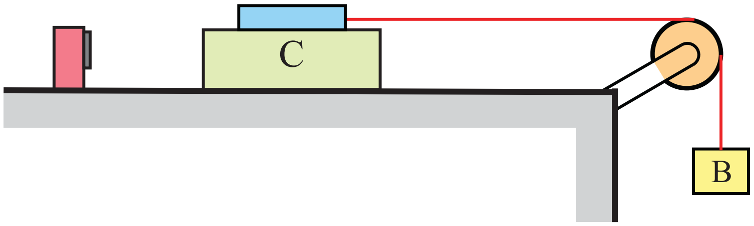


Figure 8.36 A falling block will accelerate a cart on a track via the pulling force of the string. The force sensor measures the tension in the string.

Consider a cart that is free to slide along a horizontal track (Figure 8.36). A force is applied to the cart via a string that is attached to a force sensor mounted on the cart, wrapped around a pulley and attached to a block on the other end. When the block is released the cart will begin to accelerate. The force sensor and cart together have a mass m_{C}, and the suspended block has mass m_{B}. Neglect the small mass of the string and pulley, and assume the string is inextensible. The coefficient of kinetic friction between the cart and the track is μ_{k}. Determine (i) the acceleration of the cart, and (ii) the tension in the string.

Solution: In general, we would like to draw free-body diagrams on all the individual objects (cart, sensor, pulley, rope, and block) but we can also choose a system consisting of two (or more) objects knowing that the forces of interaction between any two objects will cancel in pairs by Newton's Third Law. In this example, we shall choose the sensor/cart as one free-body, and the block as the other free-body. The free-body force diagram for the sensor/cart is shown in Figure 8.37.

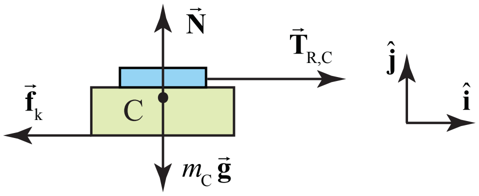


Figure 8.37 Force diagram on sensor/cart with a vector decomposition of the contact force into horizontal and vertical components

There are three forces acting on the sensor/cart: the gravitational force $m_{\mathrm{C}}\vec{\mathbf{g}}$, the pulling force $\vec{\mathbf{T}}_{\mathrm{R,C}}$ of the rope on the force sensor, and the contact force between the track and the cart. In Figure 8.34, we decompose the contact force into its two components, the kinetic frictional force $\vec{\mathbf{f}}_{\mathrm{k}} = -f_{\mathrm{k}}\hat{\mathbf{i}}$ and the normal force, $\vec{\mathbf{N}} = N\,\hat{\mathbf{j}}$.

The cart is only accelerating in the horizontal direction with $\vec{\mathbf{a}}_{\mathrm{C}} = a_{\mathrm{C},x}\,\hat{\mathbf{i}}$, so the component of the force in the vertical direction must be zero, $a_{\mathrm{C},y} = 0$. We can now apply Newton's Second Law in the horizontal and vertical directions and find that

$$\hat{\mathbf{i}}: \; T_{\mathrm{R,C}} - f_{\mathrm{k}} = m_{\mathrm{C}}a_{\mathrm{C},x} \tag{8.6.32}$$

$$\hat{\mathbf{j}}: \; N - m_{\mathrm{C}}g = 0\,. \tag{8.6.33}$$

From Eq. (8.6.33), we conclude that the normal component is

$$N = m_{\mathrm{C}}g\,. \tag{8.6.34}$$

We use Equation (8.6.34) for the normal force to find that the magnitude of the kinetic frictional force is

$$f_{\mathrm{k}} = \mu_{\mathrm{k}} N = \mu_{\mathrm{k}} m_{\mathrm{C}} g \,. \tag{8.6.35}$$

Then Equation (8.6.32) becomes

$$T_{\mathrm{R,C}} - \mu_{\mathrm{k}} m_{\mathrm{C}} g = m_{\mathrm{C}} a_{\mathrm{C},x} \,. \tag{8.6.36}$$

The force diagram for the block is shown in Figure 8.38. The two forces acting on the block are the pulling force $\vec{\mathbf{T}}_{\mathrm{R,B}}$ of the string and the gravitational force $m_{\mathrm{B}}\vec{\mathbf{g}}$. We now apply Newton's Second Law to the block and find that

$$\hat{\mathbf{j}}_{\mathrm{B}} : m_{\mathrm{B}} g - T_{\mathrm{R,B}} = m_{\mathrm{B}} a_{B,y} \,. \tag{8.6.37}$$

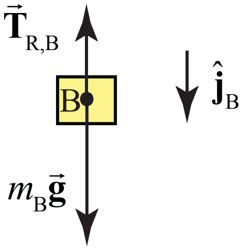


Figure 8.38 Forces acting on the block

In Equation (8.6.37), the symbol $a_{B,y}$ represents the component of the acceleration with sign determined by our choice of downward direction for the unit vector $\hat{\mathbf{j}}_{\mathrm{B}}$. Note that we made a different choice of direction for the unit vector in the vertical direction in the free-body diagram for the block shown in Figure 8.37. Each free-body diagram has an independent set of unit vectors that define a sign convention for vector decomposition of the forces acting on the free-body and the acceleration of the free-body. In our example, with the unit vector pointing downwards in Figure 8.38, if we solve for the component of the acceleration and it is positive, then we know that the direction of the acceleration is downwards.

There is a second subtle way that signs are introduced with respect to the forces acting on a free-body. In our example, the force between the string and the block acting on the block points upwards, so in the vector decomposition of the forces acting on the block that appears on the left-hand side of Equation (8.6.37), this force has a minus sign and the quantity $\vec{\mathbf{T}}_{\mathrm{R,B}} = -T_{\mathrm{R,B}}\hat{\mathbf{j}}_{\mathrm{B}}$ where $T_{\mathrm{R,B}}$ is assumed positive.

Our assumption that the mass of the rope and the mass of the pulley are negligible enables us to assert that the tension in the rope is uniform and equal in magnitude to the forces at each end of the rope,

$$T_{\mathrm{R,B}} = T_{\mathrm{R,C}} \equiv T . \tag{8.6.38}$$

We also assumed that the string is inextensible (does not stretch). This implies that the rope, block, and sensor/cart all have the same magnitude of acceleration,

$$a_{\mathrm{C},x} = a_{\mathrm{B},y} \equiv a . \tag{8.6.39}$$

Using Equations (8.6.38) and (8.6.39), we can now rewrite the equation of motion for the sensor/cart, Equation (8.6.36), as

$$T - \mu_{\mathrm{k}} m_{\mathrm{C}} g = m_{\mathrm{C}} a , \tag{8.6.40}$$

and the equation of motion (8.6.37) for the block as

$$m_{\mathrm{B}} g - T = m_{\mathrm{B}} a . \tag{8.6.41}$$

We have only two unknowns T and a, so we can now solve the two equations (8.6.40) and (8.6.41) simultaneously for the acceleration of the sensor/cart and the tension in the rope. We first solve Equation (8.6.40) for the tension

$$T = \mu_{\mathrm{k}} m_{\mathrm{C}} g + m_{\mathrm{C}} a \tag{8.6.42}$$

and then substitute Equation (8.6.42) into Equation (8.6.41) and find that

$$m_{\mathrm{B}} g - (\mu_{\mathrm{k}} m_{\mathrm{C}} g + m_{\mathrm{C}} a) = m_{\mathrm{B}} a . \tag{8.6.43}$$

We can now solve Equation (8.6.43) for the acceleration,

$$a = \frac{m_{\mathrm{B}} g - \mu_{\mathrm{k}} m_{\mathrm{C}} g}{m_{\mathrm{C}} + m_{\mathrm{B}}} . \tag{8.6.44}$$

Substitution of Equation (8.6.44) into Equation (8.6.42) gives the tension in the string,

$$\begin{aligned} T &= \mu_{\mathrm{k}} m_{\mathrm{C}} g + m_{\mathrm{C}} a \\ &= \mu_{\mathrm{k}} m_{\mathrm{C}} g + m_{\mathrm{C}} \frac{m_{\mathrm{B}} g - \mu_{\mathrm{k}} m_{\mathrm{C}} g}{m_{\mathrm{C}} + m_{\mathrm{B}}} \\ &= (\mu_{\mathrm{k}} + 1) \frac{m_{\mathrm{C}} m_{\mathrm{B}}}{m_{\mathrm{C}} + m_{\mathrm{B}}} g. \end{aligned} \tag{8.6.45}$$

In this example, we applied Newton’s Second Law to two objects, one a composite object consisting of the sensor and the cart, and the other the block. We analyzed the forces acting on each object and also any constraints imposed on the acceleration of each object. We used the force laws for kinetic friction and gravitation on each free-body system. The three equations of motion enable us to determine the forces that depend on the parameters in the example: the tension in the rope, the acceleration of the objects, and normal force between the cart and the table.

Example 8.9 Pulleys and Ropes Constraint Conditions

Consider the arrangement of pulleys and blocks shown in Figure 8.39. The pulleys are assumed massless and frictionless and the connecting strings are massless and inextensible. Denote the respective masses of the blocks as m_1, m_2 and m_3. The upper pulley in the figure is free to rotate but its center of mass does not move. Both pulleys have the same radius R. (a) How are the accelerations of the objects related? (b) Draw force diagrams on each moving object. (c) Solve for the accelerations of the objects and the tensions in the ropes.

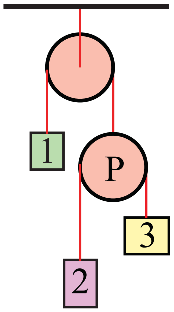


Figure 8.39 Constrained pulley system

Solution: (a) Choose an origin at the center of the upper pulley. Introduce coordinate functions for the three moving blocks, y_1, y_2 and y_3. Introduce a coordinate function y_P for the moving pulley (the pulley on the lower right in Figure 8.40). Choose downward for positive direction; the coordinate system is shown in the figure below then.

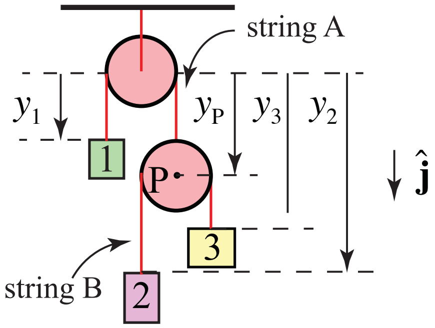


Figure 8.40 Coordinated system for pulley system

The length of string A is given by

$$l_A = y_1 + y_P + \pi R \tag{8.6.46}$$

where πR is the arc length of the rope that is in contact with the pulley. This length is constant, and so the second derivative with respect to time is zero,

$$0 = \frac{d^2 l_A}{dt^2} = \frac{d^2 y_1}{dt^2} + \frac{d^2 y_P}{dt^2} = a_{y,1} + a_{y,P}\,. \tag{8.6.47}$$

Thus block 1 and the moving pulley's components of acceleration are equal in magnitude but opposite in sign,

$$a_{y,P} = -a_{y,1}\,. \tag{8.6.48}$$

The length of string B is given by

$$l_B = (y_3 - y_P) + (y_2 - y_P) + \pi R = y_3 + y_2 - 2y_P + \pi R \tag{8.6.49}$$

where πR is the arc length of the rope that is in contact with the pulley. This length is also constant so the second derivative with respect to time is zero,

$$0 = \frac{d^2 l_B}{dt^2} = \frac{d^2 y_2}{dt^2} + \frac{d^2 y_3}{dt^2} - 2\frac{d^2 y_P}{dt^2} = a_{y,2} + a_{y,3} - 2a_{y,P}\,. \tag{8.6.50}$$

We can substitute Equation (8.6.48) for the pulley acceleration into Equation (8.6.50) yielding the *constraint relation* between the components of the acceleration of the three blocks,

$$0 = a_{y,2} + a_{y,3} + 2a_{y,1}\,. \tag{8.6.51}$$

b) Free-body Force diagrams: the forces acting on block 1 are: the gravitational force $m_1\vec{\mathbf{g}}$ and the pulling force $\vec{\mathbf{T}}_{A,1}$ of string A acting on the block 1. Denote the magnitude of this force by T_A. Because the string is assumed to be massless and the pulley is assumed to be massless and frictionless, the tension T_A in the string is uniform and equal in magnitude to the pulling force of the string on the block. The free-body diagram on block 1 is shown in Figure 8.41(a).

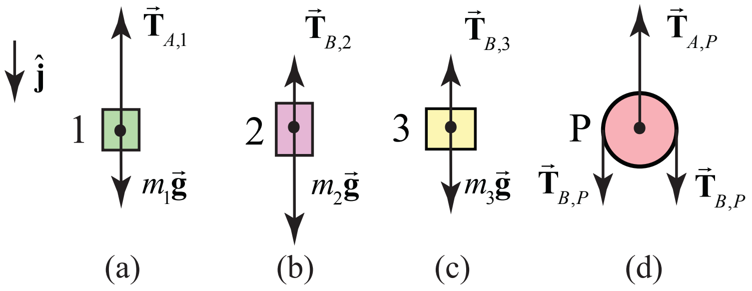


Figure 8.41 Free-body force diagram on (a) block 1; (b) block 2; (c) block 3; (d) pulley

Newton's Second Law applied to block 1 is then

$$\hat{\mathbf{j}}: \ m_1 g - T_A = m_1 \, a_{y,1} \,. \tag{8.6.52}$$

The forces on the block 2 are the gravitational force $m_2 \vec{\mathbf{g}}$ and string B holding the block, $\vec{\mathbf{T}}_{B,2}$, with magnitude T_B. The free-body diagram for the forces acting on block 2 is shown in Figure 8.41(b). Newton's second Law applied to block 2 is

$$\hat{\mathbf{j}}: \ m_2 g - T_B = m_2 \, a_{y,2} \,. \tag{8.6.53}$$

The forces on the block 3 are the gravitational force $m_3 \vec{\mathbf{g}}$ and string holding the block, $\vec{\mathbf{T}}_{B,3}$, with magnitude equal to T_B because pulley P has been assumed to be both frictionless and massless. The free-body diagram for the forces acting on block 3 is shown in Figure 8.41(c). Newton's second Law applied to block 3 is

$$\hat{\mathbf{j}}: \ m_3 g - T_B = m_3 \, a_{y,3} \,. \tag{8.6.54}$$

The forces on the moving pulley P are the gravitational force $m_P \, \vec{\mathbf{g}} = \vec{\mathbf{0}}$ (the pulley is assumed massless); string B pulls down on the pulley on each side with a force, $\vec{\mathbf{T}}_{B,P}$, which has magnitude T_B. String A holds the pulley up with a force $\vec{\mathbf{T}}_{A,P}$ with the magnitude T_A equal to the tension in string A. The free-body diagram for the forces acting on the moving pulley is shown in Figure 8.41(d). Newton's second Law applied to the pulley is

$$\hat{\mathbf{j}}: \ 2T_B - T_A = m_P \, a_{y,P} = 0 \,. \tag{8.6.55}$$

Because the pulley is assumed to be massless, we can use this last equation to determine the condition that the tension in the two strings must satisfy,

$$2T_B = T_A \tag{8.6.56}$$

We are now in position to determine the accelerations of the blocks and the tension in the two strings. We record the relevant equations as a summary.

$$0 = a_{y,2} + a_{y,3} + 2a_{y,1} \tag{8.6.57}$$

$$m_1 g - T_A = m_1 \, a_{y,1} \tag{8.6.58}$$

$$m_2 g - T_B = m_2 \, a_{y,2} \tag{8.6.59}$$

$$m_3 g - T_B = m_3 \, a_{y,3} \tag{8.6.60}$$

$$2T_B = T_A \,. \tag{8.6.61}$$

There are five equations with five unknowns, so we can solve this system. We shall first use Equation (8.6.61) to eliminate the tension T_A in Equation (8.6.58), yielding

$$m_1 g - 2T_B = m_1 a_{y,1} . \tag{8.6.62}$$

We now solve Equations (8.6.59), (8.6.60) and (8.6.62) for the accelerations,

$$a_{y,2} = g - \frac{T_B}{m_2} \tag{8.6.63}$$

$$a_{y,3} = g - \frac{T_B}{m_3} \tag{8.6.64}$$

$$a_{y,1} = g - \frac{2T_B}{m_1} . \tag{8.6.65}$$

We now substitute these results for the accelerations into the constraint equation, Equation (8.6.57),

$$0 = g - \frac{T_B}{m_2} + g - \frac{T_B}{m_3} + 2g - \frac{4T_B}{m_1} = 4g - T_B\left(\frac{1}{m_2} + \frac{1}{m_3} + \frac{4}{m_1}\right). \tag{8.6.66}$$

We can now solve this last equation for the tension in string B ,

$$T_B = \frac{4g}{\left(\frac{1}{m_2} + \frac{1}{m_3} + \frac{4}{m_1}\right)} = \frac{4g\, m_1\, m_2\, m_3}{m_1\, m_3 + m_1 m_2 + 4 m_2\, m_3} . \tag{8.6.67}$$

From Equation (8.6.61), the tension in string A is

$$T_A = 2T_B = \frac{8g\, m_1\, m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4 m_2\, m_3} . \tag{8.6.68}$$

We find the acceleration of block 1 from Equation (8.6.65), using Equation (8.6.67) for the tension in string B,

$$a_{y,1} = g - \frac{2T_B}{m_1} = g - \frac{8g\, m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4 m_2\, m_3} = g\frac{m_1\, m_3 + m_1\, m_2 - 4 m_2\, m_3}{m_1\, m_3 + m_1 m_2 + 4 m_2\, m_3} . \tag{8.6.69}$$

We find the acceleration of block 2 from Equation (8.6.63), using Equation (8.6.67) for the tension in string B,

$$a_{y,2} = g - \frac{T_B}{m_2} = g - \frac{4g\, m_1\, m_3}{m_1\, m_3 + m_1\, m_2 + 4m_2\, m_3} = g\frac{-3m_1\, m_3 + m_1\, m_2 + 4m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4m_2\, m_3}\,. \quad (8.6.70)$$

Similarly, we find the acceleration of block 3 from Equation (8.6.64), using Equation (8.6.67) for the tension in string *B*,

$$a_{y,3} = g - \frac{T_B}{m_3} = g - \frac{4g\, m_1\, m_2}{m_1\, m_3 + m_1\, m_2 + 4m_2\, m_3} = g\frac{m_1\, m_3 - 3m_1 m_2 + 4m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4m_2\, m_3}\,. \quad (8.6.71)$$

As a check on our algebra we note that

$$\begin{aligned} &2a_{1,y} + a_{2,y} + a_{3,y} = \\ &2g\frac{m_1\, m_3 + m_1\, m_2 - 4m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4m_2\, m_3} + g\frac{-3m_1\, m_3 + m_1\, m_2 + 4m_2\, m_3}{m_1\, m_3 + m_1 m_2 + 4m_2\, m_3} + g\frac{m_1\, m_3 - 3m_1\, m_2 + 4m_2\, m_3}{m_1\, m_3 + m_1 m_2 + 4m_2 m_3} \\ &= 0. \end{aligned}$$

Example 8.10 Accelerating Wedge

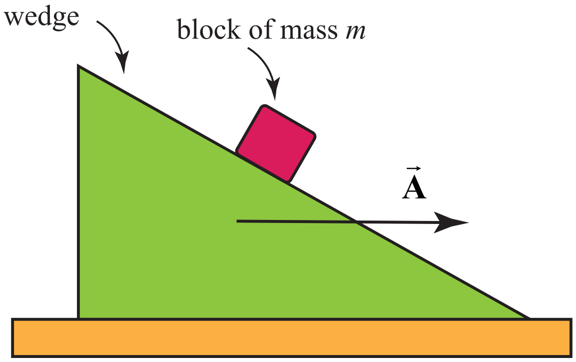


Figure 8.42 Block on accelerating wedge

A 45^o wedge is pushed along a table with constant acceleration $\vec{\mathbf{A}}$ according to an observer at rest with respect to the table. A block of mass m slides without friction down the wedge (Figure 8.42). Find its acceleration with respect to an observer at rest with respect to the table. Write down a plan for finding the magnitude of the acceleration of the block. Make sure you clearly state which concepts you plan to use to calculate any relevant physical quantities. Also clearly state any assumptions you make. Be sure you include any free-body force diagrams or sketches that you plan to use.

Solution: Choose a coordinate system for the block and wedge as shown in Figure 8.43. Then $\vec{\mathbf{A}} = A_{x,w}\,\hat{\mathbf{i}}$ where $A_{x,w}$ is the x-component of the acceleration of the wedge.

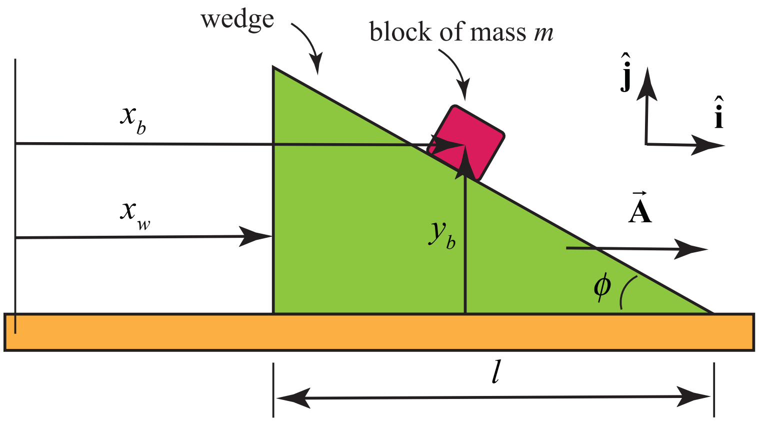


Figure 8.43 Coordinate system for block on accelerating wedge

We shall apply Newton's Second Law to the block sliding down the wedge. Because the wedge is accelerating, there is a constraint relation between the x- and y- components of the acceleration of the block. In order to find that constraint we choose a coordinate system for the wedge and block sliding down the wedge shown in the figure below. We shall find the constraint relationship between the components of the accelerations of the block and wedge by a geometric argument. From the figure above, we see that

$$\tan\phi = \frac{y_b}{l-(x_b - x_w)}. \tag{8.6.72}$$

Therefore

$$y_b = (l-(x_b - x_w))\tan\phi. \tag{8.6.73}$$

If we differentiate Eq. (8.6.73) twice with respect to time noting that

$$\frac{d^2 l}{dt^2} = 0 \tag{8.6.74}$$

we have that

$$\frac{d^2 y_b}{dt^2} = -\left(\frac{d^2 x_b}{dt^2} - \frac{d^2 x_w}{dt^2}\right)\tan\phi. \tag{8.6.75}$$

Therefore

$$a_{b,y} = -(a_{b,x} - A_{x,w})\tan\phi \tag{8.6.76}$$

where

$$A_{x,w} = \frac{d^2 x_w}{dt^2}. \tag{8.6.77}$$

We now draw a free-body force diagram for the block (Figure 8.44). Newton's Second Law in the $\hat{\mathbf{i}}$- direction becomes

$$N\sin\phi = m a_{b,x}. \tag{8.6.78}$$

and the $\hat{\mathbf{j}}$-direction becomes

$$N\cos\phi - mg = ma_{b,y} \tag{8.6.79}$$

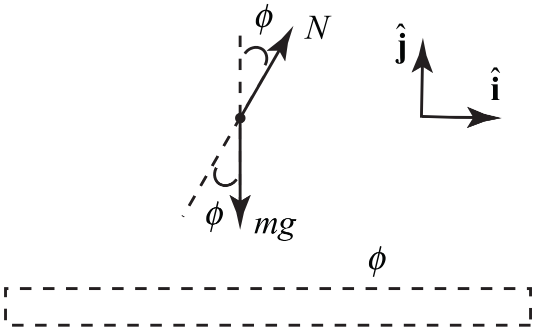


Figure 8.44 Free-body force diagram on block

We can solve for the normal force from Eq. (8.6.78):

$$N = \frac{ma_{b,x}}{\sin\phi} \tag{8.6.80}$$

We now substitute Eq. (8.6.76) and Eq. (8.6.80) into Eq. (8.6.79) yielding

$$\frac{ma_{b,x}}{\sin\phi}\cos\phi - mg = m(-(a_{b,x} - A_{w,x})\tan\phi)\,. \tag{8.6.81}$$

We now clean this up yielding

$$ma_{b,x}(\text{cotan }\phi + \tan\phi) = m(g + A_{w,x}\tan\phi) \tag{8.6.82}$$

Thus the x-component of the acceleration is then

$$a_{b,x} = \frac{g + A_{w,x}\tan\phi}{\text{cotan }\phi + \tan\phi} \tag{8.6.83}$$

From Eq. (8.6.76), the y-component of the acceleration is then

$$a_{b,y} = -(a_{b,x} - A_{w,x})\tan\phi = -\left(\frac{g + A_{w,x}\tan\phi}{\text{cotan }\phi + \tan\phi} - A_{w,x}\right)\tan\phi\,. \tag{8.6.84}$$

This simplifies to

$$a_{b,y} = \frac{A_{w,x} - g\tan\phi}{\text{cotan }\phi + \tan\phi} \tag{8.6.85}$$

When $\phi = 45^\circ$, $\text{cotan } 45^\circ = \tan 45^\circ = 1$, and so Eq. (8.6.83) becomes

$$a_{b,x} = \frac{g + A_{w,x}}{2} \tag{8.6.86}$$

and Eq. (8.6.85) becomes

$$a_{b,y} = \frac{A - g}{2}. \tag{8.6.87}$$

The magnitude of the acceleration is then

$$a = \sqrt{a_{b,x}{}^2 + a_{b,y}{}^2} = \sqrt{\left(\frac{g + A_{w,x}}{2}\right)^2 + \left(\frac{A_{w,x} - g}{2}\right)^2} \tag{8.6.88}$$

$$a = \sqrt{\left(\frac{g^2 + A_{w,x}^2}{2}\right)}.$$

Example 8.11: Capstan

A device called a capstan is used aboard ships in order to control a rope that is under great tension. The rope is wrapped around a fixed drum of radius R, usually for several turns (Figure 8.45 shows about three fourths turn as seen from overhead). The load on the rope pulls it with a force T_A, and the sailor holds the other end of the rope with a much smaller force T_B. The coefficient of static friction between the rope and the drum is μ_s. The sailor is holding the rope so that it is just about to slip. Show that $T_B = T_A e^{-\mu_s \theta_{BA}}$, where θ_{BA} is the angle subtended by the rope on the drum.

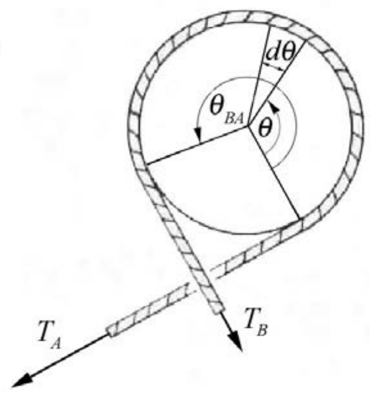


Figure 8.45 Capstan

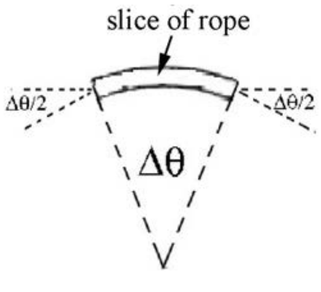


Figure 8.46 Small slice of rope

Solution: We begin by considering a small slice of rope of arc length $R\Delta\theta$, shown in the Figure 8.46. We choose unit vectors for the force diagram on this section of the rope and indicate them on Figure 8.47. The right edge of the slice is at angle θ and the left edge of the slice is at $\theta + \Delta\theta$. The angle edge end of the slice makes with the horizontal is $\Delta\theta / 2$. There are four forces acting on this section of the rope. The forces are the normal force between the capstan and the rope pointing outward, a static frictional force and the tensions at either end of the slice. The rope is held at the just slipping point, so if the load exerts a greater force the rope will slip to the right. Therefore the direction of the static frictional force between the capstan and the rope, acting on the rope, points to the left. The tension on the right side of the slice is denoted by $T(\theta) \equiv T$, while the tension on the left side of the slice is denoted by $T(\theta + \Delta\theta) \equiv T + \Delta T$. Does the tension in this slice from the right side to the left, increase, remain the same, or decrease? The tension decreases because the load on the left side is less than the load on the right side. Note that $\Delta T < 0$.

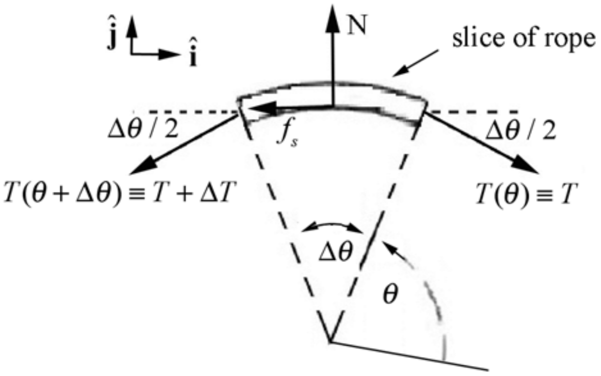


Figure 8.47 Free-body force diagram on small slice of rope

The vector decomposition of the forces is given by

$$\hat{\mathbf{i}} : T\cos(\Delta\theta / 2) - f_s - (T + \Delta T)\cos(\Delta\theta / 2) \tag{8.6.89}$$

$$\hat{\mathbf{j}} : -T\sin(\Delta\theta / 2) + N - (T + \Delta T)\sin(\Delta\theta / 2) . \tag{8.6.90}$$

For small angles $\Delta\theta$, $\cos(\Delta\theta / 2) \cong 1$ and $\sin(\Delta\theta / 2) \cong \Delta\theta / 2$. Using the small angle approximations, the vector decomposition of the forces in the x-direction (the $+\hat{\mathbf{i}}$-direction) becomes

$$\begin{aligned} T\cos(\Delta\theta / 2) - f_s - (T + \Delta T)\cos(\Delta\theta / 2) &\simeq T - f_s - (T + \Delta T) \\ &= -f_s - \Delta T \quad . \end{aligned} \tag{8.6.91}$$

By the static equilibrium condition the sum of the x-components of the forces is zero,

$$-f_s - \Delta T = 0 . \tag{8.6.92}$$

The vector decomposition of the forces in the y -direction (the $+\hat{\mathbf{j}}$ -direction) is approximately

$$\begin{aligned}-T\sin(\Delta\theta/2)+N-(T+\Delta T)\sin(\Delta\theta/2) &\simeq -T\Delta\theta/2+N-(T+\Delta T)\Delta\theta/2 \\ &= -T\Delta\theta+N-\Delta T\Delta\theta/2 \quad .\end{aligned} \tag{8.6.93}$$

In the last equation above we can ignore the terms proportional to $\Delta T\,\Delta\theta$ because these are the product of two small quantities and hence are much smaller than the terms proportional to either ΔT or $\Delta\theta$. The vector decomposition in the y -direction becomes

$$-T\Delta\theta+N\,. \tag{8.6.94}$$

Static equilibrium implies that this sum of the y -components of the forces is zero,

$$-T\Delta\theta+N=0\,. \tag{8.6.95}$$

We can solve this equation for the magnitude of the normal force

$$N=T\Delta\theta\,. \tag{8.6.96}$$

The just slipping condition is that the magnitude of the static friction attains its maximum value

$$f_{\mathrm{s}}=(f_{\mathrm{s}})_{\max}=\mu_{\mathrm{s}}N\,. \tag{8.6.97}$$

We can now combine the Equations (8.6.92) and (8.6.97) to yield

$$\Delta T=-\mu_s N\,. \tag{8.6.98}$$

Now substitute the magnitude of the normal force, Equation (8.6.96), into Equation (8.6.98), yielding

$$-\mu_s T\Delta\theta-\Delta T=0\,. \tag{8.6.99}$$

Finally, solve this equation for the ratio of the change in tension to the change in angle,

$$\frac{\Delta T}{\Delta\theta}=-\mu_{\mathrm{s}}T\,. \tag{8.6.100}$$

The derivative of tension with respect to the angle θ is defined to be the limit

$$\frac{dT}{d\theta}\equiv\lim_{\Delta\theta\to 0}\frac{\Delta T}{\Delta\theta}, \tag{8.6.101}$$

and Equation (8.6.100) becomes

$$\frac{dT}{d\theta} = -\mu_s T\,. \tag{8.6.102}$$

This is an example of a first order linear differential equation that shows that the rate of change of tension with respect to the angle θ is proportional to the negative of the tension at that angle θ. This equation can be solved by integration using the technique of separation of variables. We first rewrite Equation (8.6.102) as

$$\frac{dT}{T} = -\mu_s\, d\theta\,. \tag{8.6.103}$$

Integrate both sides, noting that when $\theta = 0$, the tension is equal to force of the load T_A, and when angle $\theta = \theta_{A,B}$ the tension is equal to the force T_B the sailor applies to the rope,

$$\int_{T=T_A}^{T=T_B} \frac{dT}{T} = -\int_{\theta=0}^{\theta=\theta_{BA}} \mu_s\, d\theta\,. \tag{8.6.104}$$

The result of the integration is

$$\ln\left(\frac{T_B}{T_A}\right) = -\mu_s \theta_{BA}\,. \tag{8.6.105}$$

Note that the exponential of the natural logarithm

$$\exp\left(\ln\left(\frac{T_B}{T_A}\right)\right) = \frac{T_B}{T_A}, \tag{8.6.106}$$

so exponentiating both sides of Equation (8.6.105) yields

$$\frac{T_B}{T_A} = e^{-\mu_s \theta_{BA}}\,; \tag{8.6.107}$$

the tension decreases exponentially,

$$T_B = T_A e^{-\mu_s \theta_{BA}}\,, \tag{8.6.108}$$

Because the tension decreases exponentially, the sailor need only apply a small force to prevent the rope from slipping.

Example 8.12 Free Fall with Air Drag

Consider an object of mass *m* that is in free fall but experiencing air resistance. The magnitude of the drag force is given by Eq. (8.6.1), where ρ is the density of air, A is the cross-sectional area of the object in a plane perpendicular to the motion, and C_D is the drag coefficient. Assume that the object is released from rest and very quickly attains speeds in which Eq. (8.6.1) applies. Determine (i) the terminal velocity, and (ii) the velocity of the object as a function of time.

Solution: Choose positive y-direction downwards with the origin at the initial position of the object as shown in Figure 8.48(a).

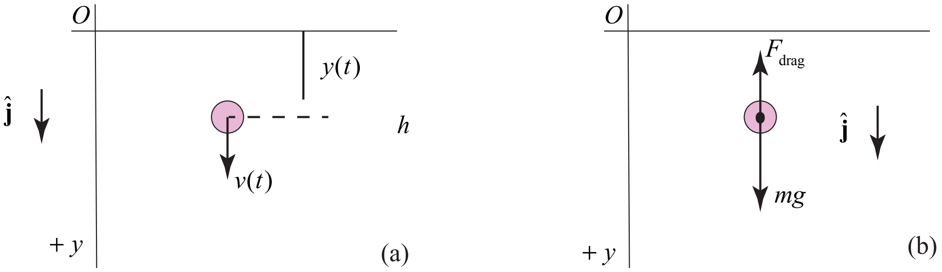


Figure 8.48 (a) Coordinate system for marble; (b) free body force diagram on marble

There are two forces acting on the object: the gravitational force, and the drag force which is given by Eq. (8.6.1). The free body diagram is shown in the Figure 8.48(b). Newton's Second Law is then

$$mg - (1/2)C_D A\rho v^2 = m\frac{dv}{dt}, \tag{8.6.109}$$

Set $\beta = (1/2)C_D A\rho$. Newton's Second Law can then be written as

$$mg - \beta v^2 = m\frac{dv}{dt}. \tag{8.6.110}$$

Initially when the object is just released with $v = 0$, the air drag is zero and the acceleration dv/dt is maximum. As the object increases its velocity, the air drag becomes larger and dv/dt decreases until the object reaches terminal velocity and $dv/dt = 0$. Set $dv/dt = 0$ in Eq. (8.6.15) and solve for the terminal velocity yielding.

$$v_\infty = \sqrt{\frac{mg}{\beta}} = \sqrt{\frac{2mg}{C_D A\rho}}. \tag{8.6.111}$$

Values for the magnitude of the terminal velocity is shown in Table 8.3 for a variety of objects with the same drag coefficient $C_D = 0.5$.

Table 8.3 Terminal Velocities for Different Sized Objects with $C_D = 0.5$

Object	Mass m (kg)	Area A (m^2)	Terminal Velocity v_∞ $(\mathrm{m \cdot s^{-1}})$
Rain drop	4×10^{-6}	3×10^{-6}	6.5
Hailstone	4×10^{-3}	3×10^{-4}	20
Osprey	20	2.5×10^{-1}	50
Human Being	7.5×10^{1}	6×10^{-1}	60

In order to integrate Eq. (8.6.15), we shall apply the technique of separation of variables and integration by partial fractions. First rewrite Eq. (8.6.15) as

$$\frac{-\beta}{m}dt = \frac{dv}{\left(v^2 - \dfrac{mg}{\beta}\right)} = \frac{dv}{\left(v^2 - v_\infty^2\right)} = \left(-\frac{1}{2v_\infty(v+v_\infty)} + \frac{1}{2v_\infty(v-v_\infty)}\right)dv\,. \qquad (8.6.112)$$

An integral expression of Eq. (8.6.112) is then

$$-\int_{v'=0}^{v'=v(t)} \frac{dv'}{2v_\infty(v'+v_\infty)} + \int_{v'=0}^{v'=v(t)} \frac{dv'}{2v_\infty(v'-v_\infty)} = -\frac{\beta}{m}\int_{t'=0}^{t'=t} dt'\,. \qquad (8.6.113)$$

Integration yields

$$\begin{gathered} -\int_{v'=0}^{v'=v(t)} \frac{dv'}{2v_\infty(v'+v_\infty)} + \int_{v'=0}^{v'=v(t)} \frac{dv'}{2v_\infty(v'-v_\infty)} = -\frac{\beta}{m}\int_{t'=0}^{t'=t} dt' \\ \frac{1}{2v_\infty}\left(-\ln\left(\frac{v(t)+v_\infty}{v_\infty}\right) + \ln\left(\frac{v_\infty - v(t)}{v_\infty}\right)\right) = -\frac{\beta}{m}t \end{gathered}\,. \qquad (8.6.114)$$

After some algebraic manipulations, Eq. (8.6.114) can be rewritten as

$$\ln\left(\frac{v_\infty - v(t)}{v(t)+v_\infty}\right) = -\frac{2v_\infty\beta}{m}t \qquad (8.6.115)$$

Exponentiate Eq. (8.6.115) yields

$$\left(\frac{v_\infty - v(t)}{v(t)+v_\infty}\right) = e^{-\frac{2v_\infty\beta}{m}t}\,. \qquad (8.6.116)$$

After some algebraic rearrangement the y-component of the velocity as a function of time is given by

$$v(t) = v_\infty \left(\frac{1 - e^{-\frac{2v_\infty \beta}{m}t}}{1 + e^{-\frac{2v_\infty \beta}{m}t}} \right) = v_\infty \tan h \left(\frac{v_\infty \beta}{m} t \right). \tag{8.6.117}$$

where $\dfrac{v_\infty \beta}{m} = \dfrac{\beta}{m}\sqrt{\dfrac{mg}{\beta}} = \sqrt{\dfrac{\beta g}{m}} = \sqrt{\dfrac{(1/2)C_D A \rho g}{m}}$.

Chapter 9 Circular Motion Dynamics

Chapter 9 Equation Chapter 9 Section 1Circular Motion Dynamics

I shall now recall to mind that the motion of the heavenly bodies is circular, since the motion appropriate to a sphere is rotation in a circle.[1]

Nicholas Copernicus

9.1 Introduction Newton's Second Law and Circular Motion

We have already shown that when an object moves in a circular orbit of radius r with angular velocity $\vec{\boldsymbol{\omega}}$, it is most convenient to choose polar coordinates to describe the position, velocity and acceleration vectors. In particular, the acceleration vector is given by

$$\vec{\mathbf{a}}(t) = -r\left(\frac{d\theta}{dt}\right)^2 \hat{\mathbf{r}}(t) + r\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}}(t). \tag{9.1.1}$$

Then Newton's Second Law, $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, can be decomposed into radial ($\hat{\mathbf{r}}$-) and tangential ($\hat{\boldsymbol{\theta}}$-) components

$$F_r = -mr\left(\frac{d\theta}{dt}\right)^2 \text{ (circular motion)}, \tag{9.1.2}$$

$$F_\theta = mr\frac{d^2\theta}{dt^2} \quad \text{(circular motion)}. \tag{9.1.3}$$

For the special case of uniform circular motion, $d^2\theta / dt^2 = 0$, and so the sum of the tangential components of the force acting on the object must therefore be zero,

$$F_\theta = 0 \quad \text{(uniform circular motion)}. \tag{9.1.4}$$

9.2 Universal Law of Gravitation and the Circular Orbit of the Moon

An important example of (approximate) circular motion is the orbit of the Moon around the Earth. We can approximately calculate the time T the Moon takes to complete one circle around the earth (a calculation of great importance to early lunar calendar systems, which became the basis for our current model.) Denote the distance from the moon to the center of the earth by $R_{\text{e, m}}$.

---

[1] *Dedicatory Letter to Pope Paul III.*

Because the Moon moves nearly in a circular orbit with angular speed $\omega = 2\pi / T$ it is accelerating towards the Earth. The radial component of the acceleration (centripetal acceleration) is

$$a_r = -\frac{4\pi^2 R_{\mathrm{e,\,m}}}{T^2}. \tag{9.2.1}$$

According to Newton's Second Law, there must be a centripetal force acting on the Moon directed towards the center of the Earth that accounts for this inward acceleration.

9.2.1 Universal Law of Gravitation

Newton's Universal Law of Gravitation describes the gravitational force between two bodies 1 and 2 with masses m_1 and m_2 respectively. This force is a radial force (always pointing along the radial line connecting the masses) and the magnitude is proportional to the inverse square of the distance that separates the bodies. Then the force on object 2 due to the gravitational interaction between the bodies is given by,

$$\vec{\mathbf{F}}_{1,2} = -G\frac{m_1 m_2}{r_{1,2}^2}\hat{\mathbf{r}}_{1,2}, \tag{9.2.2}$$

where $r_{1,2}$ is the distance between the two bodies and $\hat{\mathbf{r}}_{1,2}$ is the unit vector located at the position of object 2 and pointing from object 1 towards object 2. The Universal Gravitation Constant is $G = 6.67\times10^{-11}\mathrm{N}\cdot\mathrm{m}^2\cdot\mathrm{kg}^{-2}$. Figure 9.1 shows the direction of the forces on bodies 1 and 2 along with the unit vector $\hat{\mathbf{r}}_{1,2}$.

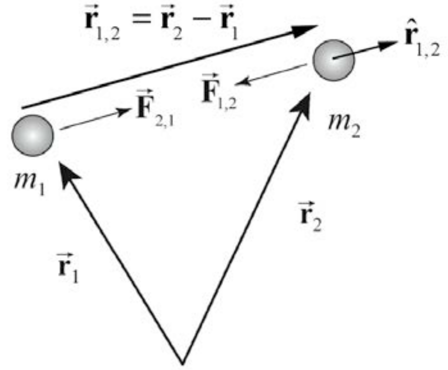


Figure 9.1 Gravitational force of interaction between two bodies

Newton realized that there were still some subtleties involved. First, why should the mass of the Earth act as if it were all placed at the center? Newton showed that for a perfect sphere with uniform mass distribution, all the mass may be assumed to be located at the center. (This calculation is difficult and can be found in Appendix 9A to this chapter.) We assume for the present calculation that the Earth and the Moon are perfect spheres with uniform mass distribution.

Second, does this gravitational force between the Earth and the Moon form an action-reaction Third Law pair? When Newton first explained the Moon's motion in 1666, he had still not formulated the Third Law, which accounted for the long delay in the publication of the *Principia*. The link between the concept of force and the concept of an action-reaction pair of forces was the last piece needed to solve the puzzle of the effect of gravity on planetary orbits. Once Newton realized that the gravitational force between any two bodies forms an action-reaction pair, and satisfies both the Universal Law of Gravitation and his newly formulated Third Law, he was able to solve the oldest and most important physics problem of his time, the motion of the planets.

The test for the Universal Law of Gravitation was performed through experimental observation of the motion of planets, which turned out to be resoundingly successful. For almost 200 years, Newton's Universal Law was in excellent agreement with observation. A sign of more complicated physics ahead, the first discrepancy only occurred when a slight deviation of the motion of Mercury was experimentally confirmed in 1882. The prediction of this deviation was the first success of Einstein's Theory of General Relativity (formulated in 1915).

We can apply this Universal Law of Gravitation to calculate the period of the Moon's orbit around the Earth. The mass of the Moon is $m_1 = 7.36\times10^{22}\ \text{kg}$ and the mass of the Earth is $m_2 = 5.98\times10^{24}\ \text{kg}$. The distance from the Earth to the Moon is $R_{\text{e, m}} = 3.82\times10^{8}\ \text{m}$. We show the force diagram in Figure 9.2.

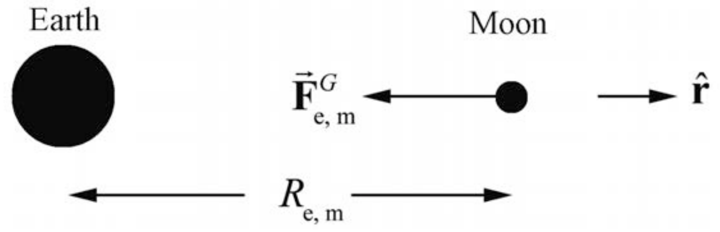


Figure 9.2 Gravitational force of moon

Newton's Second Law of motion for the radial direction becomes

$$-G\frac{m_1\, m_2}{R_{\text{e,m}}^2} = -m_1\frac{4\pi^2 R_{\text{e,m}}}{T^2}\,. \qquad (9.2.3)$$

We can solve this equation for the period of the orbit,

$$T = \sqrt{\frac{4\pi^2 R_{\mathrm{e,m}}^3}{G m_2}} \,. \tag{9.2.4}$$

Substitute the given values for the radius of the orbit, the mass of the earth, and the universal gravitational constant. The period of the orbit is

$$T = \sqrt{\frac{4\pi^2 (3.82\times 10^8 \text{ m})^3}{(6.67\times 10^{-11} \text{ N}\cdot\text{m}^2\cdot\text{kg}^{-2})(5.98\times 10^{24} \text{ kg})}} = 2.35\times 10^6 \text{ s} \,. \tag{9.2.5}$$

This period is given in days by

$$T = (2.35\times 10^6 \text{ s})\left(\frac{1 \text{ day}}{8.64\times 10^4 \text{ s}}\right) = 27.2 \text{ days}. \tag{9.2.6}$$

This period is called the *sidereal month* because it is the time that it takes for the Moon to return to a given position with respect to the stars.

The actual time T_1 between full moons, called the *synodic month* (the average period of the Moon's revolution with respect to the earth and is 29.53 days, it may range between 29.27 days and 29.83 days), is longer than the sidereal month because the Earth is traveling around the Sun. So for the next full moon, the Moon must travel a little farther than one full circle around the Earth in order to be on the other side of the Earth from the Sun (Figure 9.3).

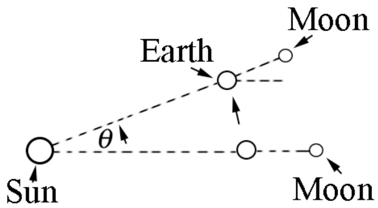


Figure 9.3: Orbital motion between full moons

Therefore the time T_1 between consecutive full moons is approximately $T_1 \simeq T + \Delta T$ where $\Delta T \simeq T / 12 = 2.3 \text{ days}$. So $T_1 \simeq 29.5 \text{ days}$.

9.2.2 Kepler's Third Law and Circular Motion

The first thing that we notice from the above solution is that the period does not depend on the mass of the Moon. We also notice that the square of the period is proportional to the cube of the distance between the Earth and the Moon,

$$T^2 = \frac{4\pi^2 R_{\text{e,m}}^3}{G m_2}. \tag{9.2.7}$$

This is an example of Kepler's Third Law, of which Newton was aware. This confirmation was convincing evidence to Newton that his Universal Law of Gravitation was the correct mathematical description of the gravitational force law, even though he still could not explain what "caused" gravity.

9.3 Worked Examples Circular Motion

Example 9.1 Geosynchronous Orbit

A geostationary satellite goes around the earth once every 23 hours 56 minutes and 4 seconds, (a sidereal day, shorter than the noon-to-noon solar day of 24 hours) so that its position appears stationary with respect to a ground station. The mass of the earth is $m_{\text{e}} = 5.98 \times 10^{24}\ \text{kg}$. The mean radius of the earth is $R_{\text{e}} = 6.37 \times 10^{6}\,\text{m}$. The universal constant of gravitation is $G = 6.67 \times 10^{-11}\,\text{N} \cdot \text{m}^2 \cdot \text{kg}^{-2}$. What is the radius of the orbit of a geostationary satellite? Approximately how many earth radii is this distance?

Solution: The satellite's motion can be modeled as uniform circular motion. The gravitational force between the earth and the satellite keeps the satellite moving in a circle (In Figure 9.4, the orbit is close to a scale drawing of the orbit). The acceleration of the satellite is directed towards the center of the circle, that is, along the radially inward direction.

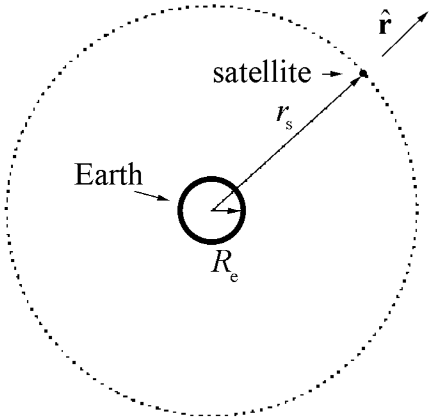


Figure 9.4 Geostationary satellite orbit (close to a scale drawing of orbit).

Choose the origin at the center of the earth, and the unit vector $\hat{\mathbf{r}}$ along the radial direction. This choice of coordinates makes sense in this problem since the direction of acceleration is along the radial direction.

Let $\vec{\mathbf{r}}$ be the position vector of the satellite. The magnitude of $\vec{\mathbf{r}}$ (we denote it as r_{s}) is the distance of the satellite from the center of the earth, and hence the radius of its circular orbit. Let ω be the angular velocity of the satellite, and the period is $T = 2\pi / \omega$. The acceleration is directed inward, with magnitude $r_{\mathrm{s}}\omega^2$; in vector form,

$$\vec{\mathbf{a}} = -r_s\omega^2\hat{\mathbf{r}}. \tag{9.3.1}$$

Apply Newton's Second Law to the satellite for the radial component. The only force in this direction is the gravitational force due to the Earth,

$$\vec{\mathrm{F}}_{\mathrm{grav}} = -m_{\mathrm{s}}\omega^2 r_{\mathrm{s}}\,\hat{\mathbf{r}}. \tag{9.3.2}$$

The inward radial force on the satellite is the gravitational attraction of the earth,

$$-G\frac{m_{\mathrm{s}}m_{\mathrm{e}}}{r_{\mathrm{s}}^2}\hat{\mathbf{r}} = -m_{\mathrm{s}}\omega^2 r_{\mathrm{s}}\,\hat{\mathbf{r}}. \tag{9.3.3}$$

Equating the $\hat{\mathbf{r}}$ components,

$$G\frac{m_{\mathrm{s}}m_{\mathrm{e}}}{r_{\mathrm{s}}^2} = m_{\mathrm{s}}\omega^2 r_{\mathrm{s}}. \tag{9.3.4}$$

Solving for the radius of orbit of the satellite r_{s},

$$r_{\mathrm{s}} = \left(\frac{G\,m_{\mathrm{e}}}{\omega^2}\right)^{1/3}. \tag{9.3.5}$$

The period T of the satellite's orbit in seconds is $86164\ \mathrm{s}$ and so the angular speed is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{86164\ \mathrm{s}} = 7.2921\times10^{-5}\ \mathrm{s}^{-1}. \tag{9.3.6}$$

Using the values of ω, G and m_{e} in Equation (9.3.5), we determine r_{s},

$$r_{\mathrm{s}} = 4.22\times10^7\,\mathrm{m} = 6.62\ R_{\mathrm{e}}. \tag{9.3.7}$$

Example 9.2 Double Star System

Consider a double star system under the influence of gravitational force between the stars. Star 1 has mass m_1 and star 2 has mass m_2. Assume that each star undergoes uniform circular motion such that the stars are always a fixed distance s apart (rotating counterclockwise in Figure 9.5). What is the period of the orbit?

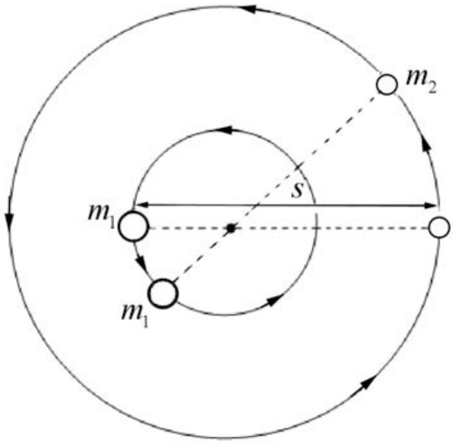


Figure 9.5 Two stars undergoing circular orbits about each other

Solution: Because the distance between the two stars doesn't change as they orbit about each other, there is a central point where the lines connecting the two objects intersect as the objects move, as can be seen in the figure above. (We will see later on in the course that central point is the center of mass of the system.) Choose radial coordinates for each star with origin at that central point. Let $\hat{\mathbf{r}}_1$ be a unit vector at Star 1 pointing radially away from the center of mass. The position of object 1 is then $\vec{\mathbf{r}}_1 = r_1\,\hat{\mathbf{r}}_1$, where r_1 is the distance from the central point. Let $\hat{\mathbf{r}}_2$ be a unit vector at Star 2 pointing radially away from the center of mass. The position of object 2 is then $\vec{\mathbf{r}}_2 = r_2\,\hat{\mathbf{r}}_2$, where r_2 is the distance from the central point. Because the distance between the two stars is fixed we have that

$$s = r_1 + r_2 .$$

The coordinate system is shown in Figure 9.6

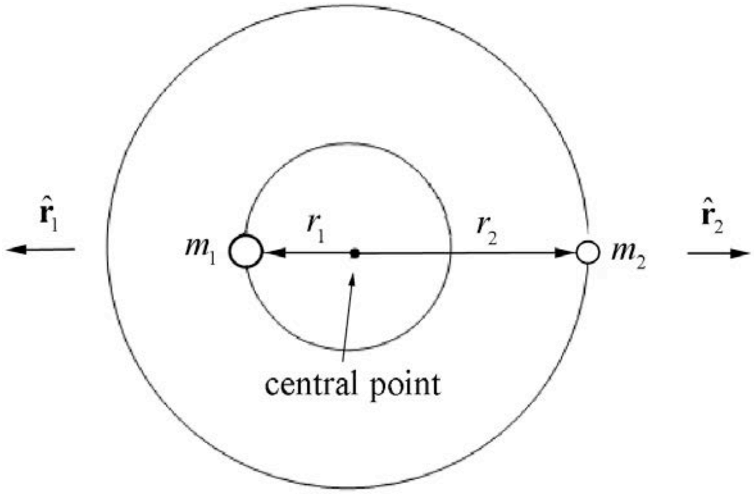


Figure 9.6 Coordinate system for double star orbits

The gravitational force on object 1 is then

$$\vec{\mathbf{F}}_{2,1} = -\frac{Gm_1m_2}{s^2}\hat{\mathbf{r}}_1 .$$

The gravitational force on object 2 is then

$$\vec{\mathbf{F}}_{1,2} = -\frac{Gm_1m_2}{s^2}\hat{\mathbf{r}}_2 .$$

The force diagrams on the two stars are shown in Figure 9.7.

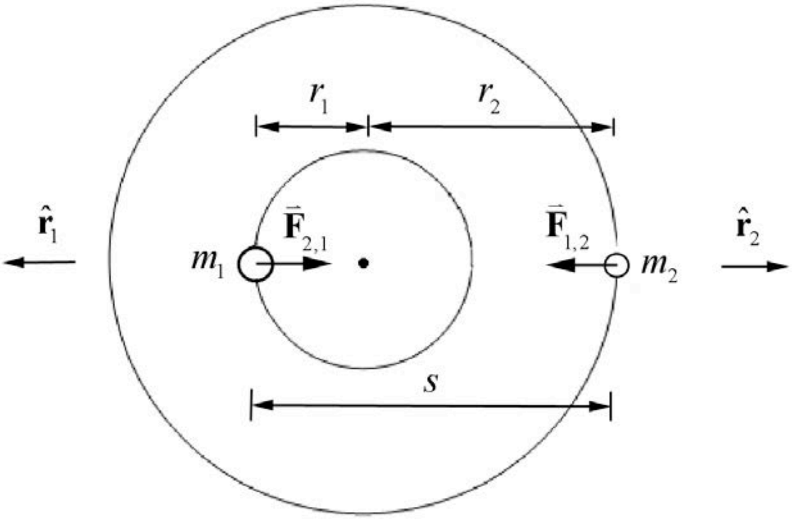


Figure 9.7 Force diagrams on objects 1 and 2

Let ω denote the magnitude of the angular velocity of each star about the central point. Then Newton's Second Law, $\vec{\mathbf{F}}_1 = m_1\vec{\mathbf{a}}_1$, for Star 1 in the radial direction $\hat{\mathbf{r}}_1$ is

$$-G\frac{m_1 m_2}{s^2} = -m_1 r_1 \omega^2 .$$

We can solve this for r_1,

$$r_1 = G\frac{m_2}{\omega^2 s^2} .$$

Newton's Second Law, $\vec{\mathbf{F}}_2 = m_2\vec{\mathbf{a}}_2$, for Star 2 in the radial direction $\hat{\mathbf{r}}_2$ is

$$-G\frac{m_1 m_2}{s^2} = -m_2 r_2 \omega^2 .$$

We can solve this for r_2,

$$r_2 = G\frac{m_1}{\omega^2 s^2} .$$

Because s, the distance between the stars, is constant

$$s = r_1 + r_2 = G\frac{m_2}{\omega^2 s^2} + G\frac{m_1}{\omega^2 s^2} = G\frac{(m_2 + m_1)}{\omega^2 s^2} .$$

Thus the magnitude of the angular velocity is

$$\omega = \left(G\frac{(m_2 + m_1)}{s^3} \right)^{1/2} ,$$

and the period is then

$$T = \frac{2\pi}{\omega} = \left(\frac{4\pi^2 s^3}{G(m_2 + m_1)} \right)^{1/2} . \qquad (9.3.8)$$

Note that both masses appear in the above expression for the period unlike the expression for Kepler's Law for circular orbits. Eq. (9.2.7). The reason is that in the argument leading up to Eq. (9.2.7), we assumed that $m_1 << m_2$, this was equivalent to assuming that the central point was located at the center of the Earth. If we used Eq. (9.3.8) instead we would find that the orbital period for the circular motion of the Earth and moon about each other is

$$T = \sqrt{\frac{4\pi^2 (3.82\times 10^8 \text{ m})^3}{(6.67\times 10^{-11} \text{ N}\cdot\text{m}^2\cdot\text{kg}^{-2})(5.98\times 10^{24} \text{ kg} + 7.36\times 10^{22} \text{ kg})}} = 2.33\times 10^6 \text{ s} ,$$

which is $1.43\times 10^4 \text{ s} = 0.17 \text{ d}$ shorter than our previous calculation.

Example 9.3 Rotating Objects

Two objects 1 and 2 of mass m_1 and m_2 are whirling around a shaft with a constant angular velocity ω. The first object is a distance d from the central axis, and the second object is a distance $2d$ from the axis (Figure 9.8). You may ignore the mass of the strings and neglect the effect of gravity. (a) What is the tension in the string between the inner object and the outer object? (b) What is the tension in the string between the shaft and the inner object?

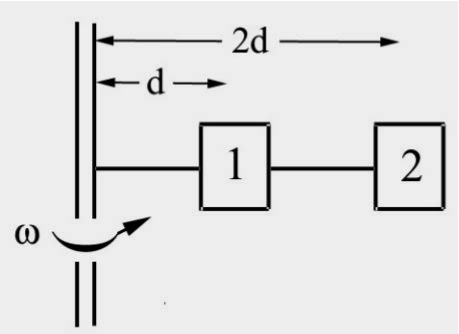
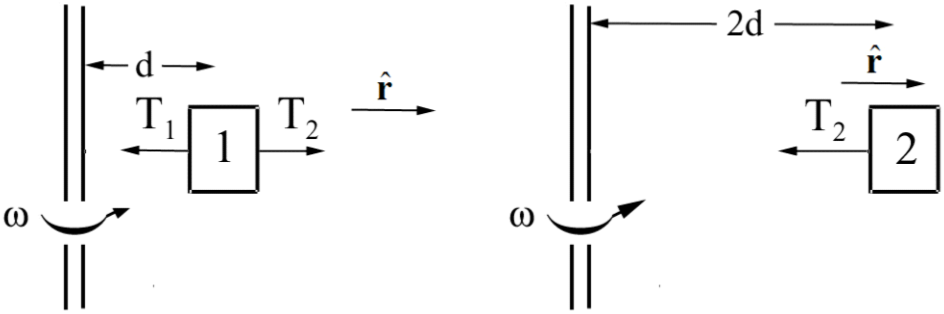


Figure 9.8 Objects attached to a rotating shaft

Solution: We begin by drawing separate force diagrams, Figure 9.9a for object 1 and Figure 9.9b for object 2.



(a) (b)

Figure 9.9 (a) and 9.9 (b) Free-body force diagrams for objects 1 and 2

Newton's Second Law, $\vec{\mathbf{F}}_1 = m_1 \vec{\mathbf{a}}_1$, for the inner object in the radial direction is

$$\hat{\mathbf{r}} : T_2 - T_1 = -m_1 d\omega^2 .$$

Newton's Second Law, $\vec{\mathbf{F}}_2 = m_2 \vec{\mathbf{a}}_2$, for the outer object in the radial direction is

$$\hat{\mathbf{r}} : -T_2 = -m_2 2d\omega^2 .$$

The tension in the string between the inner object and the outer object is therefore

$$T_2 = m_2 2d\omega^2 .$$

Using this result for T_2 in the force equation for the inner object yields

$$m_2 2d\omega^2 - T_1 = -m_1 d\omega^2 ,$$

which can be solved for the tension in the string between the shaft and the inner object

$$T_1 = d\omega^2(m_1 + 2m_2) .$$

Example 9.4 Tension in a Rope

A uniform rope of mass m and length L is attached to shaft that is rotating at constant angular velocity ω. Find the tension in the rope as a function of distance from the shaft. You may ignore the effect of gravitation.

Solution: Divide the rope into small pieces of length Δr, each of mass $\Delta m = (m / L)\Delta r$. Consider the piece located a distance r from the shaft (Figure 9.10).

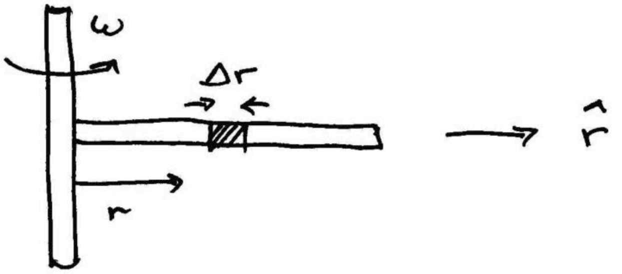


Figure 9.10 Small slice of rotating rope

The radial component of the force on that piece is the difference between the tensions evaluated at the sides of the piece, $F_r = T(r + \Delta r) - T(r)$, (Figure 9.11).

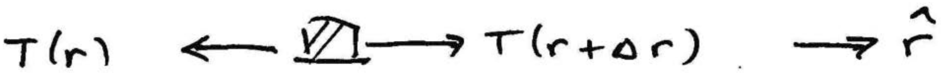


Figure 9.11 Free-body force diagram on small slice of rope

The piece is accelerating inward with a radial component $a_r = -r\omega^2$. Thus Newton's Second Law becomes

$$\begin{aligned} F_r &= -\Delta m \omega^2 r \\ T(r + \Delta r) - T(r) &= -(m / L)\Delta r \, r\omega^2 . \end{aligned} \tag{9.3.9}$$

Denote the difference in the tension by $\Delta T = T(r + \Delta r) - T(r)$. After dividing through by Δr, Eq. (9.3.9) becomes

$$\frac{\Delta T}{\Delta r} = -(m / L)\, r\omega^2 . \tag{9.3.10}$$

In the limit as $\Delta r \to 0$, Eq. (9.3.10) becomes a differential equation,

$$\frac{dT}{dr} = -(m / L)\omega^2 r . \tag{9.3.11}$$

From this, we see immediately that the tension decreases with increasing radius. We shall solve this equation by integration

$$\begin{aligned} T(r) - T(L) &= \int_{r'=L}^{r'=r} \frac{dT}{dr'} dr' = -(m\omega^2 / L)\int_L^r r'\, dr' \\ &= -(m\omega^2 / 2L)(r^2 - L^2) \\ &= (m\omega^2 / 2L)(L^2 - r^2). \end{aligned} \tag{9.3.12}$$

We use the fact that the tension, in the ideal case, will vanish at the end of the rope, $r = L$. Thus,

$$T(r) = (m\omega^2 / 2L)(L^2 - r^2). \tag{9.3.13}$$

This last expression shows the expected functional form, in that the tension is largest closest to the shaft, and vanishes at the end of the rope.

Example 9.5 Object Sliding in a Circular Orbit on the Inside of a Cone

Consider an object of mass m that slides without friction on the inside of a cone moving in a circular orbit with constant speed v_0. The cone makes an angle θ with respect to a vertical axis. The axis of the cone is vertical and gravity is directed downwards. The apex half-angle of the cone is θ as shown in Figure 9.12. Find the radius of the circular path and the time it takes to complete one circular orbit in terms of the given quantities and g.

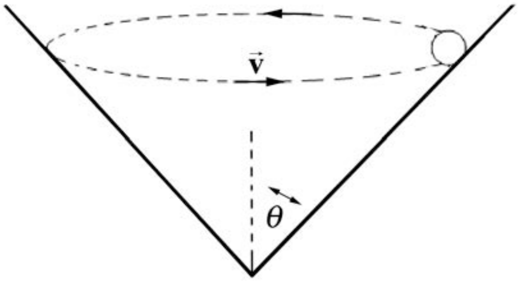


Figure 9.12 Object in a circular orbit on inside of a cone

Solution: Choose cylindrical coordinates as shown in the above figure. Choose unit vectors $\hat{\mathbf{r}}$ pointing in the radial outward direction and $\hat{\mathbf{k}}$ pointing upwards. The force diagram on the object is shown in Figure 9.13.

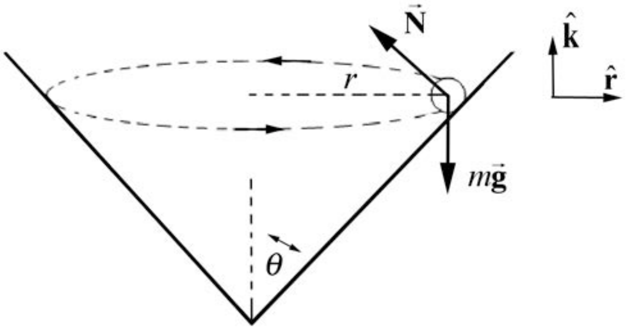


Figure 9.13 Free-body force diagram on object

The two forces acting on the object are the normal force of the wall on the object and the gravitational force. Then Newton's Second Law in the $\hat{\mathbf{r}}$-direction becomes

$$-N\cos\theta = \frac{-mv^2}{r}$$

and in the $\hat{\mathbf{k}}$-direction becomes

$$N\sin\theta - mg = 0.$$

These equations can be re-expressed as

$$N\cos\theta = m\frac{v^2}{r}$$

$$N\sin\theta = mg.$$

We can divide these two equations,

$$\frac{N\sin\theta}{N\cos\theta} = \frac{mg}{mv^2 / r}$$

yielding

$$\tan\theta = \frac{rg}{v^2}.$$

This can be solved for the radius,

$$r = \frac{v^2}{g}\tan\theta.$$

The centripetal force in this problem is the vector component of the contact force that is pointing radially inwards,

$$F_{\text{cent}} = N\cos\theta = mg\cot\theta,$$

where $N\sin\theta = mg$ has been used to eliminate N in terms of m, g and θ. The radius is independent of the mass because the component of the normal force in the vertical direction must balance the gravitational force, and so the normal force is proportional to the mass.

Example 9.6 Coin on a Rotating Turntable

A coin of mass m (which you may treat as a point object) lies on a turntable, exactly at the rim, a distance R from the center. The turntable turns at constant angular speed ω and the coin rides without slipping. Suppose the coefficient of static friction between the turntable and the coin is given by μ. Let g be the gravitational constant. What is the maximum angular speed $\omega_{\max}$ such that the coin does not slip?

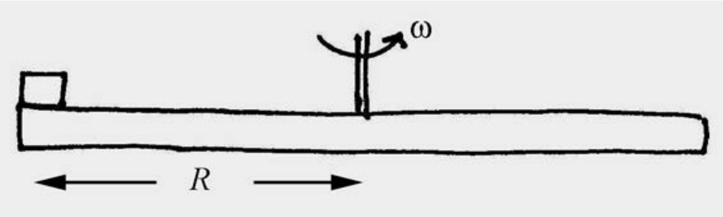


Figure 9.14 Coin on Rotating Turntable

Solution: The coin undergoes circular motion at constant speed so it is accelerating inward. The force inward is static friction and at the just slipping point it has reached its maximum value. We can use Newton's Second Law to find the maximum angular speed $\omega_{\max}$. We choose a polar coordinate system and the free-body force diagram is shown in the figure below.

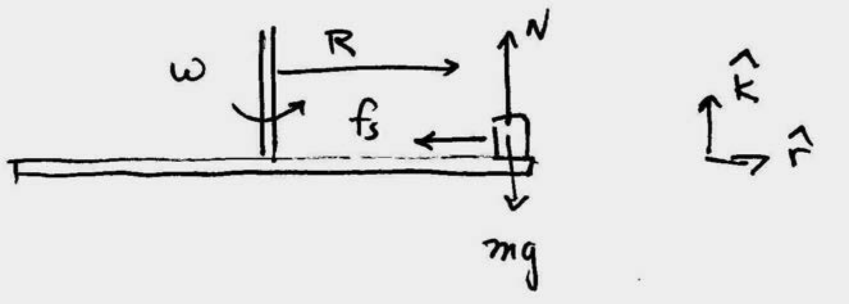


Figure 9.15 Free-body force diagram on coin

The contact force is given by

$$\vec{\mathbf{C}} = \vec{\mathbf{N}} + \vec{\mathbf{f}}_s = N\hat{\mathbf{k}} - f_s\hat{\mathbf{r}}. \tag{9.3.14}$$

The gravitational force is given by

$$\vec{\mathbf{F}}_{grav} = -mg\hat{\mathbf{k}}. \tag{9.3.15}$$

Newton's Second Law in the radial direction is given by

$$-f_{\mathrm{s}} = -m R\omega^2 . \qquad (9.3.16)$$

Newton's Second Law, $F_z = ma_z$, in the z-direction, noting that the disc is static hence $a_z = 0$, is given by

$$N - mg = 0 . \qquad (9.3.17)$$

Thus the normal force is

$$N = mg . \qquad (9.3.18)$$

As ω increases, the static friction increases in magnitude until at $\omega = \omega_{\mathrm{max}}$ and static friction reaches its maximum value (noting Eq. (9.3.18)).

$$(f_{\mathrm{s}})_{\mathrm{max}} = \mu N = \mu mg . \qquad (9.3.19)$$

At this value the disc slips. Thus substituting this value for the maximum static friction into Eq. (9.3.16) yields

$$\mu mg = mR\omega_{\mathrm{max}}^2 . \qquad (9.3.20)$$

We can now solve Eq. (9.3.20) for maximum angular speed ω_{max} such that the coin does not slip

$$\omega_{\mathrm{max}} = \sqrt{\frac{\mu g}{R}} . \qquad (9.3.21)$$

Appendix 9A The Gravitational Field of a Spherical Shell of Matter

When analyzing gravitational interactions between uniform spherical bodies we assumed we could treat each sphere as a point-like mass located at the center of the sphere and then use the Universal Law of Gravitation to determine the force between the two point-like objects. We shall now justify that assumption. For simplicity we only need to consider the interaction between a spherical object and a point-like mass. We would like to determine the gravitational force on the point-like object of mass m_1 due to the gravitational interaction with a solid uniform sphere of mass m_2 and radius R. In order to determine the force law we shall first consider the interaction between the point-like object and a uniform spherical shell of mass m_s and radius R. We will show that:

1) The gravitational force acting on a point-like object of mass m_1 located a distance $r > R$ from the center of a uniform spherical shell of mass m_s and radius R is the same force that would arise if all the mass of the shell were placed at the center of the shell.

2) The gravitational force on an object of mass m_1 placed inside a spherical shell of matter is zero.

The force law summarizes these results:

$$\vec{\mathbf{F}}_{s,1}(r) = \begin{cases} -G\dfrac{m_s m_1}{r^2}\hat{\mathbf{r}}, & r > R \\ \vec{\mathbf{0}}, & r < R \end{cases},$$

where $\hat{\mathbf{r}}$ is the unit vector located at the position of the object and pointing radially away from the center of the shell.

For a uniform spherical distribution of matter, we can divide the sphere into thin shells. Then the force between the point-like object and each shell is the same as if all the mass of the shell were placed at the center of the shell. Then we add up all the contributions of the shells (integration), the spherical distribution can be treated as point-like object located at the center of the sphere justifying our assumption.

Thus it suffices to analyze the case of the spherical shell. We shall first divide the shell into small area elements and calculate the gravitational force on the point-like object due to one element of the shell and then add the forces due to all these elements via integration.

We begin by choosing a coordinate system. Choose our z-axis to be directed from the center of the sphere to the position of the object, at position $\vec{\mathbf{r}} = z\hat{\mathbf{k}}$, so that $z \geq 0$. (Figure 9A.1 shows the object lying outside the shell with $z > R$).

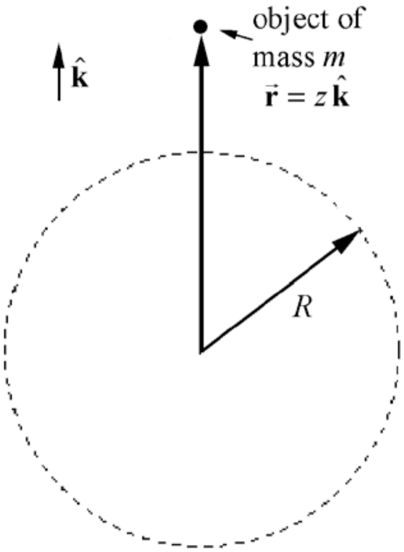


Figure 9A.1 Object lying outside shell with $z > R$.

Choose spherical coordinates as shown in Figure 9A.2.

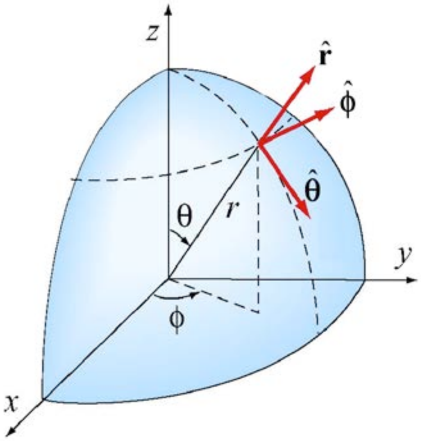


Figure 9A.2 Spherical coordinates

For a point on the surface of a sphere of radius $r = R$, the Cartesian coordinates are related to the spherical coordinates by

$$\begin{aligned} x &= R\sin\theta\cos\phi, \\ y &= R\sin\theta\sin\phi, \\ z &= R\cos\theta, \end{aligned} \tag{9.A.1}$$

where $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$.

Note that the angle θ in Figure 9A.2 and Equations (9.A.1) is not the same as that in plane polar coordinates or cylindrical coordinates. The angle θ is known as the *co-latitude*, the complement of the latitude. We now choose a small area element shown in Figure 9A.3.

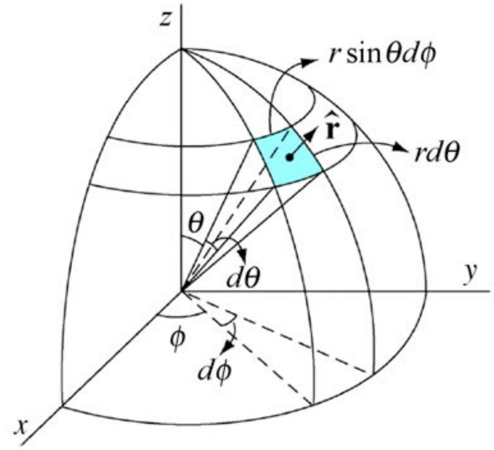


Figure 9A.3 Infinitesimal area element

The infinitesimal area element on the surface of the shell is given by

$$da = R^2 \sin\theta d\theta d\phi \, .$$

Then the mass dm contained in that element is

$$dm = \sigma da = \sigma R^2 \sin\theta d\theta d\phi \, .$$

where σ is the surface mass density given by

$$\sigma = m_s / 4\pi R^2 \, .$$

The gravitational force $\vec{\mathbf{F}}_{dm,m_1}$ on the object of mass m_1 that lies outside the shell due to the infinitesimal piece of the shell (with mass dm) is shown in Figure 9A.4.

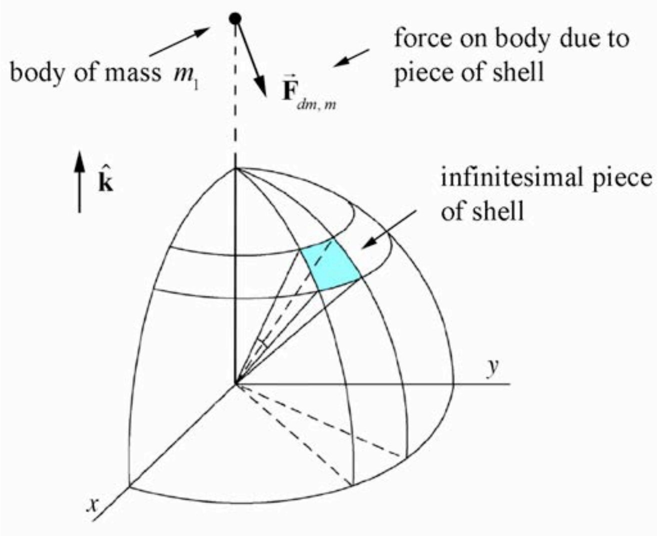


Figure 9A.4 Force on a point-like object due to piece of shell

The contribution from the piece with mass dm to the gravitational force on the object of mass m_1 that lies outside the shell has a component pointing in the negative $\hat{\mathbf{k}}$-direction and a component pointing radially away from the z-axis. By symmetry there is another mass element with the same differential mass $dm' = dm$ on the other side of the shell with same co-latitude θ but with ϕ replaced by $\phi \pm \pi$; this replacement changes the sign of x and y in Equations (9.A.1) but leaves z unchanged. This other mass element produces a gravitational force that exactly cancels the radial component of the force pointing away from the z-axis. Therefore the sum of the forces of these differential mass elements on the object has only a component in the negative $\hat{\mathbf{k}}$-direction (Figure 9A.5)

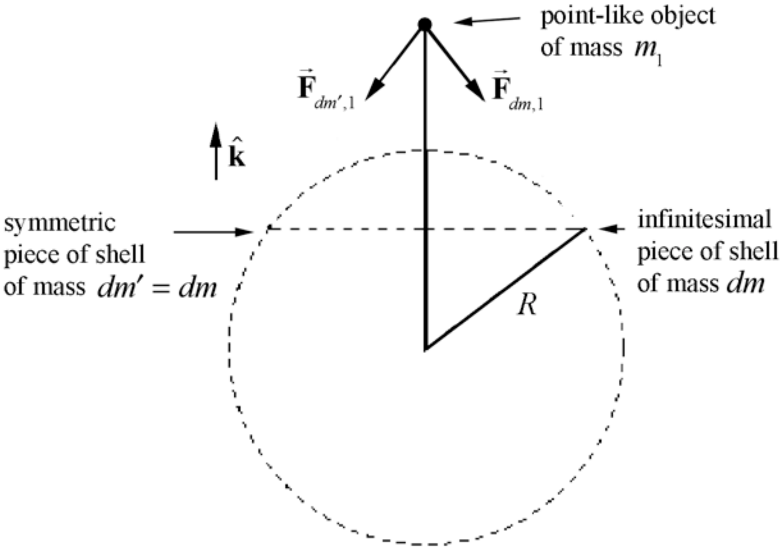


Figure 9A.5 Symmetric cancellation of components of force

Therefore we need only the z-component vector of the force due to the piece of the shell on the point-like object.

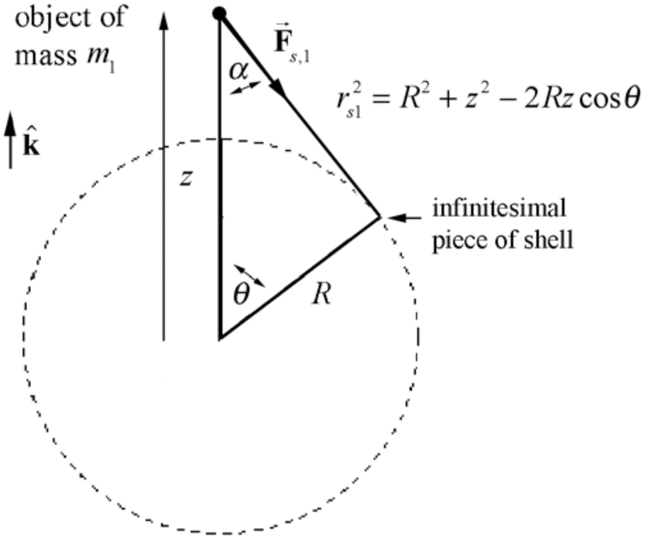


Figure 9A.6 Geometry for calculating the force due to piece of shell.

From the geometry of the set-up (Figure 9A.6) we see that

$$(d\vec{\mathbf{F}}_{s,1})_z \equiv dF_z\hat{\mathbf{k}} = -G\frac{m_1 dm}{r_{s1}^2}\cos\alpha\hat{\mathbf{k}}.$$

Thus

$$dF_z = -G\frac{m_1 dm}{r_{s1}^2}\cos\alpha = -\frac{Gm_s m_1}{4\pi}\frac{\cos\alpha\sin\theta d\theta d\phi}{r_{s1}^2}. \quad (9.A.2)$$

The integral of the force over the surface is then

$$F_z = -Gm_1\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\frac{dm\cos\alpha}{r_{s1}^2} = -\frac{Gm_s m_1}{4\pi}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\frac{\cos\alpha\sin\theta d\theta d\phi}{r_{s1}^2}. \quad (9.A.3)$$

The ϕ-integral is straightforward yielding

$$F_z = -\frac{Gm_s m_1}{2}\int_{\theta=0}^{\theta=\pi}\frac{\cos\alpha\sin\theta d\theta}{r_{s1}{}^2}. \quad (9.A.4)$$

From Figure 9A.6 we can use the law of cosines in two different ways

$$\begin{aligned} r_{s1}^2 &= R^2 + z^2 - 2Rz\cos\theta \\ R^2 &= z^2 + r_{s1}^2 - 2r_{s,1}\,z\cos\alpha. \end{aligned} \tag{9.A.5}$$

Differentiating the first expression in (9.A.5), with R and z constant yields,

$$2r_{s,1}\,dr_{s,1} = 2Rz\sin\theta\,d\theta\,. \tag{9.A.6}$$

Hence

$$\sin\theta\,d\theta = \frac{r_{s,1}}{Rz}dr_{s,1}\,. \tag{9.A.7}$$

and from the second expression in (9.A.5) we have that

$$\cos\alpha = \frac{1}{2zr_{s,1}}\left[(z^2 - R^2) + r_{s1}^2\right]. \tag{9.A.8}$$

We now have everything we need in terms of $r_{s,1}$.

For the case when $z > R$, $r_{s,1}$ varies from $z - R$ to $z + R$. Substituting Equations (9.A.7) and (9.A.8) into Eq. (9.A.3) and using the limits for the definite integral yields

$$\begin{aligned} F_z &= -\frac{Gm_sm_1}{2}\int_{\theta=0}^{\theta=\pi}\frac{\cos\alpha\sin\theta}{r^2_{\ s,1}}d\theta \\ &= -\frac{Gm_sm_1}{2}\frac{1}{2z}\int_{z-R}^{z+R}\frac{1}{r_{s,1}}\left[\left(z^2 - R^2\right) + r^2_{\ s,1}\right]\frac{1}{r^2_{\ s,1}}\frac{r_{s,1}\,dr_{s,1}}{Rz} \\ &= -\frac{Gm_sm_1}{2}\frac{1}{2Rz^2}\left[\left(z^2 - R^2\right)\int_{z-R}^{z+R}\frac{dr_{s,1}}{r^2_{\ s,1}} + \int_{z-R}^{z+R}dr_{s,1}\right]. \end{aligned} \tag{9.A.9}$$

No tables should be needed for these; the result is

$$\begin{aligned} F_z &= -\frac{Gm_sm_1}{2}\frac{1}{2Rz^2}\left[-\frac{\left(z^2 - R^2\right)}{r_{s,1}} + r_{s,1}\right]_{z-R}^{z+R} \\ &= -\frac{Gm_sm_1}{2}\frac{1}{2Rz^2}\left[-\left(z - R\right) + \left(z + R\right) + 2R\right] \\ &= -\frac{Gm_sm_1}{z^2}. \end{aligned} \tag{9.A.10}$$

For the case when $z < R$, $r_{s,1}$ varies from $R - z$ to $R + z$. Then the integral is

$$\begin{aligned}
F_z &= -\frac{Gm_s m_1}{2} \frac{1}{2Rz^2} \left[-\frac{\left(z^2 - R^2\right)}{r_{s,1}} + r_{s,1} \right]_{R-z}^{R+z} \\
&= -\frac{Gm_s m_1}{2} \frac{1}{2Rz^2} \left[-\left(z - R\right) - \left(z + R\right) + 2z \right] \qquad (9.A.11) \\
&= 0.
\end{aligned}$$

So we have demonstrated the proposition that for a point-like object located on the z-axis a distance z from the center of a spherical shell, the gravitational force on the point like object is given by

$$\vec{\mathbf{F}}_{s,1}(r) = \begin{cases} -G\dfrac{m_s m_1}{z^2}\hat{\mathbf{k}}, & z > R \\ \vec{\mathbf{0}}, & z < R \end{cases}.$$

This proves the result that the gravitational force inside the shell is zero and the gravitational force outside the shell is equivalent to putting all the mass at the center of the shell.

Chapter 10 Momentum, System of Particles, and Conservation of Momentum

Chapter 10 Momentum, System of Particles, and Conservation of Momentum

> *Law II: The change of motion is proportional to the motive force impressed, and is made in the direction of the right line in which that force is impressed.*
>
> *If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force is impressed altogether and at once or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added or subtracted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both.*[1]
>
> Isaac Newton *Principia*

10.1 Introduction

When we apply a force to an object, through pushing, pulling, hitting or otherwise, we are applying that force over a discrete interval of time, Δt. During this time interval, the applied force may be constant, or it may vary in magnitude or direction. Forces may also be applied continuously without interruption, such as the gravitational interaction between the earth and the moon. In this chapter we will investigate the relationship between forces and the time during which they are applied, and in the process learn about the quantity of momentum, the principle of conservation of momentum, and its use in solving a new set of problems involving systems of particles.

10.2 Momentum (Quantity of Motion) and Average Impulse

Consider a point-like object (particle) of mass m that is moving with velocity $\vec{\mathbf{v}}$ with respect to some fixed reference frame. The quantity of motion or the **momentum**, $\vec{\mathbf{p}}$, of the object is defined to be the product of the mass and velocity

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}} \,. \tag{10.2.1}$$

Momentum is a reference frame dependent vector quantity, with direction and magnitude. The direction of momentum is the same as the direction of the velocity. The magnitude of the momentum is the product of the mass and the instantaneous speed.

Units: In the SI system of units, momentum has units of $[\mathrm{kg \cdot m \cdot s^{-1}}]$. There is no special name for this combination of units.

---

[1] Isaac Newton. *Mathematical Principles of Natural Philosophy.* Translated by Andrew Motte (1729). Revised by Florian Cajori. Berkeley: University of California Press, 1934. p. 13.

During a time interval Δt, a non-uniform force $\vec{\mathbf{F}}$ is applied to the particle. Because we are assuming that the mass of the point-like object does not change, Newton's Second Law is then

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} = m\frac{d\vec{\mathbf{v}}}{dt} = \frac{d(m\vec{\mathbf{v}})}{dt}. \qquad (10.2.2)$$

Because we are assuming that the mass of the point-like object does not change, the Second Law can be written as

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}. \qquad (10.2.3)$$

The **impulse** of a force acting on a particle during a time interval $[t, t+\Delta t]$ is defined as the definite integral of the force from t to $t+\Delta t$,

$$\vec{\mathbf{I}} = \int_{t'=t}^{t'=t+\Delta t} \vec{\mathbf{F}}(t')\,dt'. \qquad (10.2.4)$$

The SI units of impulse are $[\mathrm{N}\cdot\mathrm{m}] = [\mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{s}^{-1}]$ which are the same units as the units of momentum.

Apply Newton's Second Law in Eq. (10.2.4) yielding

$$\vec{\mathbf{I}} = \int_{t'=t}^{t'=t+\Delta t} \vec{\mathbf{F}}(t')\,dt' = \int_{t'=t}^{t'=t+\Delta t} \frac{d\vec{\mathbf{p}}}{dt'}dt' = \int_{\vec{\mathbf{p}}'=\vec{\mathbf{p}}(t)}^{\vec{\mathbf{p}}'=\vec{\mathbf{p}}(t+\Delta t)} d\vec{\mathbf{p}}' = \vec{\mathbf{p}}(t+\Delta t) - \vec{\mathbf{p}}(t) = \Delta\vec{\mathbf{p}}. \qquad (10.2.5)$$

Eq. (10.2.5) represents the integral version of Newton's Second Law: the impulse applied by a force during the time interval $[t, t+\Delta t]$ is equal to the change in momentum of the particle during that time interval.

The average value of that force during the time interval Δt is given by the integral expression

$$\vec{\mathbf{F}}_{\text{ave}} = \frac{1}{\Delta t}\int_{t'=t}^{t'=t+\Delta t} \vec{\mathbf{F}}(t')\,dt'. \qquad (10.2.6)$$

The product of the average force acting on an object and the time interval over which it is applied is called the ***average impulse***,

$$\vec{\mathbf{I}}_{\text{ave}} = \vec{\mathbf{F}}_{\text{ave}}\,\Delta t. \qquad (10.2.7)$$

Multiply each side of Eq. (10.2.6) by Δt resulting in the statement that the average impulse applied to a particle during the time interval $[t, t+\Delta t]$ is equal to the change in momentum of the particle during that time interval,

$$\vec{\mathbf{I}}_{\text{ave}} = \Delta\vec{\mathbf{p}}. \tag{10.2.8}$$

Example 10.1 Impulse for a Non-Constant Force

Suppose you push an object for a time $\Delta t = 1.0\text{ s}$ in the $+x$-direction. For the first half of the interval, you push with a force that increases linearly with time according to

$$\vec{\mathbf{F}}(t) = bt\,\hat{\mathbf{i}},\;\; 0 \le t \le 0.5\text{s} \;\text{ with }\; b = 2.0\times10^{1}\,\text{N}\cdot\text{s}^{-1}. \tag{10.2.9}$$

Then for the second half of the interval, you push with a linearly decreasing force,

$$\vec{\mathbf{F}}(t) = (d - bt)\hat{\mathbf{i}},\;\; 0.5\text{s} \le t \le 1.0\text{s} \;\text{ with } d = 2.0\times10^{1}\,\text{N} \tag{10.2.10}$$

The force vs. time graph is shown in Figure 10.3. What is the impulse applied to the object?

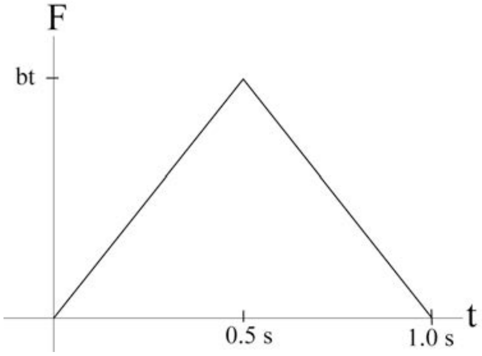


Figure 10.3 Graph of force vs. time

Solution: We can find the impulse by calculating the area under the force vs. time curve. Since the force vs. time graph consists of two triangles, the area under the curve is easy to calculate and is given by

$$\begin{aligned}\vec{\mathbf{I}} &= \left[\frac{1}{2}(b\Delta t/2)(\Delta t/2) + \frac{1}{2}(b\Delta t/2)(\Delta t/2)\right]\hat{\mathbf{i}} \\ &= \frac{1}{4}b(\Delta t)^2\,\hat{\mathbf{i}} = \frac{1}{4}(2.0\times10^{1}\,\text{N}\cdot\text{s}^{-1})(1.0\text{s})^2\,\hat{\mathbf{i}} = (5.0\,\text{N}\cdot\text{s})\hat{\mathbf{i}}.\end{aligned} \tag{10.2.11}$$

10.3 External and Internal Forces and the Change in Momentum of a System

So far we have restricted ourselves to considering how the momentum of an object changes under the action of a force. For example, if we analyze in detail the forces acting on the cart rolling down the inclined plane (Figure 10.4), we determine that there are three forces acting on the cart: the force $\vec{\mathbf{F}}_{\text{spring, cart}}$ the spring applies to the cart; the gravitational interaction $\vec{\mathbf{F}}_{\text{earth, cart}}$ between the cart and the earth; and the contact force $\vec{\mathbf{F}}_{\text{plane, cart}}$ between the inclined plane and the cart. If we define the cart as our *system*, then everything else acts as the *surroundings*. We illustrate this division of system and surroundings in Figure 10.4.

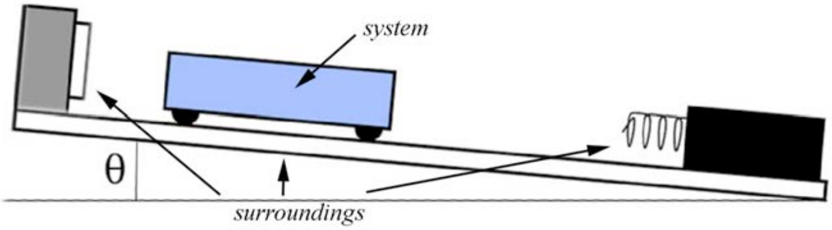


Figure 10.4 A diagram of a cart as a system and its surroundings

The forces acting on the cart are *external* forces. We refer to the vector sum of these external forces that are applied to the system (the cart) as the external force,

$$\vec{\mathbf{F}}^{\text{ext}} = \vec{\mathbf{F}}_{\text{spring, cart}} + \vec{\mathbf{F}}_{\text{earth, cart}} + \vec{\mathbf{F}}_{\text{plane, cart}} . \tag{10.3.1}$$

Then Newton's Second Law applied to the cart, in terms of impulse, is

$$\Delta\vec{\mathbf{p}}_{\text{sys}} = \int_{t_0}^{t_f} \vec{\mathbf{F}}^{\text{ext}}\, dt \equiv \vec{\mathbf{I}}_{\text{sys}} . \tag{10.3.2}$$

Let's extend our system to two interacting objects, for example the cart and the spring. The forces between the spring and cart are now *internal* forces. Both objects, the cart and the spring, experience these internal forces, which by Newton's Third Law are equal in magnitude and applied in opposite directions. So when we sum up the internal forces for the whole system, they cancel. Thus the sum of all the internal forces is always zero,

$$\vec{\mathbf{F}}^{\text{int}} = \vec{\mathbf{0}}. \tag{10.3.3}$$

External forces are still acting on our system; the gravitational force, the contact force between the inclined plane and the cart, and also a new external force, the force between the spring and the force sensor. The force acting on the system is the sum of the internal and the external forces. However, as we have shown, the internal forces cancel, so we have that

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}^{\text{ext}} + \vec{\mathbf{F}}^{\text{int}} = \vec{\mathbf{F}}^{\text{ext}}. \tag{10.3.4}$$

10.4 System of Particles

Suppose we have a system of N particles labeled by the index $i = 1, 2, 3, \cdots, N$. The force on the i^{th} particle is

$$\vec{\mathbf{F}}_i = \vec{\mathbf{F}}_i^{\text{ext}} + \sum_{j=1, j\neq i}^{j=N} \vec{\mathbf{F}}_{i,j}. \tag{10.4.1}$$

In this expression $\vec{\mathbf{F}}_{j,i}$ is the force on the i^{th} particle due to the interaction between the i^{th} and j^{th} particles. We sum over all *j* particles with $j \neq i$ since a particle cannot exert a force on itself (equivalently, we could define $\vec{\mathbf{F}}_{i,i} = \vec{\mathbf{0}}$), yielding the internal force acting on the i^{th} particle,

$$\vec{\mathbf{F}}_i^{\text{int}} = \sum_{j=1, j\neq i}^{j=N} \vec{\mathbf{F}}_{j,i}. \tag{10.4.2}$$

The force acting on the system is the sum over all i particles of the force acting on each particle,

$$\vec{\mathbf{F}} = \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i = \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i^{\text{ext}} + \sum_{i=1}^{i=N} \sum_{j=1,\, j\neq i}^{j=N} \vec{\mathbf{F}}_{j,i} = \vec{\mathbf{F}}^{\text{ext}}. \tag{10.4.3}$$

Note that the double sum vanishes,

$$\sum_{i=1}^{i=N} \sum_{j=1,\, j\neq i}^{j=N} \vec{\mathbf{F}}_{j,i} = \vec{\mathbf{0}}, \tag{10.4.4}$$

because all internal forces cancel in pairs,

$$\vec{\mathbf{F}}_{j,i} + \vec{\mathbf{F}}_{i,j} = \vec{\mathbf{0}}. \tag{10.4.5}$$

The force on the i^{th} particle is equal to the rate of change in momentum of the i^{th} particle,

$$\vec{\mathbf{F}}_i = \frac{d\vec{\mathbf{p}}_i}{dt}. \tag{10.4.6}$$

When can now substitute Equation (10.4.6) into Equation (10.4.3) and determine that that the external force is equal to the sum over all particles of the momentum change of each particle,

$$\vec{\mathbf{F}}^{\text{ext}} = \sum_{i=1}^{i=N} \frac{d\vec{\mathbf{p}}_i}{dt}. \tag{10.4.7}$$

The momentum of the system is given by the sum

$$\vec{\mathbf{p}}_{\text{sys}} = \sum_{i=1}^{i=N} \vec{\mathbf{p}}_i\,; \tag{10.4.8}$$

momenta add as vectors. We conclude that the external force causes the momentum of the system to change, and we thus restate and generalize Newton's Second Law for a system of objects as

$$\vec{\mathbf{F}}^{\text{ext}} = \frac{d\,\vec{\mathbf{p}}_{\text{sys}}}{dt}. \tag{10.4.9}$$

In terms of impulse, this becomes the statement

$$\Delta\vec{\mathbf{p}}_{\text{sys}} = \int_{t_0}^{t_f} \vec{\mathbf{F}}^{\text{ext}}\, dt \equiv \vec{\mathbf{I}}. \tag{10.4.10}$$

10.5 Center of Mass

Consider two point-like particles with masses m_1 and m_2. Choose a coordinate system with a choice of origin such that body 1 has position $\vec{\mathbf{r}}_1$ and body 2 has position $\vec{\mathbf{r}}_2$ (Figure 10.5).

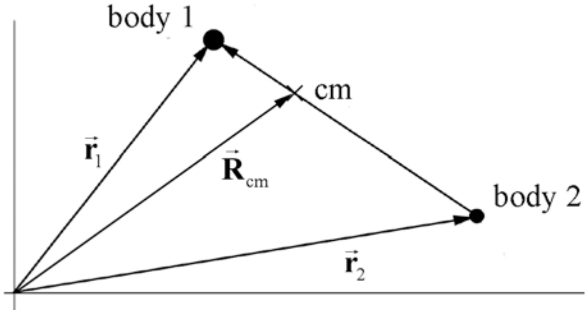


Figure 10.5 Center of mass coordinate system.

The center of mass vector, $\vec{\mathbf{R}}_{\text{cm}}$, of the two-body system is defined as

$$\vec{\mathbf{R}}_{\text{cm}} = \frac{m_1\,\vec{\mathbf{r}}_1 + m_2\,\vec{\mathbf{r}}_2}{m_1 + m_2}. \tag{10.5.1}$$

We shall now extend the concept of the center of mass to more general systems. Suppose we have a system of N particles labeled by the index $i = 1, 2, 3, \cdots, N$. Choose a coordinate system and denote the position of the i^{th} particle as $\vec{\mathbf{r}}_i$. The mass of the system is given by the sum

$$m_{\text{sys}} = \sum_{i=1}^{i=N} m_i \tag{10.5.2}$$

and the position of the center of mass of the system of particles is given by

$$\vec{\mathbf{R}}_{\text{cm}} = \frac{1}{m_{\text{sys}}} \sum_{i=1}^{i=N} m_i \vec{\mathbf{r}}_i \,. \tag{10.5.3}$$

(For a continuous rigid body, each point-like particle has mass dm and is located at the position $\vec{\mathbf{r}}'$. The center of mass is then defined as an integral over the body,

$$\vec{\mathbf{R}}_{\text{cm}} = \frac{\int\limits_{\text{body}} dm\, \vec{\mathbf{r}}'}{\int\limits_{\text{body}} dm} \,. \tag{10.5.4}$$

Example 10.2 Center of Mass of the Earth-Moon System

The mean distance from the center of the earth to the center of the moon is $r_{em} = 3.84 \times 10^8\ \text{m}$. The mass of the earth is $m_e = 5.98 \times 10^{24}\ \text{kg}$ and the mass of the moon is $m_m = 7.34 \times 10^{22}\ \text{kg}$. The mean radius of the earth is $r_e = 6.37 \times 10^6\ \text{m}$. The mean radius of the moon is $r_m = 1.74 \times 10^6\ \text{m}$. Where is the location of the center of mass of the earth-moon system? Is it inside the earth's radius or outside?

Solution: The center of mass of the earth-moon system is defined to be

$$\vec{\mathbf{R}}_{cm} = \frac{1}{m_{\text{sys}}} \sum_{i=1}^{i=N} m_i \vec{\mathbf{r}}_i = \frac{1}{m_e + m_m} (m_e \vec{\mathbf{r}}_e + m_m \vec{\mathbf{r}}_m) \,. \tag{10.5.5}$$

Choose an origin at the center of the earth and a unit vector $\hat{\mathbf{i}}$ pointing towards the moon, then $\vec{\mathbf{r}}_e = \vec{\mathbf{0}}$. The center of mass of the earth-moon system is then

$$\vec{\mathbf{R}}_{cm} = \frac{1}{m_e + m_m} (m_e \vec{\mathbf{r}}_e + m_m \vec{\mathbf{r}}_m) = \frac{m_m \vec{\mathbf{r}}_{em}}{m_e + m_m} = \frac{m_m r_{em}}{m_e + m_m} \hat{\mathbf{i}} \tag{10.5.6}$$

$$\vec{\mathbf{R}}_{cm} = \frac{(7.34 \times 10^{22}\ \text{kg})(3.84 \times 10^8\ \text{m})}{(5.98 \times 10^{24}\ \text{kg} + 7.34 \times 10^{22}\ \text{kg})} \hat{\mathbf{i}} = 4.66 \times 10^6\ \text{m}\ \hat{\mathbf{i}} \tag{10.5.7}$$

The earth's mean radius is $r_e = 6.37 \times 10^6\ \text{m}$ so the center of mass of the earth-moon system lies within the earth.

Example 10.3 Center of Mass of a Rod

A thin rod has length L and mass M.

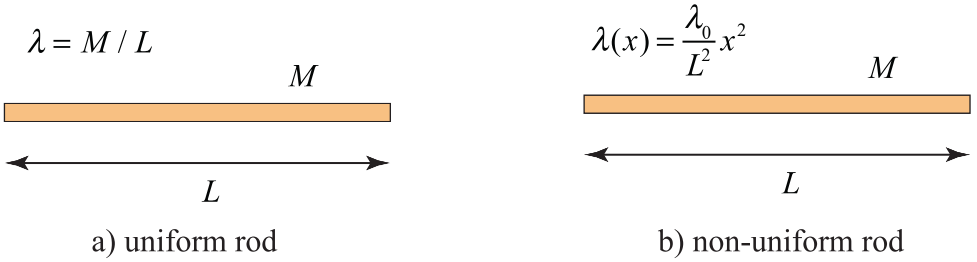


Figure 10.6 a) Uniform rod and b) non-uniform rod

(a) Suppose the rod is uniform (Figure 10.6a). Find the position of the center of mass with respect to the left end of the rod. (b) Now suppose the rod is not uniform (Figure 10.6b) with a linear mass density that varies with the distance x from the left end according to

$$\lambda(x) = \frac{\lambda_0}{L^2}x^2 \tag{10.5.8}$$

where λ_0 is a constant and has SI units $[\mathrm{kg \cdot m^{-1}}]$. Find λ_0 and the position of the center of mass with respect to the left end of the rod.

Solution: (a) Choose a coordinate system with the rod aligned along the x-axis and the origin located at the left end of the rod. The center of mass of the rod can be found using the definition given in Eq. (10.5.4). In that expression dm is an infinitesimal mass element and $\vec{\mathbf{r}}$ is the vector from the origin to the mass element dm (Figure 10.6c).

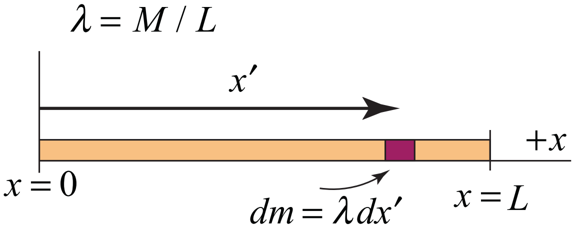


Figure 10.6c Infinitesimal mass element for rod

Choose an infinitesimal mass element dm located a distance x' from the origin. In this problem x' will be the integration variable. Let the length of the mass element be dx'. Then

$$dm = \lambda dx' \tag{10.5.9}$$

The vector $\vec{\mathbf{r}} = x'\,\hat{\mathbf{i}}$. The center of mass is found by integration

$$\vec{\mathbf{R}}_{cm} = \frac{1}{M}\int_{\text{body}} \vec{\mathbf{r}}\, dm = \frac{1}{L}\int_{x'=0}^{x} x' dx'\, \hat{\mathbf{i}} = \frac{1}{2L} x'^2 \Big|_{x'=0}^{x'=L} \hat{\mathbf{i}} = \frac{1}{2L}(L^2 - 0)\, \hat{\mathbf{i}} = \frac{L}{2}\hat{\mathbf{i}}. \quad (10.5.10)$$

(b) For a non-uniform rod (Figure 10.6d),

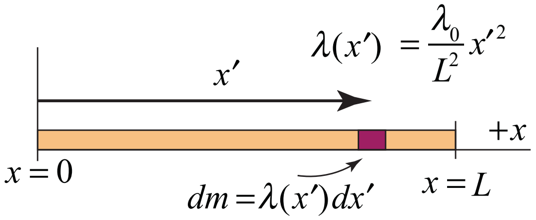


Figure 10.6d Non-uniform rod

the mass element is found using Eq. (10.5.8)

$$dm = \lambda(x')dx' = \lambda = \frac{\lambda_0}{L^2} x'^2 dx' . \quad (10.5.11)$$

The vector $\vec{\mathbf{r}} = x'\,\hat{\mathbf{i}}$. The mass is found by integrating the mass element over the length of the rod

$$M = \int_{\text{body}} dm = \int_{x'=0}^{x=L} \lambda(x')dx' = \frac{\lambda_0}{L^2}\int_{x'=0}^{x=L} x'^2 dx' = \frac{\lambda_0}{3L^2} x'^3 \Big|_{x'=0}^{x'=L} = \frac{\lambda_0}{3L^2}(L^3 - 0) = \frac{\lambda_0}{3}L. \quad (10.5.12)$$

Therefore

$$\lambda_0 = \frac{3M}{L} \quad (10.5.13)$$

The center of mass is again found by integration

$$\begin{aligned} \vec{\mathbf{R}}_{cm} &= \frac{1}{M}\int_{\text{body}} \vec{\mathbf{r}}\, dm = \frac{3}{\lambda_0 L}\int_{x'=0}^{x} \lambda(x')x'\, dx'\, \hat{\mathbf{i}} = \frac{3}{L^3}\int_{x'=0}^{x} x'^3\, dx'\, \hat{\mathbf{i}} \\ \vec{\mathbf{R}}_{cm} &= \frac{3}{4L^3} x'^4 \Big|_{x'=0}^{x'=L} \hat{\mathbf{i}} = \frac{3}{4L^3}(L^4 - 0)\, \hat{\mathbf{i}} = \frac{3}{4}L\, \hat{\mathbf{i}}. \end{aligned} \quad (10.5.14)$$

.

10.6 Translational Motion of the Center of Mass

The velocity of the center of mass is found by differentiation,

$$\vec{\mathbf{V}}_{\text{cm}} = \frac{1}{m_{\text{sys}}} \sum_{i=1}^{i=N} m_i \vec{\mathbf{v}}_i = \frac{\vec{\mathbf{p}}_{\text{sys}}}{m_{\text{sys}}}. \tag{10.6.1}$$

The momentum is then expressed in terms of the velocity of the center of mass by

$$\vec{\mathbf{p}}_{\text{sys}} = m_{\text{sys}} \vec{\mathbf{V}}_{\text{cm}}. \tag{10.6.2}$$

We have already determined that the external force is equal to the change of the momentum of the system (Equation (10.4.9)). If we now substitute Equation (10.6.2) into Equation (10.4.9), and continue with our assumption of constant masses m_i, we have that

$$\vec{\mathbf{F}}^{\text{ext}} = \frac{d\vec{\mathbf{p}}_{\text{sys}}}{dt} = m_{\text{sys}} \frac{d\vec{\mathbf{V}}_{\text{cm}}}{dt} = m_{\text{sys}} \vec{\mathbf{A}}_{\text{cm}}, \tag{10.6.3}$$

where $\vec{\mathbf{A}}_{\text{cm}}$, the derivative with respect to time of $\vec{\mathbf{V}}_{\text{cm}}$, is the acceleration of the center of mass. From Equation (10.6.3) we can conclude that in considering the linear motion of the center of mass, the sum of the external forces may be regarded as acting at the center of mass.

Example 10.4 Forces on a Baseball Bat

Suppose you push a baseball bat lying on a nearly frictionless table at the center of mass, position 2, with a force $\vec{\mathbf{F}}$ (Figure 10.7). Will the acceleration of the center of mass be *greater than, equal to, or less than* if you push the bat with the same force at either end, positions 1 and 3

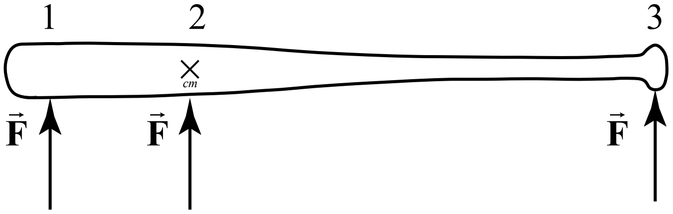


Figure 10.7 Forces acting on a baseball bat

Solution: The acceleration of the center of mass will be equal in the three cases. From our previous discussion, (Equation (10.6.3)), the acceleration of the center of mass is independent of where the force is applied. However, the bat undergoes a very different motion if we apply the force at one end or at the center of mass. When we apply the force at the center of mass all the particles in the baseball bat will undergo linear motion (Figure 10.7a).

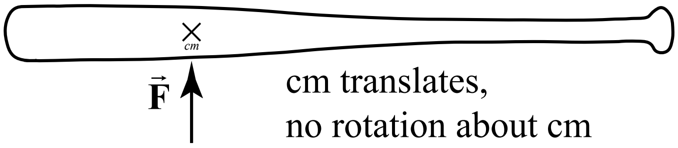


Figure 10.7a Force applied at center of mass

When we push the bat at one end, the particles that make up the baseball bat will no longer undergo a linear motion even though the center of mass undergoes linear motion. In fact, each particle will rotate about the center of mass of the bat while the center of mass of the bat accelerates in the direction of the applied force (Figure 10.7b).

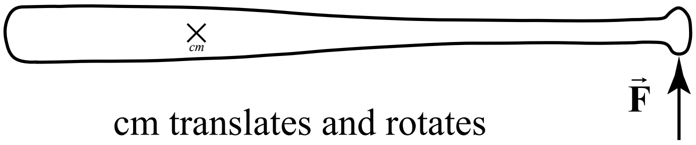


Figure 10.7b Force applied at end of bat

10.7 Constancy of Momentum and Isolated Systems

Suppose we now completely isolate our system from the surroundings. When the external force acting on the system is zero,

$$\vec{\mathbf{F}}^{\text{ext}} = \vec{\mathbf{0}}\,. \qquad (10.7.1)$$

the system is called an ***isolated system***. For an isolated system, the change in the momentum of the system is zero,

$$\Delta\vec{\mathbf{p}}_{\text{sys}} = \vec{\mathbf{0}} \quad \text{(isolated system)}, \qquad (10.7.2)$$

therefore the momentum of the isolated system is constant. The initial momentum of our system is the sum of the initial momentum of the individual particles,

$$\vec{\mathbf{p}}_{\text{sys},i} = m_1\vec{\mathbf{v}}_{1,i} + m_2\vec{\mathbf{v}}_{2,i} + \cdots. \qquad (10.7.3)$$

The final momentum is the sum of the final momentum of the individual particles,

$$\vec{\mathbf{p}}_{\text{sys},f} = m_1\vec{\mathbf{v}}_{1,f} + m_2\vec{\mathbf{v}}_{2,f} + \cdots. \qquad (10.7.4)$$

Note that the right-hand-sides of Equations. (10.7.3) and (10.7.4) are vector sums.

> *When the external force on a system is zero, then the initial momentum of the system equals the final momentum of the system,*

$$\vec{\mathbf{p}}_{\text{sys},i} = \vec{\mathbf{p}}_{\text{sys},f} \,. \qquad (10.7.5)$$

10.8 Momentum Changes and Non-isolated Systems

Suppose the external force acting on the system is not zero,

$$\vec{\mathbf{F}}^{\text{ext}} \neq \vec{\mathbf{0}}. \qquad (10.8.1)$$

and hence the system is not isolated. By Newton's Third Law, the sum of the force on the surroundings is equal in magnitude but opposite in direction to the external force acting on the system,

$$\vec{\mathbf{F}}^{\text{sur}} = -\vec{\mathbf{F}}^{\text{ext}} \,. \qquad (10.8.2)$$

It's important to note that in Equation (10.8.2), all internal forces in the surroundings sum to zero. Thus the sum of the external force acting on the system and the force acting on the surroundings is zero,

$$\vec{\mathbf{F}}^{\text{sur}} + \vec{\mathbf{F}}^{\text{ext}} = \vec{\mathbf{0}} \,. \qquad (10.8.3)$$

We have already found (Equation (10.4.9)) that the external force $\vec{\mathbf{F}}^{\text{ext}}$ acting on a system is equal to the rate of change of the momentum of the system. Similarly, the force on the surrounding is equal to the rate of change of the momentum of the surroundings. Therefore the momentum of both the system and surroundings is always conserved.

> *For a system and all of the surroundings that undergo any change of state, the change in the momentum of the system and its surroundings is zero,*

$$\Delta\vec{\mathbf{p}}_{\text{sys}} + \Delta\vec{\mathbf{p}}_{\text{sur}} = \vec{\mathbf{0}}. \qquad (10.8.4)$$

Equation (10.8.4) is referred to as the ***Principle of Conservation of Momentum.***

10.9 Worked Examples

10.9.1 Problem Solving Strategies

When solving problems involving changing momentum in a system, we shall employ our general problem solving strategy involving four basic steps:

1. Understand – get a conceptual grasp of the problem.
2. Devise a Plan - set up a procedure to obtain the desired solution.
3. Carry our your plan – solve the problem!

4. Look Back – check your solution and method of solution.

We shall develop a set of guiding ideas for the first two steps.

1. Understand – get a conceptual grasp of the problem

The first question you should ask is whether or not momentum is constant in some system that is changing its state after undergoing an interaction. First you must identify the objects that compose the system and how they are changing their state due to the interaction. As a guide, try to determine which objects change their momentum in the course of interaction. You must keep track of the momentum of these objects before and after any interaction. Second, momentum is a vector quantity so the question of whether momentum is constant or not must be answered in each relevant direction. In order to determine this, there are two important considerations. You should identify any external forces acting on the system. Remember that a non-zero external force will cause the momentum of the system to change, (Equation (10.4.9) above),

$$\vec{\mathbf{F}}^{\text{ext}} = \frac{d\vec{\mathbf{p}}_{\text{sys}}}{dt}. \tag{10.9.1}$$

Equation (10.9.1) is a vector equation; if the external force in some direction is zero, then the change of momentum in that direction is zero. In some cases, external forces may act but the time interval during which the interaction takes place is so small that the impulse is small in magnitude compared to the momentum and might be negligible. Recall that the average external impulse changes the momentum of the system

$$\vec{\mathbf{I}} = \vec{\mathbf{F}}^{\text{ext}} \Delta t_{\text{int}} = \Delta \vec{\mathbf{p}}_{\text{sys}}. \tag{10.9.2}$$

If the interaction time is small enough, the momentum of the system is constant, $\Delta\vec{\mathbf{p}} \to \vec{\mathbf{0}}$. If the momentum is not constant then you must apply either Equation (10.9.1) or Equation (10.9.2). If the momentum of the system is constant, then you can apply Equation (10.7.5),

$$\vec{\mathbf{p}}_{\text{sys},\,i} = \vec{\mathbf{p}}_{\text{sys},\,f}\,. \tag{10.9.3}$$

If there is no net external force in some direction, for example the x-direction, the component of momentum is constant in that direction, and you must apply

$$p_{\text{sys},\,x,i} = p_{\text{sys},\,x,f} \tag{10.9.4}$$

2. Devise a Plan - set up a procedure to obtain the desired solution

Draw diagrams of all the elements of your system for the two states immediately before and after the system changes its state. Choose symbols to identify each mass and velocity in the system. Identify a set of positive directions and unit vectors for each state. Choose your symbols to correspond to the state and motion (this facilitates an easy interpretation, for example $(v_{x,i})_1$ represents the x-component of the velocity of object 1 in the initial state and $(v_{x,f})_1$ represents the x-component of the velocity of object 1 in the final state). Decide whether you are using components or magnitudes for your velocity symbols. Since momentum is a vector quantity, identify the initial and final vector components of the momentum. We shall refer to these diagrams as ***momentum flow diagrams***. Based on your model you can now write expressions for the initial and final momentum of your system. As an example in which two objects are moving only in the x-direction, the initial x-component of the momentum is

$$p_{\text{sys},x,i} = m_1(v_{x,i})_1 + m_2(v_{x,i})_2 + \cdots. \tag{10.9.5}$$

The final x-component of the momentum is

$$p_{\text{sys},x,f} = m_1(v_{x,f})_1 + m_2(v_{x,f})_2 + \cdots. \tag{10.9.6}$$

If the x-component of the momentum is constant then

$$p_{\text{sys},x,i} = p_{\text{sys},x,f}\,. \tag{10.9.7}$$

We can now substitute Equations (10.9.5) and (10.9.6) into Equation (10.9.7), yielding

$$m_1(v_{x,i})_1 + m_2(v_{x,i})_2 + \cdots = m_1(v_{x,f})_1 + m_2(v_{x,f})_2 + \cdots. \tag{10.9.8}$$

Equation (10.9.8) can now be used for any further analysis required by a particular problem. For example, you may have enough information to calculate the final velocities of the objects after the interaction. If so then carry out your plan and check your solution, especially dimensions or units and any relevant vector directions.

Example 10.5 Exploding Projectile

An instrument-carrying projectile of mass m_1 accidentally explodes at the top of its trajectory. The horizontal distance between launch point and the explosion is x_i. The projectile breaks into two pieces that fly apart horizontally. The larger piece, m_3, has three times the mass of the smaller piece, m_2. To the surprise of the scientist in charge, the smaller piece returns to earth at the launching station. Neglect air resistance and effects due to the earth's curvature. How far away, $x_{3,f}$, from the original launching point does the larger piece land?

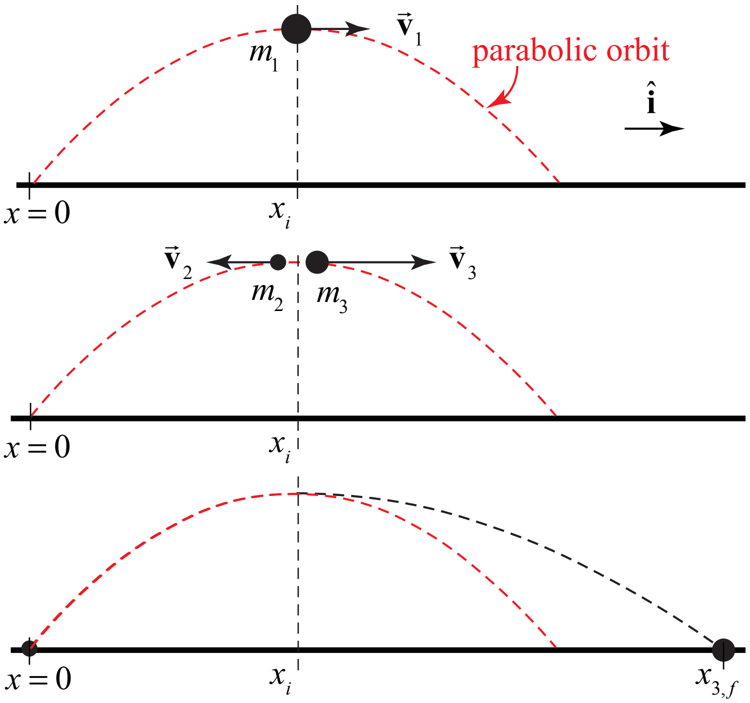


Figure 10.8 Exploding projectile trajectories

Solution: We can solve this problem two different ways. The easiest approach is utilizes the fact that the external force is the gravitational force and therefore the center of mass of the system follows a parabolic trajectory. From the information given in the problem $m_2 = m_1 / 4$ and $m_3 = 3m_1 / 4$. Thus when the two objects return to the ground the center of mass of the system has traveled a distance $R_{cm} = 2x_i$. We now use the definition of center of mass to find where the object with the greater mass hits the ground. Choose an origin at the starting point. The center of mass of the system is given by

$$\vec{\mathbf{R}}_{cm} = \frac{m_2\vec{\mathbf{r}}_2 + m_3\vec{\mathbf{r}}_3}{m_2 + m_3} .$$

So when the objects hit the ground $\vec{\mathbf{R}}_{cm} = 2x_i\,\hat{\mathbf{i}}$, the object with the smaller mass returns to the origin, $\vec{\mathbf{r}}_2 = \vec{\mathbf{0}}$, and the position vector of the other object is $\vec{\mathbf{r}}_3 = x_{3,f}\,\hat{\mathbf{i}}$. So using the definition of the center of mass,

$$2x_i\,\hat{\mathbf{i}} = \frac{(3m_1/4)x_{3,f}\,\hat{\mathbf{i}}}{m_1/4 + 3m_1/4} = \frac{(3m_1/4)x_{3,f}\,\hat{\mathbf{i}}}{m_1} = \frac{3}{4}x_{3,f}\,\hat{\mathbf{i}} .$$

Therefore

$$x_{3,f} = \frac{8}{3}x_i .$$

Note that the neither the vertical height above ground nor the gravitational acceleration g entered into our solution.

Alternatively, we can use conservation of momentum and kinematics to find the distance traveled. Because the smaller piece returns to the starting point after the collision, the velocity of the smaller piece immediately after the explosion is equal to the negative of the velocity of original object immediately before the explosion. Because the collision is instantaneous, the horizontal component of the momentum is constant during the collision. We can use this to determine the speed of the larger piece after the collision. The larger piece takes the same amount of time to return to the ground as the projectile originally takes to reach the top of the flight. We can therefore determine how far the larger piece traveled horizontally.

We begin by identifying various states in the problem.

Initial state, time $t_0 = 0$: the projectile is launched.

State 1 time t_1: the projectile is at the top of its flight trajectory immediately before the explosion. The mass is m_1 and the velocity of the projectile is $\vec{\mathbf{v}}_1 = v_1 \hat{\mathbf{i}}$.

State 2 time t_2: immediately after the explosion, the projectile has broken into two pieces, one of mass m_2 moving backwards (in the negative x-direction) with velocity $\vec{\mathbf{v}}_2 = -\vec{\mathbf{v}}_1$. The other piece of mass m_3 is moving in the positive x-direction with velocity $\vec{\mathbf{v}}_3 = v_3 \hat{\mathbf{i}}$, (Figure 10.8).

State 3: the two pieces strike the ground at time $t_f = 2t_1$, one at the original launch site and the other at a distance $x_{3,f}$ from the launch site, as indicated in Figure 10.8. The pieces take the same amount of time to reach the ground $\Delta t = t_1$ because both pieces are falling from the same height as the original piece reached at time t_1, and each has no component of velocity in the vertical direction immediately after the explosion. The momentum flow diagram with state 1 as the initial state and state 2 as the final state are shown in the upper two diagrams in Figure 10.8.

The initial momentum at time t_1 immediately before the explosion is

$$\vec{\mathbf{p}}^{sys}(t_1) = m_1 \vec{\mathbf{v}}_1 . \qquad (10.9.9)$$

The momentum at time t_2 immediately after the explosion is

$$\vec{\mathbf{p}}^{sys}(t_2) = m_2\,\vec{\mathbf{v}}_2 + m_3\,\vec{\mathbf{v}}_3 = -\frac{1}{4}m_1\vec{\mathbf{v}}_1 + \frac{3}{4}m_1\vec{\mathbf{v}}_3 \qquad (10.9.10)$$

During the duration of the instantaneous explosion, impulse due to the external gravitational force may be neglected and therefore the momentum of the system is constant. In the horizontal direction, we have that

$$m_1\,\vec{\mathbf{v}}_1 = -\frac{1}{4}m_1\vec{\mathbf{v}}_1 + \frac{3}{4}m_1\vec{\mathbf{v}}_3\,. \qquad (10.9.11)$$

Equation (10.9.11) can now be solved for the velocity of the larger piece immediately after the collision,

$$\vec{\mathbf{v}}_3 = \frac{5}{3}\vec{\mathbf{v}}_1\,. \qquad (10.9.12)$$

The larger piece travels a distance

$$x_{3,f} = v_3\,t_1 = \frac{5}{3}v_1\,t_1 = \frac{5}{3}x_i\,. \qquad (10.9.13)$$

Therefore the total distance the larger piece traveled from the launching station is

$$x_f = x_i + \frac{5}{3}x_i = \frac{8}{3}x_i\,, \qquad (10.9.14)$$

in agreement with our previous approach.

Example 10.6 Landing Plane and Sandbag

Figure 10.9 Plane and sandbag

A light plane of mass 1000 kg makes an emergency landing on a short runway. With its engine off, it lands on the runway at a speed of $40\ \mathrm{m\cdot s^{-1}}$. A hook on the plane snags a cable attached to a 120 kg sandbag and drags the sandbag along. If the coefficient of friction between the sandbag and the runway is $\mu_k = 0.4$, and if the plane's brakes give an additional retarding force of magnitude 1400 N, how far does the plane go before it comes to a stop?

Solution: We shall assume that when the plane snags the sandbag, the collision is instantaneous so the momentum in the horizontal direction remains constant,

$$p_{x,i} = p_{x,1} . \tag{10.9.15}$$

We then know the speed of the plane and the sandbag immediately after the collision. After the collision, there are two external forces acting on the system of the plane and sandbag, the friction between the sandbag and the ground and the braking force of the runway on the plane. So we can use the Newton's Second Law to determine the acceleration and then one-dimensional kinematics to find the distance the plane traveled since we can determine the change in kinetic energy.

The momentum of the plane immediately before the collision is

$$\vec{\mathbf{p}}_i = m_p v_{p,i}\, \hat{\mathbf{i}} \tag{10.9.16}$$

The momentum of the plane and sandbag immediately after the collision is

$$\vec{\mathbf{p}}_1 = (m_p + m_s) v_{p,1}\, \hat{\mathbf{i}} \tag{10.9.17}$$

Because the x- component of the momentum is constant, we can substitute Eqs. (10.9.16) and (10.9.17) into Eq. (10.9.15) yielding

$$m_p v_{p,i} = (m_p + m_s) v_{p,1} . \tag{10.9.18}$$

The speed of the plane and sandbag immediately after the collision is

$$v_{p,1} = \frac{m_p v_{p,i}}{m_p + m_s} \tag{10.9.19}$$

The forces acting on the system consisting of the plane and the sandbag are the normal force on the sandbag,

$$\vec{\mathbf{N}}_{g,s} = N_{g,s}\hat{\mathbf{j}} , \tag{10.9.20}$$

the frictional force between the sandbag and the ground

$$\vec{\mathbf{f}}_k = -f_k\hat{\mathbf{i}} = -\mu_k N_{g,s}\hat{\mathbf{i}} , \tag{10.9.21}$$

the braking force on the plane

$$\vec{\mathbf{F}}_{g,p} = -F_{g,p}\hat{\mathbf{i}} , \tag{10.9.22}$$

and the gravitational force on the system,

$$(m_p + m_s)\vec{\mathbf{g}} = -(m_p + m_s)g\hat{\mathbf{j}}. \tag{10.9.23}$$

Newton's Second Law in the $\hat{\mathbf{i}}$-direction becomes

$$-F_{g,p} - f_k = (m_p + m_s)a_x. \tag{10.9.24}$$

If we just look at the vertical forces on the sandbag alone then Newton's Second Law in the $\hat{\mathbf{j}}$-direction becomes

$$N - m_s g = 0.$$

The frictional force on the sandbag is then

$$\vec{\mathbf{f}}_k = -\mu_k N_{g,s}\hat{\mathbf{i}} = -\mu_k m_s g\hat{\mathbf{i}}. \tag{10.9.25}$$

Newton's Second Law in the $\hat{\mathbf{i}}$-direction becomes

$$-F_{g,p} - \mu_k m_s g = (m_p + m_s)a_x.$$

The x-component of the acceleration of the plane and the sand bag is then

$$a_x = \frac{-F_{g,p} - \mu_k m_s g}{m_p + m_s} \tag{10.9.26}$$

We choose our origin at the location of the plane immediately after the collision, $x_p(0) = 0$. Set $t = 0$ immediately after the collision. The x-component of the velocity of the plane immediately after the collision is $v_{x,0} = v_{p,1}$. Set $t = t_f$ when the plane just comes to a stop. Because the acceleration is constant, the kinematic equations for the change in velocity is

$$v_{x,f}(t_f) - v_{p,1} = a_x t_f.$$

We can solve this equation for $t = t_f$, where $v_{x,f}(t_f) = 0$

$$t_f = -v_{p,1} / a_x t.$$

Then the position of the plane when it first comes to rest is

$$x_p(t_f) - x_p(0) = v_{p,1}t_f + \frac{1}{2}a_x t_f^2 = -\frac{1}{2}\frac{v_{p,1}^2}{a_x}. \tag{10.9.27}$$

Then using $x_p(0) = 0$ and substituting Eq. (10.9.26) into Eq. (10.9.27) yields

$$x_p(t_f) = \frac{1}{2}\frac{(m_p + m_s)v_{p,1}^2}{(F_{g,p} + \mu_k m_s g)} . \tag{10.9.28}$$

We now use the condition from conservation of the momentum law during the collision, Eq. (10.9.19) in Eq. (10.9.28) yielding

$$x_p(t_f) = \frac{m_p^2 v_{p,i}^2}{2(m_p + m_s)(F_{g,p} + \mu_k m_s g)} . \tag{10.9.29}$$

Substituting the given values into Eq. (10.9.28) yields

$$x_p(t_f) = \frac{(1000\text{ kg})^2(40\text{ m}\cdot\text{s}^{-1})^2}{2(1000\text{ kg} + 120\text{ kg})(1400\text{ N} + (0.4)(120\text{ kg})(9.8\text{m}\cdot\text{s}^{-2}\ ))} = 3.8\times10^2\text{ m} . \tag{10.9.30}$$

Chapter 11 Reference Frames

Chapter 11 Reference Frames

> *Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the "light medium" suggest that the phenomena of electromagnetism as well as mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that, ..., the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. We will raise this conjecture (the purport of which will hereafter be called the "Principle of Relativity") to the status of a postulate, and also introduce another postulate, ..., namely that light is always propagated in empty space with a definite velocity c, which is independent of the state of motion of the emitting body.* [1]
>
> *Albert Einstein*

11.1 Introduction

In order to describe physical events that occur in space and time such as the motion of bodies, we introduced a coordinate system. Its spatial and temporal coordinates can now specify a ***space-time event***. In particular, the position of a moving body can be described by space-time events specified by its space-time coordinates. You can place an *observer* at the origin of coordinate system. The coordinate system with your observer acts as a ***reference frame*** for describing the position, velocity, and acceleration of bodies. The position vector of the body depends on the choice of origin (location of your observer) but the displacement, velocity, and acceleration vectors are independent of the location of the observer.

You can always choose a second reference frame that is moving with respect to the first reference frame. Then the position, velocity and acceleration of bodies as seen by the different observers do depend on the relative motion of the two reference frames. The relative motion can be described in terms of the relative position, velocity, and acceleration of the observer at the origin, O, in reference frame S with respect to a second observer located at the origin, O', in reference frame S'.

11.2 Galilean Coordinate Transformations

Let the vector $\vec{\mathbf{R}}$ point from the origin of frame S to the origin of reference frame S'. Suppose an object is located at a point 1. Denote the position vector of the object with respect to origin of reference frame S by $\vec{\mathbf{r}}$. Denote the position vector of the object with respect to origin of reference frame S' by $\vec{\mathbf{r}}'$ (Figure 11.1).

---

[1] A. Einstein, *Zur Elektrodynamik begetter Körper*, (*On the Electrodynamics of Moving Bodies*), Ann. Physik, **17**, 891 (1905); translated by W. Perrett and G.B. Jeffrey, 19223, in *The Principle of Relativity*, Dover, New York.

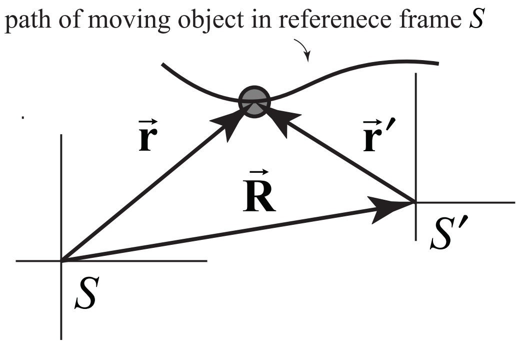


Figure 11.1 Two reference frames S and S'.

The position vectors are related by

$$\vec{\mathbf{r}}' = \vec{\mathbf{r}} - \vec{\mathbf{R}} \tag{11.2.1}$$

These coordinate transformations are called the ***Galilean Coordinate Transformations.*** They enable the observer in frame S to predict the position vector in frame S', based only on the position vector in frame S and the relative position of the origins of the two frames.

The relative velocity between the two reference frames is given by the time derivative of the vector $\vec{\mathbf{R}}$, defined as the limit as of the displacement of the two origins divided by an interval of time, as the interval of time becomes infinitesimally small,

$$\vec{\mathbf{V}} = \frac{d\vec{\mathbf{R}}}{dt}. \tag{11.2.2}$$

11.2.1 Relatively Inertial Reference Frames and the Principle of Relativity

If the relative velocity between the two reference frames is constant, then the relative acceleration between the two reference frames is zero,

$$\vec{\mathbf{A}} = \frac{d\vec{\mathbf{V}}}{dt} = \vec{\mathbf{0}}. \tag{11.2.3}$$

When two reference frames are moving with a constant velocity relative to each other as above, the reference frames are called ***relatively inertial reference frames***.

We can reinterpret Newton's First Law

> *Law 1: Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.*

as the Principle of Relativity:

In relatively inertial reference frames, if there is no net force impressed on an object at rest in frame S, then there is also no net force impressed on the object in frame S'.

11.3 Law of Addition of Velocities: Newtonian Mechanics

Suppose the object in Figure 11.1 is moving; then observers in different reference frames will measure different velocities. Denote the velocity of the object in frame S by $\vec{\mathbf{v}} = d\vec{\mathbf{r}}/dt$, and the velocity of the object in frame S' by $\vec{\mathbf{v}}' = d\vec{\mathbf{r}}'/dt'$. Since the derivative of the position is velocity, the velocities of the object in two different reference frames are related according to

$$\frac{d\vec{\mathbf{r}}'}{dt'} = \frac{d\vec{\mathbf{r}}}{dt} - \frac{d\vec{\mathbf{R}}}{dt}, \tag{11.3.1}$$

$$\vec{\mathbf{v}}' = \vec{\mathbf{v}} - \vec{\mathbf{V}}. \tag{11.3.2}$$

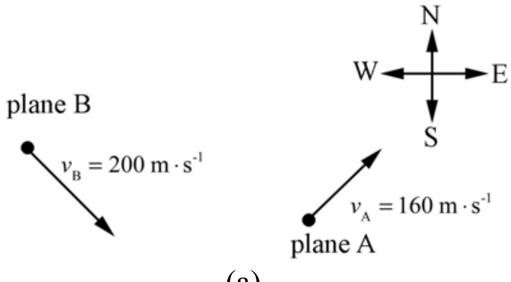
This is called the ***Law of Addition of Velocities.***

11.4 Worked Examples

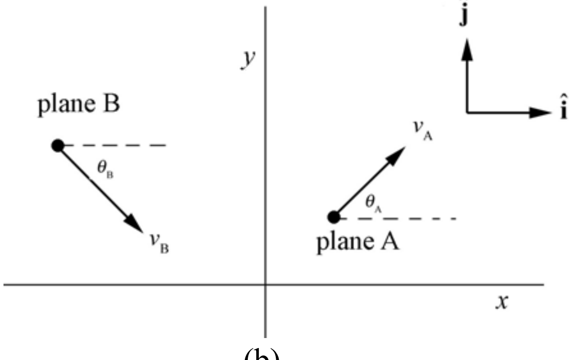
Example 11.1 Relative Velocities of Two Moving Planes

An airplane A is traveling northeast with a speed of $v_{\text{A}} = 160\ \text{m}\cdot\text{s}^{-1}$. A second airplane B is traveling southeast with a speed of $v_{\text{B}} = 200\ \text{m}\cdot\text{s}^{-1}$. (a) Choose a coordinate system and write down an expression for the velocity of each airplane as vectors, $\vec{\mathbf{v}}_{\mathbf{A}}$ and $\vec{\mathbf{v}}_B$. Carefully use unit vectors to express your answer. (b) Sketch the vectors $\vec{\mathbf{v}}_{\mathbf{A}}$ and $\vec{\mathbf{v}}_B$ on your coordinate system. (c) Find a vector expression that expresses the velocity of aircraft A as seen from an observer flying in aircraft B. Calculate this vector. What is its magnitude and direction? Sketch it on your coordinate system.

Solution: From the information given in the problem we draw the velocity vectors of the airplanes as shown in Figure 11.2a.



(a)



(b)

Figure 11.2 (a): Motion of two planes

Figure 11.2 (b): Coordinate System

An observer at rest with respect to the ground defines a reference frame S. Choose a coordinate system shown in Figure 11.2b. According to this observer, airplane A is moving with velocity $\vec{\mathbf{v}}_{\mathrm{A}} = v_{\mathrm{A}} \cos\theta_{\mathrm{A}} \hat{\mathbf{i}} + v_{\mathrm{A}} \sin\theta_{\mathrm{A}} \hat{\mathbf{j}}$, and airplane B is moving with velocity $\vec{\mathbf{v}}_{\mathrm{B}} = v_{\mathrm{B}} \cos\theta_{\mathrm{B}} \hat{\mathbf{i}} + v_{\mathrm{B}} \sin\theta_{\mathrm{B}} \hat{\mathbf{j}}$. According to the information given in the problem airplane A flies northeast so $\theta_{\mathrm{A}} = \pi / 4$ and airplane B flies southeast east so $\theta_{\mathrm{B}} = -\pi / 4$. Thus $\vec{\mathbf{v}}_{\mathrm{A}} = (80\sqrt{2} \text{ m} \cdot \text{s}^{-1})\hat{\mathbf{i}} + (80\sqrt{2} \text{ m} \cdot \text{s}^{-1})\hat{\mathbf{j}}$ and $\vec{\mathbf{v}}_{\mathrm{B}} = (100\sqrt{2} \text{ m} \cdot \text{s}^{-1})\hat{\mathbf{i}} - (100\sqrt{2} \text{ m} \cdot \text{s}^{-1})\hat{\mathbf{j}}$

Consider a second observer moving along with airplane B, defining reference frame S'. What is the velocity of airplane A according to this observer moving in airplane B? The velocity of the observer moving along in airplane B with respect to an observer at rest on the ground is just the velocity of airplane B and is given by $\vec{\mathbf{V}} = \vec{\mathbf{v}}_{\mathrm{B}} = v_{\mathrm{B}} \cos\theta_{\mathrm{B}} \hat{\mathbf{i}} + v_{\mathrm{B}} \sin\theta_{\mathrm{B}} \hat{\mathbf{j}}$. Using the Law of Addition of Velocities, Equation (11.3.2), the velocity of airplane A with respect to an observer moving along with Airplane B is given by

$$\begin{aligned}
\vec{\mathbf{v}}'_{\mathrm{A}} &= \vec{\mathbf{v}}_{\mathrm{A}} - \vec{\mathbf{V}} = (v_{\mathrm{A}} \cos\theta_{\mathrm{A}} \hat{\mathbf{i}} + v_{\mathrm{A}} \sin\theta_{\mathrm{A}} \hat{\mathbf{j}}) - (v_{\mathrm{B}} \cos\theta_{\mathrm{B}} \hat{\mathbf{i}} + v_{\mathrm{B}} \sin\theta_{\mathrm{B}} \hat{\mathbf{j}}) \\
&= (v_{\mathrm{A}} \cos\theta_{\mathrm{A}} - v_{\mathrm{B}} \cos\theta_{\mathrm{B}})\hat{\mathbf{i}} + (v_{\mathrm{A}} \sin\theta_{\mathrm{A}} - v_{\mathrm{B}} \sin\theta_{\mathrm{B}})\hat{\mathbf{j}} \\
&= ((80\sqrt{2} \text{ m} \cdot \text{s}^{-1}) - (100\sqrt{2} \text{ m} \cdot \text{s}^{-1}))\hat{\mathbf{i}} + ((80\sqrt{2} \text{ m} \cdot \text{s}^{-1}) + (100\sqrt{2} \text{ m} \cdot \text{s}^{-1}))\hat{\mathbf{j}}. \qquad (11.4.1) \\
&= -(20\sqrt{2} \text{ m} \cdot \text{s}^{-1})\hat{\mathbf{i}} + (180\sqrt{2} \text{ m} \cdot \text{s}^{-1})\hat{\mathbf{j}} \\
&= v'_{Ax} \hat{\mathbf{i}} + v'_{Ay} \hat{\mathbf{j}}
\end{aligned}$$

Figure 11.3 shows the velocity of airplane A with respect to airplane B in reference frame S'.

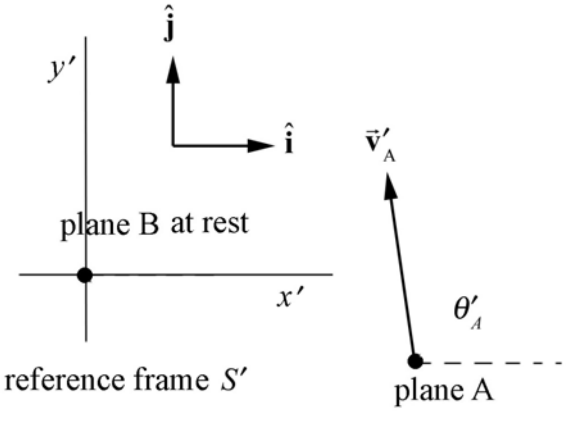


Figure 11.3 Airplane A as seen from observer in airplane B

The magnitude of velocity of airplane A as seen by an observer moving with airplane B is given by

$$\left|\vec{v}_{\mathrm{A}}'\right| = ({v_{Ax}'}^{2} + {v_{Ay}'}^{2})^{1/2} = ((-20\sqrt{2}\ \mathrm{m}\cdot\mathrm{s}^{-1})^{2} + (180\sqrt{2}\ \mathrm{m}\cdot\mathrm{s}^{-1})^{2})^{1/2} = 256\ \mathrm{m}\cdot\mathrm{s}^{-1}. \tag{11.4.2}$$

The angle of velocity of airplane A as seen by an observer moving with airplane B is given by,

$$\begin{aligned}\theta_A' &= \tan^{-1}(v_{Ay}' / v_{Ax}') = \tan^{-1}((180\sqrt{2}\ \mathrm{m}\cdot\mathrm{s}^{-1}) / (-20\sqrt{2}\ \mathrm{m}\cdot\mathrm{s}^{-1})) \\ &= \tan^{-1}(-9) = 180^{\circ} - 83.7^{\circ} = 96.3^{\circ}\end{aligned}. \tag{11.4.3}$$

Example 11.2 Relative Motion and Polar Coordinates

By relative velocity we mean velocity with respect to a specified coordinate system. (The term velocity, alone, is understood to be relative to the observer's coordinate system.) (a) A point is observed to have velocity $\vec{\mathbf{v}}_A$ relative to coordinate system A. What is its velocity relative to coordinate system B, which is displaced from system A by distance $\vec{\mathbf{R}}$? ($\vec{\mathbf{R}}$ can change in time.) (b) Particles a and b move in opposite directions around a circle with the magnitude of the angular velocity ω, as shown in Figure 11.4. At $t = 0$ they are both at the point $\vec{\mathbf{r}} = l\hat{\mathbf{j}}$, where l is the radius of the circle. Find the velocity of a relative to b.

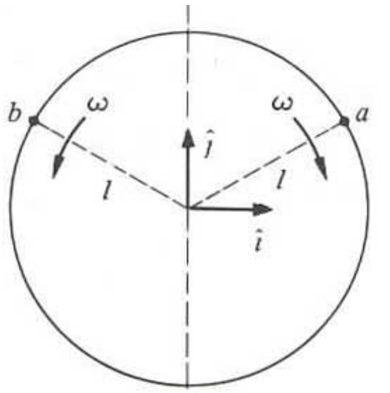


Figure 11.4 Particles a and b moving relative to each other

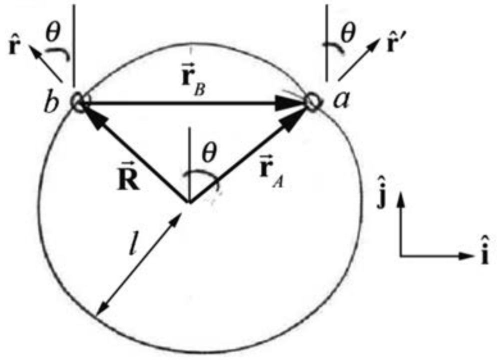


Figure 11.5 Particles a and b moving relative to each other

Solution: (a) The position vectors are related by

$$\vec{\mathbf{r}}_B = \vec{\mathbf{r}}_A - \vec{\mathbf{R}}\,. \tag{11.4.4}$$

The velocities are related by the taking derivatives, (law of addition of velocities Eq. (11.3.2))

$$\vec{\mathbf{v}}_B = \vec{\mathbf{v}}_A - \vec{\mathbf{V}}\,. \tag{11.4.5}$$

(b) Let's choose two reference frames; frame B is centered at particle b, and frame A is centered at the center of the circle in Figure 11.5. Then the relative position vector between the origins of the two frames is given by

$$\vec{\mathbf{R}} = l\,\hat{\mathbf{r}}\,. \tag{11.4.6}$$

The position vector of particle a relative to frame A is given by

$$\vec{\mathbf{r}}_A = l\,\hat{\mathbf{r}}'\,. \tag{11.4.7}$$

The position vector of particle b in frame B can be found by substituting Eqs. (11.4.7) and (11.4.6) into Eq. (11.4.4),

$$\vec{\mathbf{r}}_B = \vec{\mathbf{r}}_A - \vec{\mathbf{R}} = l\,\hat{\mathbf{r}}' - l\,\hat{\mathbf{r}}\,. \tag{11.4.8}$$

We can decompose each of the unit vectors $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$ with respect to the Cartesian unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ (see Figure 11.5),

$$\hat{\mathbf{r}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}} \tag{11.4.9}$$

$$\hat{\mathbf{r}}' = \sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}\,. \tag{11.4.10}$$

Then Eq. (11.4.8) giving the position vector of particle b in frame B becomes

$$\vec{\mathbf{r}}_B = l\,\hat{\mathbf{r}}' - l\,\hat{\mathbf{r}} = l\,(\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}) - l\,(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}) = 2l\sin\theta\,\hat{\mathbf{i}}\,. \tag{11.4.11}$$

In order to find the velocity vector of particle a in frame B (i.e. with respect to particle b), differentiate Eq. (11.4.11)

$$\vec{\mathbf{v}}_B = \frac{d}{dt}(2l\sin\theta)\,\hat{\mathbf{i}} = (2l\cos\theta)\frac{d\theta}{dt}\,\hat{\mathbf{i}} = 2\omega l\cos\theta\,\hat{\mathbf{i}}\,. \tag{11.4.12}$$

Example 11.3 Recoil in Different Frames

A person of mass m_1 is standing on a cart of mass m_2. Assume that the cart is free to move on its wheels without friction. The person throws a ball of mass m_3 at an angle of θ with respect to the horizontal as measured by the person in the cart. The ball is thrown with a speed v_0 with respect to the cart (Figure 11.6). (a) What is the final velocity of the ball as seen by an observer fixed to the ground? (b) What is the final velocity of the cart as seen by an observer fixed to the ground? (c) With respect to the horizontal, what angle the fixed observer see the ball leave the cart?

Figure 11.6 Recoil of a person on cart due to thrown ball

Solution: a), b) Our reference frame will be that fixed to the ground. We shall take as our initial state that before the ball is thrown (cart, ball, throwing person stationary) and our final state that after the ball is thrown. We are assuming that there is no friction, and so there are no external forces acting in the horizontal direction. The initial x-component of the total momentum is zero,

$$p_{x,0}^{\text{total}} = 0\,. \tag{11.4.13}$$

After the ball is thrown, the cart and person have a final momentum

$$\vec{\mathbf{p}}_{f,\text{cart}} = -(m_2 + m_1)v_{f,\text{cart}}\,\hat{\mathbf{i}} \tag{11.4.14}$$

as measured by the person on the ground, where $v_{f,\text{cart}}$ is the speed of the person and cart. (The person's center of mass will move with respect to the cart while the ball is being thrown, but since we're interested in velocities, not positions, we need only assume that the person is at rest with respect to the cart after the ball is thrown.)

The ball is thrown with a speed v_0 and at an angle θ with respect to the horizontal as measured by the person in the cart. Therefore the person in the cart throws the ball with velocity

$$\vec{\mathbf{v}}'_{f,\text{ball}} = v_0\cos\theta\,\hat{\mathbf{i}} + v_0\sin\theta\,\hat{\mathbf{j}}\,. \qquad (11.4.15).$$

Because the cart is moving in the negative x-direction with speed $v_{f,\text{cart}}$ just as the ball leaves the person's hand, the x-component of the velocity of the ball as measured by an observer on the ground is given by

$$v_{xf,\text{ball}} = v_0 \cos\theta - v_{f,\text{cart}} . \tag{11.4.16}$$

The ball appears to have a smaller x-component of the velocity according to the observer on the ground. The velocity of the ball as measured by an observer on the ground is

$$\vec{\mathbf{v}}_{f,\text{ball}} = (v_0 \cos\theta - v_{f,\text{cart}})\,\hat{\mathbf{i}} + v_0 \sin\theta\,\hat{\mathbf{j}} . \tag{11.4.17}$$

The final momentum of the ball according to an observer on the ground is

$$\vec{\mathbf{p}}_{f,\text{ball}} = m_3\left[(v_0 \cos\theta - v_{f,\text{cart}})\,\hat{\mathbf{i}} + v_0 \sin\theta\,\hat{\mathbf{j}}\right]. \tag{11.4.18}$$

The momentum flow diagram is shown in (Figure 11.7).

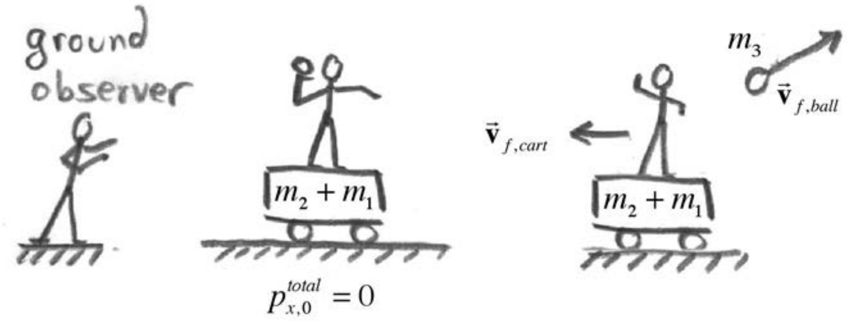


Figure 11.7 Momentum flow diagram for recoil

Because the x -component of the momentum of the system is constant, we have that

$$\begin{aligned} 0 &= (p_{x,f})_{\text{cart}} + (p_{x,f})_{\text{ball}} \\ &= -(m_2 + m_1)v_{f,\text{cart}} + m_3(v_0 \cos\theta - v_{f,\text{cart}}). \end{aligned} \tag{11.4.19}$$

We can solve Equation (11.4.19) for the final speed and velocity of the cart as measured by an observer on the ground,

$$v_{f,\text{cart}} = \frac{m_3\, v_0 \cos\theta}{m_2 + m_1 + m_3} , \tag{11.4.20}$$

$$\vec{\mathbf{v}}_{f,\text{ cart}} = v_{f,\text{ cart}}\,\hat{\mathbf{i}} = \frac{m_3\, v_0 \cos\theta}{m_2 + m_1 + m_3}\hat{\mathbf{i}} . \tag{11.4.21}$$

Note that the y-component of the momentum is not constant because as the person is throwing the ball he or she is pushing off the cart and the normal force with the ground exceeds the gravitational force so the net external force in the y-direction is non-zero.

Substituting Equation (11.4.20) into Equation (11.4.17) gives

$$\begin{aligned}\vec{\mathbf{v}}_{f,\text{ball}} &= (v_0\cos\theta - v_{f,\text{cart}})\,\hat{\mathbf{i}} + v_0\sin\theta\,\hat{\mathbf{j}}\\ &= \frac{m_1+m_2}{m_1+m_2+m_3}(v_0\cos\theta)\hat{\mathbf{i}} + (v_0\sin\theta)\,\hat{\mathbf{j}}.\end{aligned} \tag{11.4.22}$$

As a check, note that in the limit $m_3 << m_1 + m_2$, $\vec{\mathbf{v}}_{f,\text{ball}}$ has speed v_0 and is directed at an angle θ above the horizontal; the fact that the much more massive person-cart combination is free to move doesn't affect the flight of the ball as seen by the fixed observer. Also note that in the unrealistic limit $m >> m_1 + m_2$ the ball is moving at a speed much smaller than v_0 as it leaves the cart.

c) The angle ϕ at which the ball is thrown as seen by the observer on the ground is given by

$$\begin{aligned}\phi &= \tan^{-1}\frac{(v_{f,\text{ball}})_y}{(v_{f,\text{ball}})_x} = \tan^{-1}\frac{v_0\sin\theta}{\left[(m_1+m_2)/(m_1+m_2+m_3)\right]v_0\cos\theta}\\ &= \tan^{-1}\left[\left(\frac{m_1+m_2+m_3}{m_1+m_2}\right)\tan\theta\right].\end{aligned} \tag{11.4.23}$$

For arbitrary values for the masses, the above expression will not reduce to a simplified form. However, we can see that $\tan\phi > \tan\theta$ for arbitrary masses, and that in the limit $m_3 << m_1 + m_2$, $\phi \to \theta$ and in the unrealistic limit $m_3 >> m_1 + m_2$, $\phi \to \pi/2$. Can you explain this last odd prediction?

Chapter 12 Momentum and the Flow of Mass

Chapter 12 Momentum and the Flow of Mass

> *Even though the release was pulled, the rocket did not rise at first, but the flame came out, and there was a steady roar. After a number of seconds it rose, slowly until it cleared the flame, and then at express-train speed, curving over to the left, and striking the ice and snow, still going at a rapid rate. It looked almost magical as it rose, without any appreciably greater noise or flame, as if it said, "I've been here long enough; I think I'll be going somewhere else, if you don't mind."* [1]
>
> *Robert Goddard*

Preface: The *Challenger* Flight

When the Rogers Commission in 1986 investigated the *Challenger* disaster, a commission member, physicist Richard Feynman, made an extraordinary demonstration during the hearings.

"He (Feynman) also learned that rubber used to seal the solid rocket booster joints using O-rings, failed to expand when the temperature was at or below 32 degrees F (0 degrees C). The temperature at the time of the *Challenger* liftoff was 32 degrees F. Feynman now believed that he had the solution, but to test it, he dropped a piece of the O-ring material, squeezed with a C-clamp to simulate the actual conditions of the shuttle, into a glass of ice water. Ice, of course, is 32 degrees F. At this point one needs to understand exactly what role the O-rings play in the solid rocket booster (SRB) joints. When the material in the SRB start to heat up, it expands and pushes against the sides of the SRB. If there is an opening in a joint in the SRB, the gas tries to escape through that opening (think of it like water in a tea kettle escaping through the spout.) This leak in the *Challenger's* SRB was easily visible as a small flicker in a launch photo. This flicker turned into a flame and began heating the fuel tank, which then ruptured. When this happened, the fuel tank released liquid hydrogen into the atmosphere where it exploded. As Feynman explained, because the O-rings cannot expand in 32 degree weather, the gas finds gaps in the joints, which led to the explosion of the booster and then the shuttle itself."[2]

In the Report of the Presidential Commission on the Space Shuttle *Challenger* Accident (1986), *Appendix F - Personal observations on the reliability of the Shuttle*, Feynman wrote

The *Challenger* flight is an excellent example. … The O-rings of the Solid Rocket Boosters were not designed to erode. Erosion was a clue that something was wrong. Erosion was not something from which safety can be inferred. There was no way, without

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[1] describing the first rocket flight using liquid propellants at Aunt Effie's farm, 17 March 1926.

[2] http://www.fotuva.org/online/frameload.htm?/online/challenger.htm.

full understanding, that one could have confidence that conditions the next time might not produce erosion three times more severe than the time before. Nevertheless, officials fooled themselves into thinking they had such understanding and confidence, in spite of the peculiar variations from case to case. A mathematical model was made to calculate erosion. This was a model based not on physical understanding but on empirical curve fitting. To be more detailed, it was supposed a stream of hot gas impinged on the O-ring material, and the heat was determined at the point of stagnation (so far, with reasonable physical, thermodynamic laws). But to determine how much rubber eroded it was assumed this depended only on this heat by a formula suggested by data on a similar material. A logarithmic plot suggested a straight line, so it was supposed that the erosion varied as the .58 power of the heat, the .58 being determined by a nearest fit. At any rate, adjusting some other numbers, it was determined that the model agreed with the erosion (to depth of one-third the radius of the ring). There is nothing much so wrong with this as believing the answer! Uncertainties appear everywhere. How strong the gas stream might be was unpredictable, it depended on holes formed in the putty. Blow-by showed that the ring might fail even though not, or only partially eroded through. The empirical formula was known to be uncertain, for it did not go directly through the very data points by which it was determined. There were a cloud of points some twice above, and some twice below the fitted curve, so erosions twice predicted were reasonable from that cause alone. Similar uncertainties surrounded the other constants in the formula, etc., etc. When using a mathematical model careful attention must be given to uncertainties in the model. …

In any event this has had very unfortunate consequences, the most serious of which is to encourage ordinary citizens to fly in such a dangerous machine, as if it had attained the safety of an ordinary airliner. The astronauts, like test pilots, should know their risks, and we honor them for their courage. Who can doubt that McAuliffe was equally a person of great courage, who was closer to an awareness of the true risk than NASA management would have us believe? Let us make recommendations to ensure that NASA officials deal in a world of reality in understanding technological weaknesses and imperfections well enough to be actively trying to eliminate them. …. For a successful technology, reality must take precedence over public relations, for nature cannot be fooled.[3]

12.1 Introduction

So far we have restricted ourselves to considering systems consisting of discrete objects or point-like objects that have fixed amounts of mass. We shall now consider systems in which material flows between the objects in the system, for example we shall consider coal falling from a hopper into a moving railroad car, sand leaking from railroad car fuel, grain moving forward into a railroad car, and fuel ejected from the back of a rocket, In each of these examples material is continuously flows into or out of an object. We have already shown that the total external force causes the momentum of a system to change,

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[3] R. P. Feynman, *Appendix F - Personal observations on the reliability of the Shuttle*, Report of the PRESIDENTIAL COMMISSION on the Space Shuttle Challenger Accident (1986), http://history.nasa.gov/rogersrep/genindex.htm.

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \frac{d\vec{\mathbf{p}}_{\text{system}}}{dt}. \tag{12.2.1}$$

We shall analyze how the momentum of the constituent elements our system change over a time interval $[t, t+\Delta t]$, and then consider the limit as $\Delta t \to 0$. We can then explicit calculate the derivative on the right hand side of Eq. (12.2.1) and Eq. (12.2.1) becomes

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \frac{d\vec{\mathbf{p}}_{\text{system}}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\mathbf{p}}_{\text{system}}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{p}}_{\text{system}}(t+\Delta t) - \vec{\mathbf{p}}_{\text{system}}(t)}{\Delta t}. \tag{12.2.2}$$

We need to be very careful how we apply this generalized version of Newton's Second Law to systems in which mass flows between constituent objects. In particular, when we isolate elements as part of our system we must be careful to identify the mass Δm of the material that continuous flows in or out of an object that is part of our system during the time interval Δt under consideration.

We shall consider four categories of mass flow problems that are characterized by the momentum transfer of the material of mass Δm.

12.1.1 Transfer of Material into an Object, but no Transfer of Momentum

Consider for example rain falling vertically downward with speed u into car of mass m moving forward with speed v. A small amount of falling rain Δm_r has no component of momentum in the direction of motion of the car. There is a transfer of rain into the car but no transfer of momentum in the direction of motion of the car (Figure 12.1).

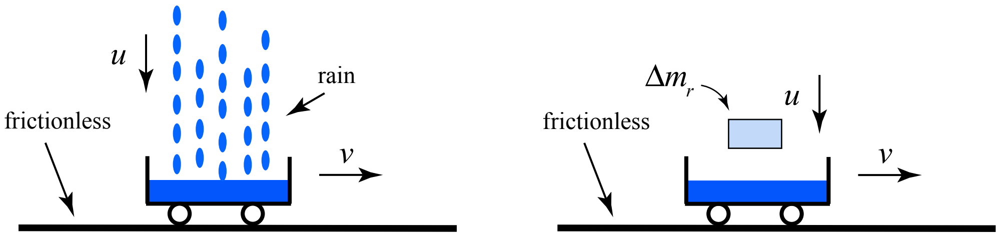


Figure 12.1 Transfer of rain mass into the car but no transfer of momentum in direction of motion

12.1.2 Transfer of Material Out of an Object, but no Transfer of Momentum

The material continually leaves the object but it does not transport any momentum away from the object in the direction of motion of the object (Figure 12.2). Consider an ice skater gliding on ice at speed v holding a bag of sand that is leaking straight down with respect to the moving skater. The sand continually leaves the bag but it does not transport any momentum away from the bag in the direction of motion of the object. In Figure 12.2, sand of mass Δm_s leaves the bag.

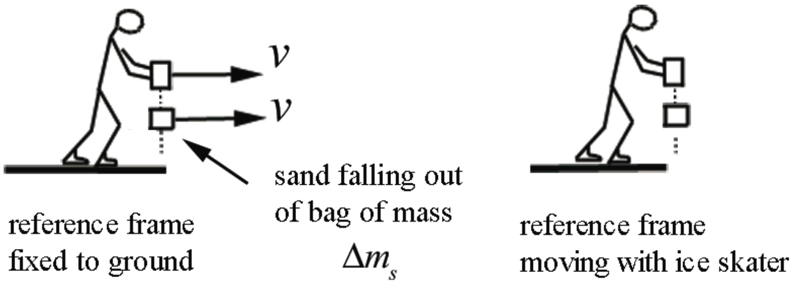


Figure 12.2 Transfer of mass out of object but no transfer of momentum in direction of motion

12.1.3 Transfer of Material Impulses Object Via Transfer of Momentum

Suppose a fire hose is used to put out a fire on a boat of mass m_b. Assume the column of water moves horizontally with speed u. The incoming water continually hits the boat propelling it forward. During the time interval Δt, a column of water of mass Δm_s will hit the boat that is moving forward with speed v increasing it's speed (Figure 12.3).

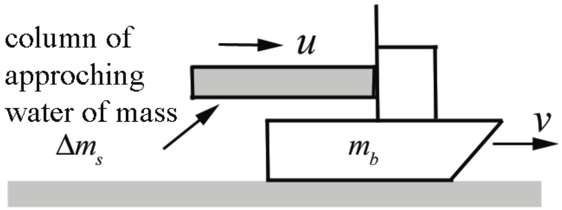


Figure 12.3 Transfer of mass of water increases speed of boat

12.1.4 Material Continually Ejected From Object results in Recoil of Object

When fuel of mass Δm_f is ejected from the back of a rocket with speed u relative to the rocket, the rocket of mass m_r recoils forward. Figure 12.4a shows the recoil of the rocket in the reference frame of the rocket. The rocket recoils forward with speed Δv_r. In a reference frame in which the rocket is moving forward with speed v_r, then the speed after recoil is $v_r + \Delta v_r$. The speed of the backwardly ejected fuel is $u - v_r$ (Figure 12.4b).

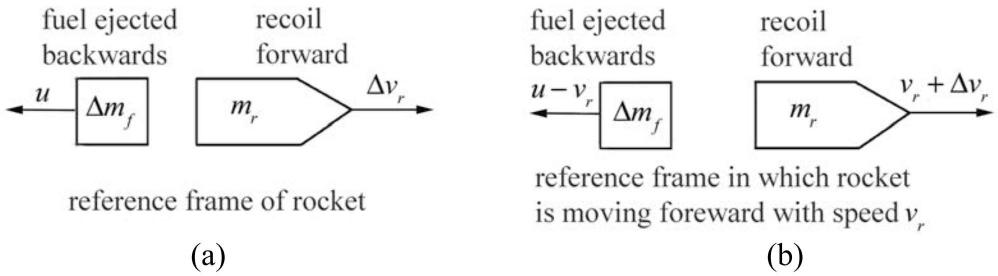


Figure 12.4 Transfer of mass out of rocket provides impulse on rocket in (a) reference frame of rocket, (b) reference frame in which rocket moves with speed v_r

We must carefully identify the momentum of the object and the material transferred at time t in order to determine $\vec{\mathbf{p}}_{\text{system}}(t)$. We must also identify the momentum of the object and the material transferred at time $t+\Delta t$ in order to determine $\vec{\mathbf{p}}_{\text{system}}(t+\Delta t)$ as well. Recall that when we defined the momentum of a system, we assumed that the mass of the system remain constant. Therefore we cannot ignore the momentum of the transferred material at time $t+\Delta t$ even though it may have left the object; it is still part of our system (or at time t even though it has not flowed into the object yet).

12.2 Worked Examples

Example 12.1 Filling a Coal Car

An empty coal car of mass m_0 starts from rest under an applied force of magnitude F. At the same time coal begins to run into the car at a steady rate b from a coal hopper at rest along the track (Figure 12.5). Find the speed when a mass m_c of coal has been transferred.

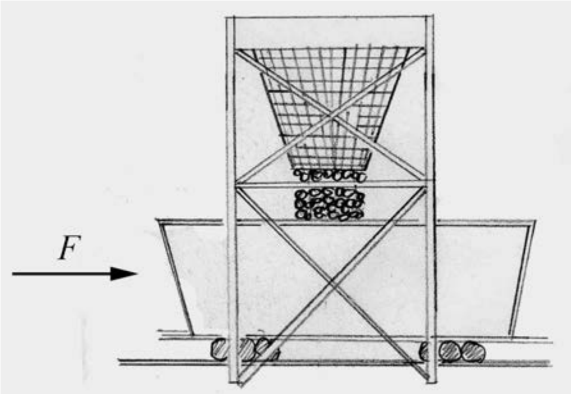


Figure 12.5 Filling a coal car

Solution: We shall analyze the momentum changes in the horizontal direction, which we call the x-direction. Because the falling coal does not have any horizontal velocity, the falling coal is not transferring any momentum in the x-direction to the coal car. So we shall take as our system the empty coal car and a mass m_c of coal that has been transferred. Our initial state at $t=0$ is when the coal car is empty and at rest before any coal has been transferred. The x-component of the momentum of this initial state is zero,

$$p_x(0)=0\,. \qquad (12.3.1)$$

Our final state at $t = t_f$ is when all the coal of mass $m_c = bt_f$ has been transferred into the car that is now moving at speed v_f. The x-component of the momentum of this final state is

$$p_x(t_f) = (m_0 + m_c)v_f = (m_0 + bt_f)v_f \,. \tag{12.3.2}$$

There is an external constant force $F_x = F$ applied through the transfer. The momentum principle applied to the x-direction is

$$\int_0^{t_f} F_x dt = \Delta p_x = p_x(t_f) - p_x(0) \,. \tag{12.3.3}$$

Because the force is constant, the integral is simple and the momentum principle becomes

$$Ft_f = (m_0 + bt_f)v_f \,. \tag{12.3.4}$$

So the final speed is

$$v_f = \frac{Ft_f}{(m_0 + bt_f)} \,. \tag{12.3.5}$$

Example 12.2 Emptying a Freight Car

A freight car of mass m_c contains sand of mass m_s. At $t = 0$ a constant horizontal force of magnitude F is applied in the direction of rolling and at the same time a port in the bottom is opened to let the sand flow out at the constant rate $b = dm_s / dt$. Find the speed of the freight car when all the sand is gone (Figure 12.6). Assume that the freight car is at rest at $t = 0$.

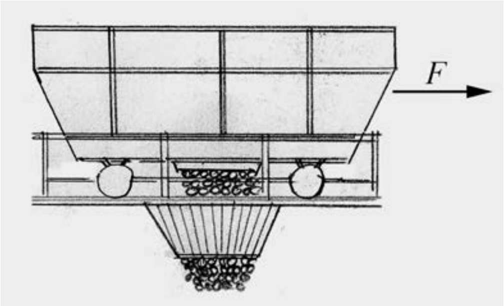


Figure 12.6 Emptying a freight car

Solution: Choose the positive x-direction to point in the direction that the car is moving. Choose for the system the amount of sand in the fright car at time t, $m_c(t)$. At time t, the car is moving with velocity $\vec{\mathbf{v}}_c(t) = v_c(t)\hat{\mathbf{i}}$. The momentum diagram for the system at time t is shown in the diagram on the left in Figure 12.7.

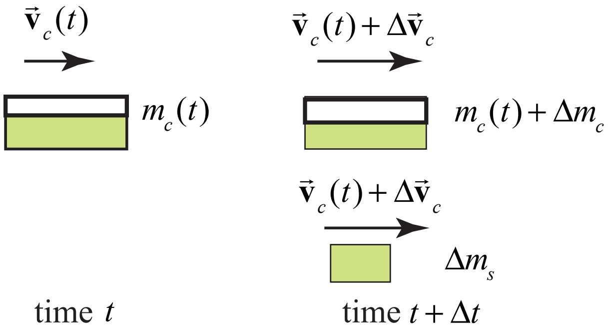


Figure 12.7 Momentum diagram at time t and at time $t+\Delta t$

The momentum of the system at time t is given by

$$\vec{\mathbf{p}}_{sys}(t) = m_c(t)\vec{\mathbf{v}}_c(t). \tag{12.3.6}$$

During the time interval $[t, t+\Delta t]$, an amount of sand of mass Δm_s leaves the freight car and the mass of the freight car changes by $m_c(t+\Delta t) = m_c(t) + \Delta m_c$, where $\Delta m_c = -\Delta m_s$. At the end of the interval the car is moving with velocity $\vec{\mathbf{v}}_c(t+\Delta t) = \vec{\mathbf{v}}_c(t) + \Delta\vec{\mathbf{v}}_c = (v_c(t) + \Delta v_c)\hat{\mathbf{i}}$. The momentum diagram for the system at time $t+\Delta t$ is shown in the diagram on the right in Figure 12.7. The momentum of the system at time $t+\Delta t$ is given by

$$\vec{\mathbf{p}}_{sys}(t+\Delta t) = (\Delta m_s + m_c(t) + \Delta m_c)(\vec{\mathbf{v}}_c(t) + \Delta\vec{\mathbf{v}}_c) = m_c(t)(\vec{\mathbf{v}}_c(t) + \Delta\vec{\mathbf{v}}_c). \tag{12.3.7}$$

Note that the sand that leaves the car is shown with velocity $\vec{\mathbf{v}}_c(t) + \Delta\vec{\mathbf{v}}_c$. This implies that all the sand leaves the car with the velocity of the car at the end of the interval. This is an approximation. Because the sand leaves continuous, the velocity will vary from $\vec{\mathbf{v}}_c(t)$ to $\vec{\mathbf{v}}_c(t) + \Delta\vec{\mathbf{v}}_c$ but so does the change in mass of the car and these two contributions to the system's moment exactly cancel. The change in momentum of the system is then

$$\Delta\vec{\mathbf{p}}_{sys} = \vec{\mathbf{p}}_{sys}(t+\Delta t) - \vec{\mathbf{p}}_{sys}(t) = m_c(t)(\vec{\mathbf{v}}_c(t) + \Delta\vec{\mathbf{v}}_c) - m_c(t)\vec{\mathbf{v}}_c(t) = m_c(t)\Delta\vec{\mathbf{v}}_c. \tag{12.3.8}$$

Throughout the interval a constant force $\vec{\mathbf{F}} = F\hat{\mathbf{i}}$ is applied to the system so the momentum principle becomes

$$\vec{\mathbf{F}} = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{p}}_{sys}(t+\Delta t) - \vec{\mathbf{p}}_{sys}(t)}{\Delta t} = \lim_{\Delta t \to 0} m_c(t)\frac{\Delta \vec{\mathbf{v}}_c}{\Delta t} = m_c(t)\frac{d\vec{\mathbf{v}}_c}{dt}. \qquad (12.3.9)$$

Because the motion is one-dimensional, Eq. (12.3.9) written in terms of x-components becomes

$$F = m_c(t)\frac{dv_c}{dt}. \qquad (12.3.10)$$

Denote by initial mass of the car by $m_{c,0} = m_c + m_s$ where m_c is the mass of the car and m_s is the mass of the sand in the car at $t = 0$. The mass of the sand that has left the car at time t is given by

$$m_s(t) = \int_0^t \frac{dm_s}{dt}dt = \int_0^t bdt = bt. \qquad (12.3.11)$$

Thus

$$m_c(t) = m_{c,0} - bt = m_c + m_s - bt. \qquad (12.3.12)$$

Therefore Eq. (12.3.10) becomes

$$F = (m_c + m_s - bt)\frac{dv_c}{dt}. \qquad (12.3.13)$$

This equation can be solved for the x-component of the velocity at time t, $v_c(t)$, (which in this case is the speed) by the method of separation of variables. Rewrite Eq. (12.3.13) as

$$dv_c = \frac{Fdt}{(m_c + m_s - bt)}. \qquad (12.3.14)$$

Then integrate both sides of Eq. (12.3.14) with the limits as shown

$$\int_{v'_c=0}^{v'_c=v_c(t)} dv'_c = \int_{t'=0}^{t'=t} \frac{Fdt'}{m_c + m_s - bt'}. \qquad (12.3.15)$$

Integration yields the speed of the car as a function of time

$$v_c(t) = -\frac{F}{b}\ln(m_c + m_s - bt')\Big|_{t'=0}^{t'=t} = -\frac{F}{b}\ln\left(\frac{m_c + m_s - bt}{m_c + m_s}\right) = \frac{F}{b}\ln\left(\frac{m_c + m_s}{m_c + m_s - bt}\right). \qquad (12.3.16)$$

In writing Eq. (12.3.16), we used the property that $\ln(a)-\ln(b)=\ln(a/b)$ and therefore $\ln(a/b)=-\ln(b/a)$. Note that $m_c+m_s \geq m_c+m_s-bt$, so the term $\ln\left(\dfrac{m_c+m_s}{m_c+m_s-bt}\right) \geq 0$, and the speed of the car increases as we expect.

Example 12.3 Filling a Freight Car

Grain is blown into car A from car B at a rate of b kilograms per second. The grain leaves the chute vertically downward, so that it has the same horizontal velocity, u as car B, (Figure 12.8). Car A is initially at rest before any grain is transferred in and has mass $m_{A,0}$. At the moment of interest, car A has mass m_A and speed v. Determine an expression for the speed car A as a function of time t.

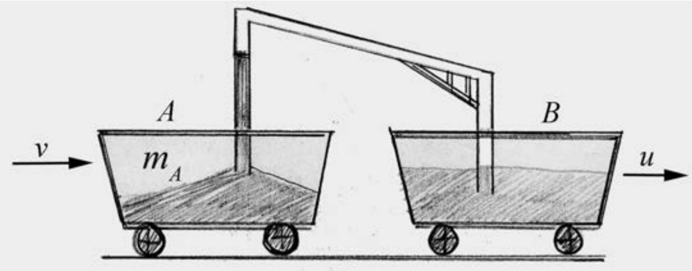


Figure 12.8 Filling a freight car

Solution: Choose positive x-direction to the right in the direction the cars are moving. Define the system at time t to be the car and grain that is already in it, which together has mass $m_A(t)$, and the small amount of material of mass Δm_g that is blown into car A during the time interval $[t, t+\Delta t]$. At time that is moving with x-component of the velocity v_A. At time t, car A is moving with velocity $\vec{\mathbf{v}}_A(t)=v_A(t)\hat{\mathbf{i}}$, and the material blown into car is moving with velocity $\vec{\mathbf{u}}=u\hat{\mathbf{i}}$ At time $t+\Delta t$, car A is moving with velocity $\vec{\mathbf{v}}_A(t)+\Delta\vec{\mathbf{v}}_A=(v_A(t)+\Delta v_A)\hat{\mathbf{i}}$, and the mass of car A is $m_A(t+\Delta t)=m_A(t)+\Delta m_A$, where $\Delta m_A=\Delta m_g$. The momentum diagram for times t and for $t+\Delta t$ is shown in Figure 12.9.

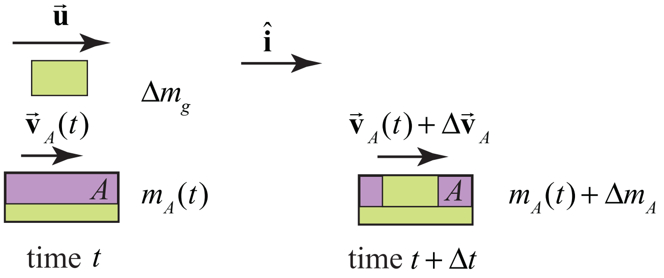


Figure 12.9 Momentum diagram at times t and $t+\Delta t$

The momentum at time t is

$$\vec{\mathbf{P}}_{sys}(t) = m_A(t)\vec{\mathbf{v}}_A(t) + \Delta m_g \vec{\mathbf{u}} \ . \tag{12.3.17}$$

The momentum at time $t + \Delta t$ is

$$\vec{\mathbf{P}}_{sys}(t + \Delta t) = (m_A(t) + \Delta m_A)(\vec{\mathbf{v}}_A(t) + \Delta\vec{\mathbf{v}}_A) \, . \tag{12.3.18}$$

There are no external forces acting on the system in the x-direction and the external forces acting on the system perpendicular to the motion sum to zero, so the momentum principle becomes

$$\vec{\mathbf{0}} = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{P}}_{sys}(t + \Delta t) - \vec{\mathbf{P}}_{sys}(t)}{\Delta t} \, . \tag{12.3.19}$$

Using the results above (Eqs. (12.3.17) and (12.3.18), the momentum principle becomes

$$\vec{\mathbf{0}} = \lim_{\Delta t \to 0} \frac{(m_A(t) + \Delta m_A)(\vec{\mathbf{v}}_A(t) + \Delta\vec{\mathbf{v}}_A) - (m_A(t)\vec{\mathbf{v}}_A(t) + \Delta m_g \vec{\mathbf{u}})}{\Delta t} \, . \tag{12.3.20}$$

which after using the condition that $\Delta m_A = \Delta m_g$ and some rearrangement becomes

$$\vec{\mathbf{0}} = \lim_{\Delta t \to 0} \frac{m_A(t)\Delta\vec{\mathbf{v}}_A}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_A(\vec{\mathbf{v}}_A(t) - \vec{\mathbf{u}})}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_A \Delta\vec{\mathbf{v}}_A}{\Delta t} \, . \tag{12.3.21}$$

In the limit as , the product $\Delta m_A \Delta\vec{\mathbf{v}}_A$ is a second order differential (the product of two first order differentials) and the term $\Delta m_A \Delta\vec{\mathbf{v}}_A / \Delta t$ approaches zero, therefore the momentum principle yields the differential equation

$$\vec{\mathbf{0}} = m_A(t)\frac{d\vec{\mathbf{v}}_A}{dt} + \frac{dm_A}{dt}(\vec{\mathbf{v}}_A(t) - \vec{\mathbf{u}}) \, . \tag{12.3.22}$$

The x-component of Eq. (12.3.22) is then

$$0 = m_A(t)\frac{dv_A}{dt} + \frac{dm_A}{dt}(v_A(t) - u) \, . \tag{12.3.23}$$

Rearranging terms and using the fact that the material is blown into car A at a constant rate $b \equiv dm_A / dt$, we have that the rate of change of the x-component of the velocity of car A is given by

$$\frac{dv_A(t)}{dt} = \frac{b(u - v_A(t))}{m_A(t)} \, . \tag{12.3.24}$$

We cannot directly integrate Eq. (12.3.24) with respect to dt because the mass of the car A is a function of time. In order to find the x-component of the velocity of car A we need to know the relationship between the mass of car A and the x-component of the velocity of the car A. There are two approaches. In the first approach we separate variables in Eq. (12.3.24) where we have suppressed the dependence on t in the expressions for m_A and v_A yielding

$$\frac{dv_A}{u - v_A} = \frac{dm_A}{m_A}, \tag{12.3.25}$$

which becomes the integral equation

$$\int_{v'_A=0}^{v'_A=v_A(t)} \frac{dv'_A}{u - v'_A} = \int_{m'_A=m_{A,0}}^{m'_A=m_A(t)} \frac{dm'_A}{m'_A}, \tag{12.3.26}$$

where $m_{A,0}$ is the mass of the car before any material has been blown in. After integration we have that

$$\ln\frac{u}{u - v_A(t)} = \ln\frac{m_A(t)}{m_{A,0}}. \tag{12.3.27}$$

Exponentiate both side yields

$$\frac{u}{u - v_A(t)} = \frac{m_A(t)}{m_{A,0}}. \tag{12.3.28}$$

We can solve this equation for the x-component of the velocity of the car

$$v_A(t) = \frac{m_A(t) - m_{A,0}}{m_A(t)} u. \tag{12.3.29}$$

Because the material is blown into the car at a constant rate $b \equiv dm_A / dt$, the mass of the car as a function of time is given by

$$m_A(t) = m_{A,0} + bt. \tag{12.3.30}$$

Therefore substituting Eq. (12.3.30) into Eq. (12.3.29) yields the x-component of the velocity of the car as a function of time

$$v_A(t) = \frac{bt}{m_{A,0} + bt} u. \tag{12.3.31}$$

In a second approach, we substitute Eq. (12.3.30) into Eq. (12.3.24) yielding

$$\frac{dv_A}{dt} = \frac{b(u - v_A)}{m_{A,0} + bt}. \qquad (12.3.32)$$

Separate variables in Eq. (12.3.32):

$$\frac{dv_A}{u - v_A} = \frac{bdt}{m_{A,0} + bt}, \qquad (12.3.33)$$

which then becomes the integral equation

$$\int_{v_A'=0}^{v_A'=v_A(t)} \frac{dv_A'}{u - v_A'} = \int_{t'=0}^{t'=t} \frac{dt'}{m_{A,0} + bt'}. \qquad (12.3.34)$$

Integration yields

$$\ln\frac{u}{u - v_A(t)} = \ln\frac{m_{A,0} + bt}{m_{A,0}}. \qquad (12.3.35)$$

Again exponentiate both sides resulting in

$$\frac{u}{u - v_A(t)} = \frac{m_{A,0} + bt}{m_{A,0}}. \qquad (12.3.36)$$

After some algebraic manipulation we can find the speed of the car as a function of time

$$v_A(t) = \frac{bt}{m_{A,0} + bt} u. \qquad (12.3.37)$$

in agreement with Eq. (12.3.31).

Check result:

We can rewrite Eq. (12.3.37) as

$$(m_{A,0} + bt)v_A(t) = btu, \qquad (12.3.38)$$

which illustrates the point that the momentum of the system at time t is equal to the momentum of the grain that has been transferred to the system during the interval $[0,t]$.

Example 12.4 Boat and Fire Hose

A burning boat of mass m_0 is initially at rest. A fire fighter stands on a bridge and sprays water onto the boat. The water leaves the fire hose with a speed u at a rate α (measured in $\text{kg}\cdot\text{s}^{-1}$). Assume that the motion of the boat and the water jet are horizontal, that gravity does not play any role, and that the river can be treated as a frictionless surface. Also assume that the change in the mass of the boat is only due to the water jet and that all the water from the jet is added to the boat, (Figure 12.10).

Figure 12.10 Example 12.4

a) In a time interval $[t, t+\Delta t]$, an amount of water Δm hits the boat. Choose a system. Is the total momentum constant in your system? Write down a differential equation that results from the analysis of the momentum changes inside your system.

b) Integrate the differential equation you found in part a), to find the velocity $v(m)$ as a function of the increasing mass m of the boat, m_0, and u.

Solution: Let's take as our system the boat, the amount of water of mass Δm_w that enters the boat during the time interval $[t, t+\Delta t]$ and whatever water is in the boat at time t. The water from the fire hose has a speed u. Denote the mass of the boat (including some water) at time t by $m_b \equiv m_b(t)$, and the speed of the boat by $v \equiv v_b(t)$. At time $t+\Delta t$ the speed of the boat is $v+\Delta v$. Choose the positive x- direction in the direction that the boat is moving. Then the x-components of the momentum of the system at time t and $t+\Delta t$ are shown in Figure 12.11.

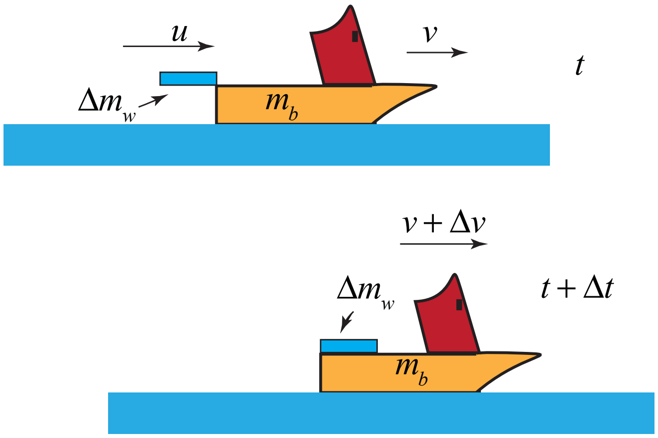


Figure 12.11 Momentum diagrams for burning boat

Because we are assuming that the burning boat slides with negligible resistance and that gravity has a negligible effect on the arc of the water jet, there are no external forces acting on the system in the x-direction. Therefore the x-component of the momentum of the system is constant during the interval $[t, t+\Delta t]$ and so

$$0 = \lim_{\Delta t \to 0} \frac{p_x(t + \Delta t) - p_x(t)}{\Delta t}. \tag{12.3.39}$$

Using the information from the figure above, Eq. (12.3.39) becomes

$$0 = \lim_{\Delta t \to 0} \frac{(m_b + \Delta m_w)(v + \Delta v) - (\Delta m_w u + m_b v)}{\Delta t}. \tag{12.3.40}$$

Eq. (12.3.40) simplifies to

$$0 = \lim_{\Delta t \to 0} m_b \frac{\Delta v}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_w}{\Delta t} v + \lim_{\Delta t \to 0} \frac{\Delta m_w \Delta v}{\Delta t} - \lim_{\Delta t \to 0} \frac{\Delta m_w}{\Delta t} u. \tag{12.3.41}$$

The third term vanishes when we take the limit $\Delta t \to 0$ because it is of second order in the infinitesimal quantities (in this case $\Delta m_w \Delta v$) and so when dividing by Δt the quantity is of first order and hence vanishes since both $\Delta m_w \to 0$ and $\Delta v \to 0$. Eq. (12.3.41) becomes

$$0 = \lim_{\Delta t \to 0} m_b \frac{\Delta v}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_w}{\Delta t} v - \lim_{\Delta t \to 0} \frac{\Delta m_w}{\Delta t} u. \tag{12.3.42}$$

We now use the definition of the derivatives:

$$\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}; \; \lim_{\Delta t \to 0} \frac{\Delta m_w}{\Delta t} = \frac{dm_w}{dt}. \tag{12.3.43}$$

in Eq. (12.3.42) to fund the differential equation describing the relation between the acceleration of the boat and the time rate of change of the mass of water entering the boat

$$0 = m_b \frac{dv}{dt} + \frac{dm_w}{dt}(v - u). \tag{12.3.44}$$

The mass of the boat is increasing due to the addition of the water. Let $m_w(t)$ denote the mass of the water that is in the boat at time t. Then the mass of the boat can be written as

$$m_b(t) = m_0 + m_w(t), \tag{12.3.45}$$

where m_0 is the mass of the boat before any water entered. Note we are neglecting the effect of the fire on the mass of the boat. Differentiating Eq. (12.3.45) with respect to time yields

$$\frac{dm_b}{dt} = \frac{dm_w}{dt}, \tag{12.3.46}$$

Then Eq. (12.3.44) becomes

$$0 = m_b \frac{dv}{dt} + \frac{dm_b}{dt}(v - u). \tag{12.3.47}$$

(b) We can integrate this equation through the separation of variable technique. Rewrite Eq. (12.3.47) as (cancel the common factor dt)

$$\frac{dv}{v-u} = -\frac{dm_b}{m_b}. \tag{12.3.48}$$

We can then integrate both sides of Eq. (12.3.48) with the limits as shown

$$\int_{v=0}^{v(t)} \frac{dv}{v-u} = -\int_{m_0}^{m_b(t)} \frac{dm_b}{m_b} \tag{12.3.49}$$

Integration yields

$$\ln\left(\frac{v(t)-u}{-u}\right) = -\ln\left(\frac{m_b(t)}{m_0}\right) \tag{12.3.50}$$

Recall that $\ln(a/b) = -\ln(b/a)$ so Eq. (12.3.50) becomes

$$\ln\left(\frac{v(t)-u}{-u}\right) = \ln\left(\frac{m_0}{m_b(t)}\right) \tag{12.3.51}$$

Also recall that $\exp(\ln(a/b)) = a/b$ and so exponentiating both sides of Eq. (12.3.51) yields

$$\frac{v(t)-u}{-u} = \frac{m_0}{m_b(t)} \tag{12.3.52}$$

So the speed of the boat at time t can be expressed as

$$v(t) = u\left(1-\frac{m_0}{m_b(t)}\right) \tag{12.3.53}$$

Check result:

We can rewrite Eq. (12.3.52) as

$$m_b(t)(v(t)-u) = -m_0 u \Rightarrow m_b(t)v(t) = (m_b(t)-m_0)u \ . \tag{12.3.54}$$

Recall that the mass of the water that enters the car during the interval $[0,t]$ is $m_w(t) = m_b(t) - m_0$. Therefore Eq. (12.3.54) becomes

$$m_b(t)v(t) = m_w(t)u \ . \tag{12.3.55}$$

During the interaction between the jet of water and the boat, the water transfers an amount of momentum $m_w(t)u$ to the boat and car producing a momentum $m_b(t)v(t)$. Because all the water that collides with the boat ends up in the boat, all the interaction forces between the jet of water and the boat are internal forces. The boat recoils forward and the water recoils backward and through collisions with the boat stays in the boat. Therefore if we choose as our system, all of the water that eventually ends up in the boat and the boat then the momentum principle states

$$p_{sys}(t) = p_{sys}(0) \ , \tag{12.3.56}$$

where $p_{sys}(0) = m_w(t)u$ is the momentum of all of the water that eventually ends up in the boat.

Note that the problem didn't ask to find the speed of the boat as a function t. We shall now show how to find that. We begin by observing that

$$\frac{dm_b}{dt} = \frac{dm_w}{dt} \neq \alpha \tag{12.3.57}$$

where the constant α is measured in $\mathrm{kg \cdot s^{-1}}$ and is specified as a given constant according to the information in the problem statement. The reason is that α is the rate that the water is ejected from the hose but not the rate that the water enters the boat.

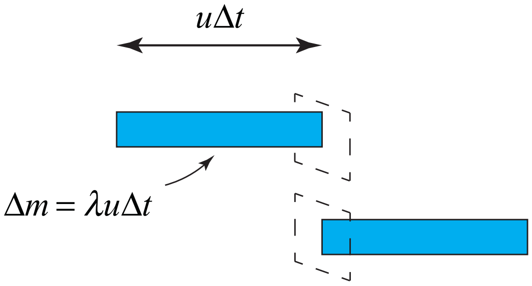


Figure 12.12 Mass per unit length of water jet

Consider a small amount of water that is moving with speed u that, in a time interval Δt, flows through a cross sectional area oriented perpendicular to the flow (see Figure 12.12). The area is larger than the cross sectional area of the jet of water. The amount of water that floes through the area element $\Delta m = \lambda u \Delta t$, where λ is the mass per unit length of the jet and $u\Delta t$ is the length of the jet that flows through the area in the interval Δt. The mass rate of water that flows through the cross sectional area element is then

$$\alpha = \frac{\Delta m}{\Delta t} = \lambda u \ . \tag{12.3.58}$$

In the Figure 12.13 we consider a small length $u\Delta t$ of the water jet that is just behind the boat at time t. During the time interval $[t,t+\Delta t]$, the boat moves a distance $v\Delta t$.

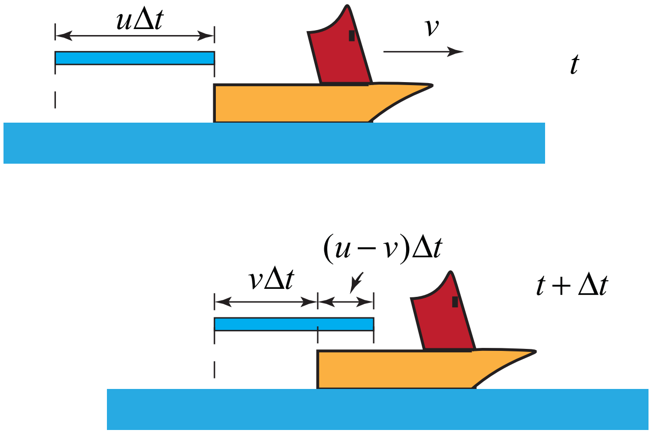


Figure 12.13 Amount of water that enter boat in time interval $[t,t+\Delta t]$

Only a fraction of the length $u\Delta t$ of water enters the boat and is given by

$$\Delta m_{w} = \lambda(u-v)\Delta t = \frac{\alpha}{u}(u-v)\Delta t \tag{12.3.59}$$

Dividing Eq. (12.3.59) through by Δt and taking limits we have that

$$\frac{dm_{w}}{dt} = \lim_{\Delta t\to 0}\frac{\Delta m_{w}}{\Delta t} = \frac{\alpha}{u}(u-v) = \alpha(1-\frac{v}{u})\,. \tag{12.3.60}$$

Substituting Eq. (12.3.53) and Eq. (12.3.46) into Eq. (12.3.60) yields

$$\frac{dm_{b}}{dt} = \alpha(1-\frac{v}{u}) = \alpha\frac{m_{0}}{m_{b}(t)}\,. \tag{12.3.61}$$

We can integrate this equation by separating variables to find an integral expression for the mass of the boat as a function of time

$$\int_{m_0}^{m_b(t)} m_b\, dm_b = \alpha m_0 \int_{t=0}^{t} dt\,. \tag{12.3.62}$$

We can easily integrate both sides of Eq. (12.3.62) yielding

$$\frac{1}{2}\left(m_b(t)^2 - m_0{}^2\right) = \alpha m_{b,0} t \,. \qquad (12.3.63)$$

The mass of the boat as a function of time is then

$$m_b(t) = m_0 \sqrt{1 + 2\frac{\alpha t}{m_0}} \,. \qquad (12.3.64)$$

We now substitute Eq. (12.3.64) into Eq. (12.3.65)yielding the speed of the burning boat as a function of time

$$v(t) = u\left(1 - \frac{1}{\sqrt{1 + 2\dfrac{\alpha t}{m_{b,0}}}}\right) \qquad (12.3.66)$$

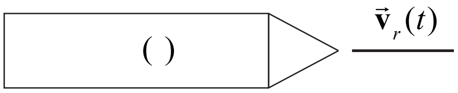
12.3 Rocket Propulsion

A rocket at time $t = t_i$ is moving with velocity $\vec{\mathbf{v}}_{r,i}$ with respect to a fixed reference frame. During the time interval $[t_i, t_f]$ the rocket continuously burns fuel that is continuously ejected backwards with velocity $\vec{\mathbf{u}}$ relative to the rocket. This exhaust velocity is independent of the velocity of the rocket. The rocket must exert a force to accelerate the ejected fuel backwards and therefore by Newton's Third law, the fuel exerts a force that is equal in magnitude but opposite in direction accelerating the rocket forward. The rocket velocity is a function of time, $\vec{\mathbf{v}}_r(t)$. Because fuel is leaving the rocket, the mass of the rocket is also a function of time, $m_r(t)$, and is decreasing at a rate dm_r / dt. Let $\vec{\mathbf{F}}_{ext}$ denote the total external force acting on the rocket. We shall use the momentum principle, to determine a differential equation that relates $d\vec{\mathbf{v}}_r / dt$, dm_r / dt, $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}_r(t)$, and $\vec{\mathbf{F}}_{ext}$, an equation known as the rocket equation.

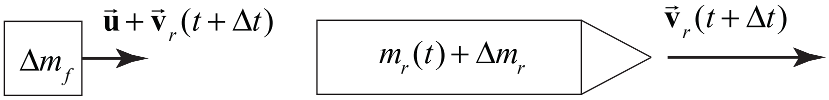
We shall apply the momentum principle during the time interval $[t, t + \Delta t]$ with Δt taken to be a small interval (we shall eventually consider the limit that $\Delta t \to 0$), and $t_i < t < t_f$. During this interval, choose as our system the mass of the rocket at time t,

$$m_{sys} = m_r(t) = m_{r,d} + m_f(t) \,, \qquad (12.3.67)$$

where $m_{r,d}$ is the dry mass of the rocket and $m_f(t)$ is the mass of the fuel in the rocket at time t. During the time interval $[t, t + \Delta t]$, a small amount of fuel of mass Δm_f (in the limit that $\Delta t \to 0$, $\Delta m_f \to 0$) is ejected backwards with velocity $\vec{\mathbf{u}}$ to the rocket. Before the fuel is ejected, it is traveling at the velocity of the rocket and so during the time interval $[t, t+\Delta t]$, the elected fuel undergoes a change in momentum and the rocket recoils forward. At time $t+\Delta t$ the rocket has velocity $\vec{\mathbf{v}}_r(t+\Delta t)$. Although the ejected fuel continually changes its velocity, we shall assume that the fuel is all ejected at the instant $t+\Delta t$ and then consider the limit as $\Delta t \to 0$. Therefore the velocity of the ejected fuel with respect to the fixed reference frame is the vector sum of the relative velocity of the fuel with respect to the rocket and the velocity of the rocket, $\vec{\mathbf{u}}+\vec{\mathbf{v}}_r(t+\Delta t)$. Figure 12.14 represents momentum diagrams for our system at time t and $t+\Delta t$ relative to a fixed inertial reference frame in which velocity of the rocket at time t is $\vec{\mathbf{v}}_r(t)$.



time



time $t+\Delta t$

Figure 12.14 Momentum diagrams for system at time t and $t+\Delta t$

The momentum of the system at time t is

$$\vec{\mathbf{p}}_{sys}(t) = m_r(t)\vec{\mathbf{v}}_r(t). \tag{12.3.68}$$

Note that the mass of the system at time t is

$$m_{sys} = m_r(t). \tag{12.3.69}$$

The momentum of the system at time $t+\Delta t$ is

$$\vec{\mathbf{p}}_{sys}(t+\Delta t) = m_r(t+\Delta t)\vec{\mathbf{v}}_r(t+\Delta t) + \Delta m_f(\vec{\mathbf{u}}+\vec{\mathbf{v}}_r(t+\Delta t)), \tag{12.3.70}$$

where $m_r(t+\Delta t) = m_r(t)+\Delta m_r$. With this notation the mass of the system at time $t+\Delta t$ is given by

$$m_{sys} = m_r(t+\Delta t)+\Delta m_f = m_r(t)+\Delta m_r+\Delta m_f. \tag{12.3.71}$$

Because the mass of the system is constant, setting Eq. (12.3.69) equal to Eq. (12.3.71) requires that

$$\Delta m_r = -\Delta m_f \,. \tag{12.3.72}$$

The momentum of the system at time $t + \Delta t$ (Eq. (12.3.70)) can be rewritten as

$$\begin{aligned} \vec{\mathbf{p}}_{sys}(t+\Delta t) &= (m_r(t) + \Delta m_r)\vec{\mathbf{v}}_r(t+\Delta t) - \Delta m_r(\vec{\mathbf{u}} + \vec{\mathbf{v}}_r(t+\Delta t)) \\ \vec{\mathbf{p}}_{sys}(t+\Delta t) &= m_r(t)\vec{\mathbf{v}}_r(t+\Delta t) - \Delta m_r\vec{\mathbf{u}} \end{aligned} \,, \tag{12.3.73}$$

We can now apply Newton’s Second Law in the form of the momentum principle,

$$\begin{aligned} \vec{\mathbf{F}}_{ext} &= \lim_{\Delta t \to 0} \frac{(m_r(t)\vec{\mathbf{v}}_r(t+\Delta t) - \Delta m_r\vec{\mathbf{u}}) - m_r(t)\vec{\mathbf{v}}_r(t)}{\Delta t} \\ &= m_r(t) \lim_{\Delta t \to 0} \frac{\vec{\mathbf{v}}_r(t+\Delta t) - \vec{\mathbf{v}}_r(t)}{\Delta t} - \lim_{\Delta t \to 0} \frac{\Delta m_r}{\Delta t}\vec{\mathbf{u}} \end{aligned} \,.. \tag{12.3.74}$$

We now take the limit as

$$\vec{\mathbf{F}}_{ext} = m_r(t)\frac{d\vec{\mathbf{v}}_r}{dt} - \frac{dm_r}{dt}\vec{\mathbf{u}} \,. \tag{12.3.75}$$

Eq. (12.3.75) is known as the ***rocket equation***.

Suppose the rocket is moving in the positive x-direction with an external force given by $\vec{\mathbf{F}}_{ext} = F_{ext,x}\hat{\mathbf{i}}$ Then $\vec{\mathbf{u}} = -u\hat{\mathbf{i}}$, where $u > 0$ is the relative speed of the fuel and it is moving in the negative x-direction, $\vec{\mathbf{v}}_r = v_{r,x}\hat{\mathbf{i}}$. Then the rocket equation (Eq. (12.3.75)) becomes

$$F_{ext,x} = m_r(t)\frac{dv_{r,x}}{dt} + \frac{dm_r}{dt}u \,. \tag{12.3.76}$$

Note that the rate of decrease of the mass of the rocket, dm_r / dt, is equal to the negative of the rate of increase of the exhaust fuel

$$\frac{dm_r}{dt} = -\frac{dm_f}{dt} \,. \tag{12.3.77}$$

We can rewrite Eq. (12.3.76) as

$$F_{ext,x} - \frac{dm_r}{dt}u = m_r(t)\frac{dv_{r,x}}{dt} \,. \tag{12.3.78}$$

The second term on the left-hand-side of Eq. (12.3.78) is called the ***thrust***

$$F_{thrust,x} = -\frac{dm_r}{dt}u = \frac{dm_f}{dt}u \,. \tag{12.3.79}$$

Note that this is not an extra force but the result of the forward recoil due to the ejection of the fuel. Because we are burning fuel at a positive rate $dm_f / dt > 0$ and the speed $u > 0$, the direction of the thrust is in the positive x-direction.

12.3.1 Rocket Equation in Gravity-free Space

We shall first consider the case in which there are no external forces acting on the system, then Eq. (12.3.78) becomes

$$-\frac{dm_r}{dt}u = m_r(t)\frac{dv_{r,x}}{dt} \,. \tag{12.3.80}$$

In order to solve this equation, we separate the variable quantities $v_{r,x}(t)$ and $m_r(t)$ and multiply both sides by dt yielding

$$dv_{r,x} = -u\frac{dm_r}{m_r(t)} \,. \tag{12.3.81}$$

We now integrate both sides of Eq. (12.3.81) with limits corresponding to the values of the x-component of the velocity and mass of the rocket at times t_i when the ejection of the burned fuel began and the time t_f when the process stopped,

$$\int_{v'_{r,x}=v_{r,x,i}}^{v'_{r,x}=v_{r,x,f}} dv'_{r,x} = -\int_{m'_r=m_{r,i}}^{m'_r=m_{r,f}} \frac{u}{m'_r}dm'_r \,. \tag{12.3.82}$$

Performing the integration and substituting in the values at the endpoints yields

$$v_{r,x,f} - v_{r,x,i} = -u\ln\left(\frac{m_{r,f}}{m_{r,i}}\right). \tag{12.3.83}$$

Because the rocket is losing fuel, $m_{r,f} < m_{r,i}$, we can rewrite Eq. (12.3.83) as

$$v_{r,x,f} - v_{r,x,i} = u\ln\left(\frac{m_{r,i}}{m_{r,f}}\right). \tag{12.3.84}$$

We note $\ln(m_{r,i} / m_{r,f}) > 1$. Therefore $v_{r,x,f} > v_{r,x,i}$, as we expect. After a slight rearrangement of Eq. (12.3.84), we have an expression for the x-component of the velocity of the rocket as a function of the mass m_r of the rocket

$$v_{r,x,f} = v_{r,x,i} + u \ln\left(\frac{m_{r,i}}{m_{r,f}} \right). \tag{12.3.85}$$

Let's examine our result. First, let's suppose that all the fuel was burned and ejected. Then $m_{r,f} \equiv m_{r,d}$ is the final dry mass of the rocket (empty of fuel). The ratio

$$R = \frac{m_{r,i}}{m_{r,d}} \tag{12.3.86}$$

is the ratio of the initial mass of the rocket (including the mass of the fuel) to the final dry mass of the rocket (empty of fuel). The final velocity of the rocket is then

$$v_{r,x,f} = v_{r,x,i} + u \ln R . \tag{12.3.87}$$

This is why multistage rockets are used. You need a big container to store the fuel. Once all the fuel is burned in the first stage, the stage is disconnected from the rocket. During the next stage the dry mass of the rocket is much less and so R is larger than the single stage, so the next burn stage will produce a larger final speed then if the same amount of fuel were burned with just one stage (more dry mass of the rocket). In general rockets do not burn fuel at a constant rate but if we assume that the burning rate is constant where

$$b = \frac{dm_f}{dt} = -\frac{dm_r}{dt} \tag{12.3.88}$$

then we can integrate Eq. (12.3.88)

$$\int_{m'_r = m_{r,i}}^{m'_r = m_r(t)} dm'_r = -b \int_{t' = t_i}^{t' = t} dt' \tag{12.3.89}$$

and find an equation that describes how the mass of the rocket changes in time

$$m_r(t) = m_{r,i} - b(t - t_i). \tag{12.3.90}$$

For this special case, if we set $t_f = t$ in Eq. (12.3.85), then the velocity of the rocket as a function of time is given by

$$v_{r,x,f} = v_{r,x,i} + u \ln\left(\frac{m_{r,i}}{m_{r,i} - bt} \right). \tag{12.3.91}$$

Example 12.4 Single-Stage Rocket

Before a rocket begins to burn fuel, the rocket has a mass of $m_{r,i} = 2.81\times 10^{7}\,\text{kg}$, of which the mass of the fuel is $m_{f,i} = 2.46\times 10^{7}\,\text{kg}$. The fuel is burned at a constant rate with total burn time is 510 s and ejected at a speed $u = 3000\text{ m/s}$ relative to the rocket. If the rocket starts from rest in empty space, what is the final speed of the rocket after all the fuel has been burned?

Solution: The dry mass of the rocket is $m_{r,d} \equiv m_{r,i} - m_{f,i} = 0.35\times 10^{7}\,\text{kg}$, hence $R = m_{r,i} / m_{r,d} = 8.03$. The final speed of the rocket after all the fuel has burned is

$$v_{r,f} = \Delta v_{r} = u\ln R = 6250\text{ m/s} \ . \tag{12.3.92}$$

Example 12.5 Two-Stage Rocket

Now suppose that the same rocket in Example 12.4 burns the fuel in two stages ejecting the fuel in each stage at the same relative speed. In stage one, the available fuel to burn is $m_{f,1,i} = 2.03\times 10^{7}\,\text{kg}$ with burn time 150 s. Then the empty fuel tank and accessories from stage one are disconnected from the rest of the rocket. These disconnected parts have a mass $m = 1.4\times 10^{6}\,\text{kg}$. All the remaining fuel with mass is burned during the second stage with burn time of 360 s. What is the final speed of the rocket after all the fuel has been burned?

Solution: The mass of the rocket after all the fuel in the first stage is burned is $m_{r,1,d} = m_{r,1,i} - m_{f,1,i} = 0.78\times 10^{7}\,\text{kg}$ and $R_1 = m_{r,1,i} / m_{r,1,d} = 3.60$. The change in speed after the first stage is complete is

$$\Delta v_{r,1} = u\ln R_1 = 3840\text{ m/s}. \tag{12.3.93}$$

After the empty fuel tank and accessories from stage one are disconnected from the rest of the rocket, the remaining mass of the rocket is $m_{r,2,d} = 2.1\times 10^{6}\,\text{kg}$. The remaining fuel has mass $m_{f,2,i} = 4.3\times 10^{6}\,\text{kg}$. The mass of the rocket plus the unburned fuel at the beginning of the second stage is $m_{r,2,i} = 6.4\times 10^{6}\,kg$. Then $R_2 = m_{r,2,i} / m_{r,2,d} = 3.05$. Therefore the rocket increases its speed during the second stage by an amount

$$\Delta v_{r,2} = u\ln R_2 = 3340\text{ m/s}. \tag{12.3.94}$$

The final speed of the rocket is the sum of the change in speeds due to each stage,

$$v_f = \Delta v_r = u \ln R_1 + u \ln R_2 = u \ln(R_1 R_2) = 7190 \text{ m/s}, \qquad (12.3.95)$$

which is greater than if the fuel were burned in one stage. Plots of the speed of the rocket as a function time for both one-stage and two-stage burns are shown Figure 12.15.

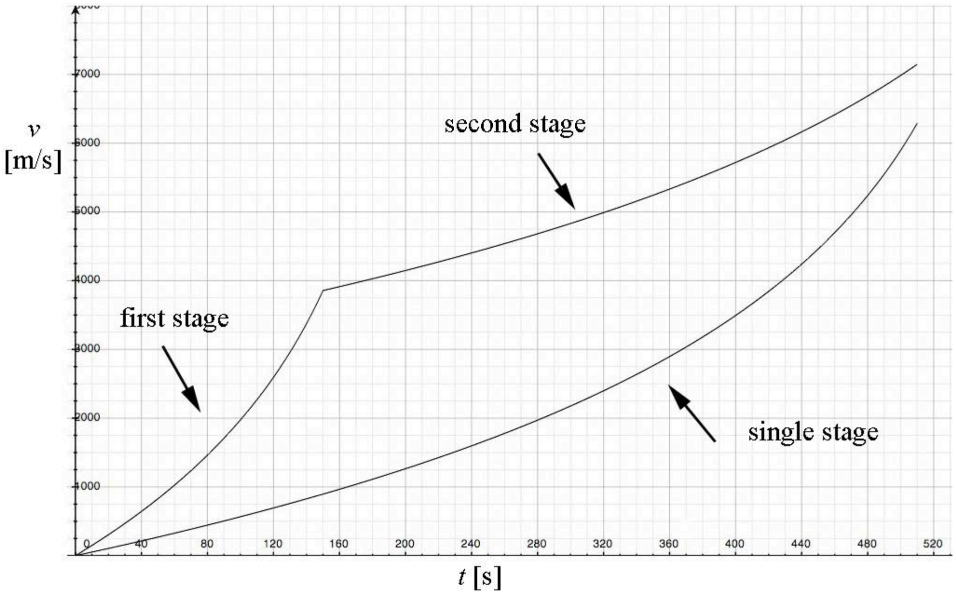


Figure 12.15 Plots of speed of rocket for both one-stage burn and two-stage burn

12.3.2 Rocket in a Constant Gravitational Field:

Now suppose that the rocket takes off from rest at time $t = 0$ in a constant gravitational field then the external force is

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = m_r \vec{\mathbf{g}}. \qquad (12.3.96)$$

Choose the positive x-axis in the upward direction then $F_{ext,x}(t) = -m_r(t)g$. Then the rocket equation (Eq. (12.3.75) becomes

$$-m_r(t)g - \frac{dm_r}{dt}u = m_r(t)\frac{dv_{r,x}}{dt}. \qquad (12.3.97)$$

Multiply both sides of Eq. (12.3.97) by dt, and divide both sides by $m_r(t)$. Then Eq. (12.3.97) can be written as

$$dv_{r,x} = -gdt - \frac{dm_r}{m_r(t)}u. \qquad (12.3.98)$$

We now integrate both sides

$$\int_{v_{r,x,i}=0}^{v_{r,x}(t)} dv'_{r,x} = -u \int_{m_{r,i}}^{m_r(t)} \frac{dm'_r}{m'_r} - g\int_0^t dt' , \qquad (12.3.99)$$

where $m_{r,i}$ is the initial mass of the rocket and the fuel. Integration yields

$$v_{r,x}(t) = -u\ln\left(\frac{m_r(t)}{m_{r,i}}\right) - gt = u\ln\left(\frac{m_{r,i}}{m_r(t)}\right) - gt . \qquad (12.3.100)$$

After all the fuel is burned at $t = t_f$, the mass of the rocket is equal to the dry mass $m_{r,f} = m_{r,d}$ and so

$$v_{r,x}(t_f) = u\ln R - gt_f . \qquad (12.3.101)$$

The first term on the right hand side is independent of the burn time. However the second term depends on the burn time. The shorter the burn time, the smaller the negative contribution from the third turn, and hence the rocket ends up with a larger final speed. So the rocket engine should burn the fuel as fast as possible in order to obtain the maximum possible speed.

Chapter 13 The Concept of Energy and Conservation of Energy

Chapter 13 Energy, Kinetic Energy, and Work

> *Acceleration of the expansion of the universe is one of the most exciting and significant discoveries in physics, with implications that could revolutionize theories of quantum physics, gravitation, and cosmology. With its revelation that close to the three-quarters of the energy density of the universe, given the name dark energy, is of a new, unknown origin and that its exotic gravitational "repulsion" will govern the fate of the universe, dark energy and the accelerating universe becomes a topic not just of great interest to research physicists but to science students at all levels.* [1]
>
> *Eric Linder*

13.1 The Concept of Energy and Conservation of Energy

The transformation of energy is a powerful concept that enables us to describe a vast number of processes:

> Falling water releases stored *gravitational potential energy*, which can become the *kinetic energy* associated with a *coherent motion* of matter. The harnessed *mechanical energy* can be used to spin turbines and alternators, doing *work* to generate *electrical energy*, transmitted to consumers along power lines. When you use any electrical device, the electrical energy is transformed into other forms of energy. In a refrigerator, electrical energy is used to compress a gas into a liquid. During the compression, some of the internal energy of the gas is transferred to the *random motion* of molecules in the outside environment. The liquid flows from a high-pressure region into a low-pressure region where the liquid evaporates. During the evaporation, the liquid absorbs energy from the *random motion* of molecules inside of the refrigerator. The gas returns to the compressor.
>
> "Human beings transform the stored chemical energy of food into various forms necessary for the maintenance of the functions of the various organ system, tissues and cells in the body."[2] A person can do work on their surroundings – for example, by pedaling a bicycle – and transfer energy to the surroundings in the form of increasing random motion of air molecules, by using this catabolic energy.
>
> Burning gasoline in car engines converts *chemical energy*, stored in the molecular bonds of the constituent molecules of gasoline, into coherent (ordered) motion of the molecules that constitute a piston. With the use of gearing and tire/road friction, this motion is converted into kinetic energy of the car; the automobile moves.

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[1] Eric Linder, *Resource Letter: Dark Energy and the Accelerating Universe*, Am.J.Phys.76: 197-204, 2008; p. 197.

[2] George B. Benedek and Felix M.H. Villars, *Physics with Illustrative Examples from Medicine and Biology, Volume 1: Mechanics,* Addison-Wesley, Reading, 1973, p. 115-6.

> Stretching or compressing a spring stores *elastic potential energy* that can be released as kinetic energy.
>
> The process of vision begins with stored *atomic energy* released as electromagnetic radiation (light), which is detected by exciting photoreceptors in the eye, releasing chemical energy.
>
> When a proton fuses with deuterium (a hydrogen atom with a neutron and proton for a nucleus), helium-three is formed (with a nucleus of two protons and one neutron) along with radiant energy in the form of photons. The combined *internal energy* of the proton and deuterium are greater than the internal energy of the helium-three. This difference in internal energy is carried away by the photons as light energy.

There are many such processes involving different forms of energy: kinetic energy, gravitational energy, thermal energy, elastic energy, electrical energy, chemical energy, electromagnetic energy, nuclear energy and more. The total energy is always conserved in these processes, although different forms of energy are converted into others.

Any physical process can be characterized by two states, initial and final, between which energy transformations can occur. Each form of energy E_j, where "j" is an arbitrary label identifying one of the N forms of energy, may undergo a change during this transformation,

$$\Delta E_j \equiv E_{\text{final},j} - E_{\text{initial},j} . \qquad (13.1.1)$$

Conservation of energy means that the sum of these changes is zero,

$$\Delta E_1 + \Delta E_2 + \cdots + \Delta E_N = \sum_{j=1}^{N} \Delta E_j = 0 . \qquad (13.1.2)$$

Two important points emerge from this idea. First, we are interested primarily in changes in energy and so we search for relations that describe how each form of energy changes. Second, we must account for all the ways energy can change. If we observe a process, and the sum of the changes in energy is not zero, either our expressions for energy are incorrect, or there is a new type of change of energy that we had not previously discovered. This is our first example of the importance of conservation laws in describing physical processes, as energy is a key quantity conserved in all physical processes. If we can quantify the changes of different forms of energy, we have a very powerful tool to understand nature.

We will begin our analysis of conservation of energy by considering processes involving only a few forms of changing energy. We will make assumptions that greatly simplify our description of these processes. At first we shall only consider processes acting on bodies in which the atoms move in a coherent fashion, ignoring processes in which energy is transferred into the random motion of atoms. Thus we will initially ignore the effects of friction. We shall then treat processes involving friction between consider rigid bodies. We will later return to processes in which there is an energy transfer resulting in an increase or decrease in random motion when we study the First Law of Thermodynamics.

Energy is always conserved but we often prefer to restrict our attention to a set of objects that we define to be our *system*. The rest of the universe acts as the *surroundings*. We illustrate this division of system and surroundings in Figure 13.1.

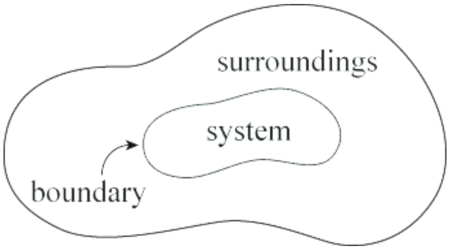

Figure 13.1 A diagram of a system and its surroundings with boundary

Because energy is conserved, any energy that leaves the system must cross through the boundary and enter the surroundings. Consider any physical process in which energy transformations occur between initial and final states. We assert that

> *when a system and its surroundings undergo a transition from an initial state to a final state, the change in energy is zero,*

$$\Delta E = \Delta E_{\text{system}} + \Delta E_{\text{surroundings}} = 0\,. \qquad (13.1.3)$$

Eq. (13.1.3) is called ***conservation of energy*** and is our operating definition for energy. We will sometime refer to Eq. (13.1.3) as the *energy principle*. In any physical application, we first identify our system and surroundings, and then attempt to quantify changes in energy. In order to do this, we need to identify every type of change of energy in every possible physical process. When there is no change in energy in the surroundings then the system is called a ***closed system***, and consequently the energy of a closed system is constant.

$$\Delta E_{\text{system}} = 0, \qquad (\text{closed system})\,. \qquad \mathit{(13.1.4)}$$

If we add up all known changes in energy in the system and surroundings and do not arrive at a zero sum, we have an open scientific problem. By searching for the missing changes in energy, we may uncover some new physical phenomenon. Recently, one of the most exciting open problems in cosmology is the apparent acceleration of the expansion of the universe, which has been attributed to *dark energy* that resides in space itself, an energy type without a clearly known source.[3]

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[3] http://www-supernova.lbl.gov/~evlinder/sci.html

13.2 Kinetic Energy

The first form of energy that we will study is an energy associated with the coherent motion of molecules that constitute a body of mass m; this energy is called the *kinetic energy* (from the Greek word *kinetikos* which translates as *moving*). Let us consider a car moving along a straight road (along which we will place the x-axis). For an observer at rest with respect to the ground, the car has velocity $\vec{\mathbf{v}} = v_x \hat{\mathbf{i}}$. The speed of the car is the magnitude of the velocity, $v \equiv |v_x|$.

> *The* ***kinetic energy*** *K of a non-rotating body of mass m moving with speed v is defined to be the positive scalar quantity*

$$K \equiv \frac{1}{2} m v^2 \tag{13.2.1}$$

The kinetic energy is proportional to the square of the speed. The SI units for kinetic energy are $[\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}]$. This combination of units is defined to be a joule and is denoted by $[\mathrm{J}]$, thus $1\,\mathrm{J} \equiv 1\,\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$. (The SI unit of energy is named for James Prescott Joule.) The above definition of kinetic energy does not refer to any direction of motion, just the speed of the body.

Let's consider a case in which our car changes velocity. For our initial state, the car moves with an initial velocity $\vec{\mathbf{v}}_i = v_{x,i}\, \hat{\mathbf{i}}$ along the x-axis. For the final state (at some later time), the car has changed its velocity and now moves with a final velocity $\vec{\mathbf{v}}_f = v_{x,f}\, \hat{\mathbf{i}}$. Therefore the change in the kinetic energy is

$$\Delta K = \frac{1}{2} m v_f^{\,2} - \frac{1}{2} m v_i^{\,2}. \tag{13.2.2}$$

Example 13.1 Change in Kinetic Energy of a Car

Suppose car *A* increases its speed from 10 to 20 mph and car *B* increases its speed from 50 to 60 mph. Both cars have the same mass m. (a) What is the ratio of the change of kinetic energy of car *B* to the change of kinetic energy of car *A*? In particular, which car has a greater change in kinetic energy? (b) What is the ratio of the change in kinetic energy of car *B* to car *A* as seen by an observer moving with the initial velocity of car *A*?

Solution: (a) The ratio of the change in kinetic energy of car *B* to car *A* is

$$\frac{\Delta K_B}{\Delta K_A}=\frac{\frac{1}{2}m(v_{B,f})^2-\frac{1}{2}m(v_{B,i})^2}{\frac{1}{2}m(v_{A,f})^2-\frac{1}{2}m(v_{A,i})^2}=\frac{(v_{B,f})^2-(v_{B,i})^2}{(v_{A,f})^2-(v_{A,i})^2}$$

$$=\frac{(60\text{ mph})^2-(50\text{ mph})^2}{(20\text{ mph})^2-(10\text{ mph})^2}=11/3.$$

Thus car *B* has a much greater increase in its kinetic energy than car *A*.

(b) In a reference moving with the speed of car *A* , car *A* increases its speed from rest to 10 mph and car *B* increases its speed from 40 to 50 mph. The ratio is now

$$\frac{\Delta K_B}{\Delta K_A}=\frac{\frac{1}{2}m(v_{B,f})^2-\frac{1}{2}m(v_{B,0})^2}{\frac{1}{2}m(v_{A,f})^2-\frac{1}{2}m(v_{A,0})^2}=\frac{(v_{B,f})^2-(v_{B,0})^2}{(v_{A,f})^2-(v_{A,0})^2}$$

$$=\frac{(50\text{ mph})^2-(40\text{ mph})^2}{(10\text{ mph})^2}=9.$$

The ratio is greater than that found in part a). Note that from the new reference frame both car *A* and car *B* have smaller increases in kinetic energy.

13.3 Kinematics and Kinetic Energy in One Dimension

13.3.1 Constant Accelerated Motion

Let's consider a constant accelerated motion of a *rigid body* in one dimension in which we treat the rigid body as a point mass. Suppose at $t=0$ the body has an initial x-component of the velocity given by $v_{x,i}$. If the acceleration is in the direction of the displacement of the body then the body will increase its speed. If the acceleration is opposite the direction of the displacement then the acceleration will decrease the body's speed. The displacement of the body is given by

$$\Delta x=v_{x,i}\,t+\frac{1}{2}a_x\,t^2\,. \tag{13.3.1}$$

The product of acceleration and the displacement is

$$a_x\Delta x=a_x(v_{x,i}\;t+\frac{1}{2}a_x\,t^2)\,. \tag{13.3.2}$$

The acceleration is given by

$$a_x = \frac{\Delta v_x}{\Delta t} = \frac{(v_{x,f} - v_{x,i})}{t}. \qquad (13.3.3)$$

Therefore

$$a_x \Delta x = \frac{(v_{x,f} - v_{x,i})}{t}\left(v_{x,i}\, t + \frac{1}{2}\frac{(v_{x,f} - v_{x,i})}{t} t^2 \right). \qquad (13.3.4)$$

Equation (13.3.4) becomes

$$a_x \Delta x = (v_{x,f} - v_{x,i})(v_{x,i}) + \frac{1}{2}(v_{x,f} - v_{x,i})(v_{x,f} - v_{x,i}) = \frac{1}{2} v_{x,f}{}^2 - \frac{1}{2} v_{x,i}^2 . \qquad (13.3.5)$$

If we multiply each side of Equation (13.3.5) by the mass m of the object this kinematical result takes on an interesting interpretation for the motion of the object. We have

$$m\, a_x \Delta x = \frac{1}{2} m v_{x,f}^2 - m \frac{1}{2} v_{x,i}^2 = K_f - K_i . \qquad (13.3.6)$$

Recall that for one-dimensional motion, Newton's Second Law is $F_x = m\, a_x$, for the motion considered here, Equation (13.3.6) becomes

$$F_x\, \Delta x = K_f - K_i . \qquad (13.3.7)$$

13.3.2 Non-constant Accelerated Motion

If the acceleration is not constant, then we can divide the displacement into N intervals indexed by $j = 1$ to N. It will be convenient to denote the displacement intervals by Δx_j, the corresponding time intervals by Δt_j and the x-components of the velocities at the beginning and end of each interval as $v_{x,j-1}$ and $v_{x,j}$. Note that the x-component of the velocity at the beginning and end of the first interval $j = 1$ is then $v_{x,1} = v_{x,i}$ and the velocity at the end of the last interval, $j = N$ is $v_{x,N} = v_{x,j}$. Consider the sum of the products of the average acceleration $(a_{x,j})_{ave}$ and displacement Δx_j in each interval,

$$\sum_{j=1}^{j=N} (a_{x,j})_{\text{ave}} \Delta x_j . \qquad (13.3.8)$$

The average acceleration over each interval is equal to

$$(a_{x,j})_{\text{ave}} = \frac{\Delta v_{x,j}}{\Delta t_j} = \frac{(v_{x,j+1} - v_{x,j})}{\Delta t_j}, \qquad (13.3.9)$$

and so the contribution in each integral can be calculated as above and we have that

$$(a_{x,j})_{\text{ave}}\,\Delta x_j = \frac{1}{2}v_{x,j}{}^2 - \frac{1}{2}v_{x,j-1}{}^2 . \tag{13.3.10}$$

When we sum over all the terms only the last and first terms survive, all the other terms cancel in pairs, and we have that

$$\sum_{j=1}^{j=N}(a_{x,j})_{\text{ave}}\,\Delta x_j = \frac{1}{2}v_{x,f}^2 - \frac{1}{2}v_{x,i}^2 . \tag{13.3.11}$$

In the limit as $N \to \infty$ *and* $\Delta x_j \to 0$ for all j (both conditions must be met!), the limit of the sum is the definition of the definite integral of the acceleration with respect to the position,

$$\lim_{\substack{N\to\infty \\ \Delta x_j \to 0}} \sum_{j=1}^{j=N}(a_{x,j})_{\text{ave}}\,\Delta x_j \equiv \int_{x=x_i}^{x=x_f} a_x(x)\,dx . \tag{13.3.12}$$

Therefore In the limit as $N \to \infty$ *and* $\Delta x_j \to 0$ for all j, with $v_{x,N} \to v_{x,f}$, Eq. (13.3.11) becomes

$$\int_{x=x_i}^{x=x_f} a_x(x)\,dx = \frac{1}{2}(v_{x,f}^2 - v_{x,i}^2) \tag{13.3.13}$$

This integral result is consequence of the definition that $a_x \equiv dv_x / dt$. The integral in Eq. (13.3.13) is an integral with respect to space, while our previous integral

$$\int_{t=t_i}^{t=t_f} a_x(t)\,dt = v_{x,f} - v_{x,i} . \tag{13.3.14}$$

requires integrating acceleration with respect to time. Multiplying both sides of Eq. (13.3.13) by the mass m yields

$$\int_{x=x_i}^{x=x_f} ma_x(x)\,dx = \frac{1}{2}m(v_{x,f}^2 - v_{x,i}^2) = K_f - K_i . \tag{13.3.15}$$

When we introduce Newton's Second Law in the form $F_x = ma_x$, then Eq. (13.3.15) becomes

$$\int_{x=x_i}^{x=x_f} F_x(x)\,dx = K_f - K_i . \tag{13.3.16}$$

The integral of the x-component of the force with respect to displacement in Eq. (13.3.16) applies to the motion of a point-like object. For extended bodies, Eq. (13.3.16) applies to the center of mass motion because the external force on a rigid body causes the center of mass to accelerate.

13.4 Work done by Constant Forces

We will begin our discussion of the concept of work by analyzing the motion of an object in one dimension acted on by constant forces. Let's consider the following example: push a cup forward with a constant force along a desktop. When the cup changes velocity (and hence kinetic energy), the sum of the forces acting on the cup must be non-zero according to Newton's Second Law. There are three forces involved in this motion: the applied pushing force $\vec{\mathbf{F}}^a$; the contact force $\vec{\mathbf{C}} \equiv \vec{\mathbf{N}} + \vec{\mathbf{f}}_k$; and gravity $\vec{\mathbf{F}}^g = m\vec{\mathbf{g}}$. The force diagram on the cup is shown in Figure 13.2.

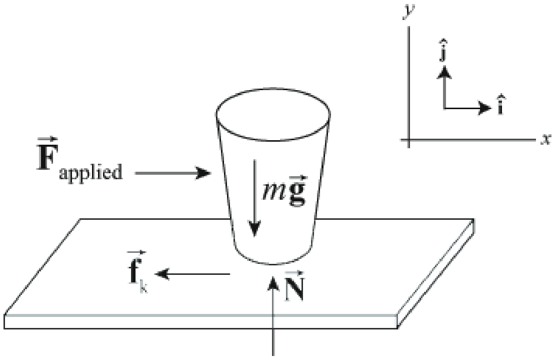


Figure 13.2 Force diagram for cup.

Let's choose our coordinate system so that the $+x$-direction is the direction of the forward motion of the cup. The pushing force can then be described by

$$\vec{\mathbf{F}}^a = F_x^a \,\hat{\mathbf{i}} . \tag{13.4.1}$$

Suppose a body moves from an initial point x_i to a final point x_f so that the displacement of the point the force acts on is $\Delta x \equiv x_f - x_i$. The ***work done by a constant force*** $\vec{\mathbf{F}}^a = F_x^a \hat{\mathbf{i}}$ acting on the body is the product of the component of the force F_x^a and the displacement Δx,

$$W^a = F_x^a \Delta x . \tag{13.4.2}$$

Work is a scalar quantity; it is not a vector quantity. The SI unit for work is

$$[1\ \mathrm{N} \cdot \mathrm{m}] = [1\ \mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}][1\ \mathrm{m}] = [1\ \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}] = [1\ \mathrm{J}]. \qquad (13.4.3)$$

Note that work has the same dimension and the same SI unit as kinetic energy. Because our applied force is along the direction of motion, both $F_x^a > 0$ and $\Delta x > 0$. In this example, the work done is just the product of the magnitude of the applied force and the distance through which that force acts and is positive. In the definition of work done by a force, the force can act at any point on the body. The displacement that appears in Equation (13.4.2) is not the displacement of the body but the *displacement of the point of application of the force*. For point-like objects, the displacement of the point of application of the force is equal to the displacement of the body. However for an extended body, we need to focus on where the force acts and whether or not that point of application undergoes any displacement in the direction of the force as the following example illustrates.

Example 13.2 Work Done by Static Fiction

Suppose you are initially standing and you start walking by pushing against the ground with your feet and your feet do not slip. What is the work done by the static friction force acting on you?

Solution: When you apply a contact force against the ground, the ground applies an equal and opposite contact force on you. The tangential component of this constant force is the force of static friction acting on you. Since your foot is at rest while you are pushing against the ground, there is no displacement of the point of application of this static friction force. Therefore static friction does zero work on you while you are accelerating. You may be surprised by this result but if you think about energy transformation, chemical energy stored in your muscle cells is being transformed into kinetic energy of motion and thermal energy.

When forces are opposing the motion, as in our example of pushing the cup, the kinetic friction force is given by

$$\vec{\mathbf{F}}^f = f_{k,x}\,\hat{\mathbf{i}} = -\mu_k N\,\hat{\mathbf{i}} = -\mu_k mg\,\hat{\mathbf{i}}. \qquad (13.4.4)$$

Here the component of the force is in the opposite direction as the displacement. The work done by the kinetic friction force is negative,

$$W^f = -\mu_k mg\Delta x. \qquad (13.4.5)$$

Since the gravitation force is perpendicular to the motion of the cup, the gravitational force has no component along the line of motion. Therefore the gravitation force does zero work on the cup when the cup is slid forward in the horizontal direction. The normal force is also perpendicular to the motion, and hence does no work.

We see that the pushing force does positive work, the kinetic friction force does negative work, and the gravitation and normal force does zero work.

Example 13.3 Work Done by Force Applied in the Direction of Displacement

Push a cup of mass 0.2 kg along a horizontal table with a force of magnitude 2.0 N for a distance of 0.5 m. The coefficient of friction between the table and the cup is $\mu_k = 0.10$. Calculate the work done by the pushing force and the work done by the friction force.

Solution: The work done by the pushing force is

$$W^a = F_x^a \Delta x = (2.0\text{ N})(0.5\text{ m}) = 1.0\text{ J}\,. \qquad (13.4.6)$$

The work done by the friction force is

$$W^f = -\mu_k mg\Delta x = -(0.1)(0.2\text{ kg})(9.8\text{ m}\cdot\text{s}^{-2})(0.5\text{ m}) = -\,0.10\text{ J}\,. \qquad (13.4.7)$$

Example 13.4 Work Done by Force Applied at an Angle to the Direction of Displacement

Suppose we push the cup in the previous example with a force of the same magnitude but at an angle $\theta = 30^o$ upwards with respect to the table. Calculate the work done by the pushing force. Calculate the work done by the kinetic friction force.

Solution: The force diagram on the cup and coordinate system is shown in Figure 13.3.

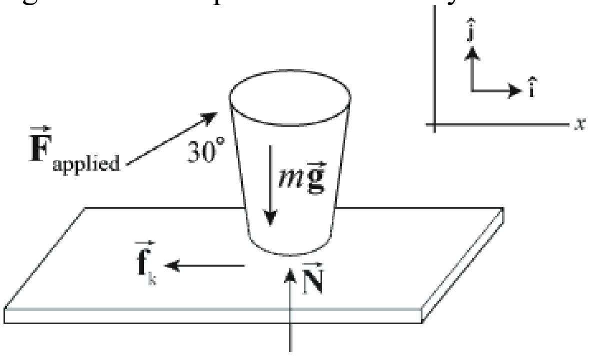


Figure 13.3 Force diagram on cup.

The x-component of the pushing force is now

$$F_x^a = F^a\cos(\theta) = (2.0\text{ N})(\cos(30^\circ)) = 1.7\text{ N}\,. \qquad (13.4.8)$$

The work done by the pushing force is

$$W^a = F_x^a \Delta x = (1.7\ \text{N})(0.5\ \text{m}) = 8.7 \times 10^{-1}\ \text{J}\,. \tag{13.4.9}$$

The kinetic friction force is

$$\vec{\mathbf{F}}^f = -\mu_k N\,\hat{\mathbf{i}}\,. \tag{13.4.10}$$

In this case, the magnitude of the normal force is not simply the same as the weight of the cup. We need to find the y-component of the applied force,

$$F_y^a = F^a \sin(\theta) = (2.0\ \text{N})(\sin(30^o) = 1.0\ \text{N}\,. \tag{13.4.11}$$

To find the normal force, we apply Newton's Second Law in the y-direction,

$$F_y^a + N - mg = 0\,. \tag{13.4.12}$$

Then the normal force is

$$N = mg - F_y^a = (0.2\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2}) - (1.0\ \text{N}) = 9.6 \times 10^{-1}\ \text{N}\,. \tag{13.4.13}$$

The work done by the kinetic friction force is

$$W^f = -\mu_k N \Delta x = -(0.1)(9.6 \times 10^{-1}\ \text{N})(0.5\ \text{m}) = 4.8 \times 10^{-2}\ \text{J}\,. \tag{13.4.14}$$

Example 13.5 Work done by Gravity Near the Surface of the Earth

Consider a point-like body of mass m near the surface of the earth falling directly towards the center of the earth. The gravitation force between the body and the earth is nearly constant, $\vec{\mathbf{F}}_{grav} = m\vec{\mathbf{g}}$. Let's choose a coordinate system with the origin at the surface of the earth and the $+y$-direction pointing away from the center of the earth Suppose the body starts from an initial point y_i and falls to a final point y_f closer to the earth. How much work does the gravitation force do on the body as it falls?

Solution: The displacement of the body is negative, $\Delta y \equiv y_f - y_i < 0$. The gravitation force is given by

$$\vec{\mathbf{F}}^g = m\vec{\mathbf{g}} = F_y^g\,\hat{\mathbf{j}} = -mg\,\hat{\mathbf{j}}\,. \tag{13.4.15}$$

The work done on the body is then

$$W^g = F_y^g \Delta y = -mg\Delta y\,. \tag{13.4.16}$$

For a falling body, the displacement of the body is negative, $\Delta y \equiv y_f - y_i < 0$; therefore the work done by gravity is positive, $W^g > 0$. The gravitation force is pointing in the same direction as the displacement of the falling object so the work should be positive.

When an object is rising while under the influence of a gravitation force, $\Delta y \equiv y_f - y_i > 0$. The work done by the gravitation force for a rising body is negative, $W^g < 0$, because the gravitation force is pointing in the opposite direction from that in which the object is displaced.

It's important to note that the choice of the positive direction as being away from the center of the earth ("up") does not make a difference. If the downward direction were chosen positive, the falling body would have a positive displacement and the gravitational force as given in Equation (13.4.15) would have a positive downward component; the product $F_y^g \Delta y$ would still be positive.

13.5 Work done by Non-Constant Forces

Consider a body moving in the x-direction under the influence of a non-constant force in the x-direction, $\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}}$. The body moves from an initial position x_i to a final position x_f. In order to calculate the work done by a non-constant force, we will divide up the displacement of the point of application of the force into a large number N of small displacements Δx_j where the index j marks the j^{th} displacement and takes integer values from *1* to N. Let $(F_{x,j})_{ave}$ denote the average value of the x-component of the force in the displacement interval $[x_{j-1}, x_j]$. For the j^{th} displacement interval we calculate the contribution to the work

$$W_j = (F_{x,j})_{\text{ave}} \Delta x_j \tag{13.5.1}$$

This contribution is a scalar so we add up these scalar quantities to get the total work

$$W_N = \sum_{j=1}^{j=N} W_j = \sum_{j=1}^{j=N} (F_{x,j})_{\text{ave}} \Delta x_j\ . \tag{13.5.2}$$

The sum in Equation (13.5.2) depends on the number of divisions N and the width of the intervals Δx_j. In order to define a quantity that is independent of the divisions, we take the limit as $N \to \infty$ and $\left|\Delta x_j\right| \to 0$ for all j. The work is then

$$W = \lim_{\substack{N\to\infty \\ |\Delta x_j|\to 0}} \sum_{j=1}^{j=N} (F_{x,j})_{\text{ave}} \Delta x_j = \int_{x=x_i}^{x=x_f} F_x(x)\,dx \qquad (13.5.3)$$

This last expression is the definite integral of the x-component of the force with respect to the parameter x. In Figure 13.5 we graph the x-component of the force as a function of the parameter x. The work integral is the area under this curve between $x = x_i$ and $x = x_f$.

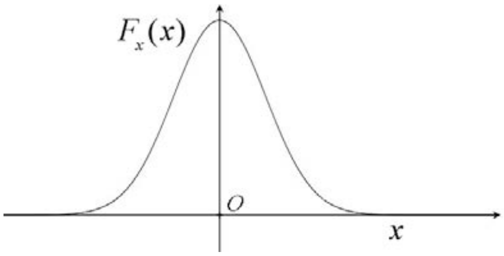


Figure 13.5 Plot of x-component of a sample force $F_x(x)$ as a function of x.

Example 13.6 Work done by the Spring Force

Connect one end of an unstretched spring of length l_0 with spring constant k to an object resting on a smooth frictionless table and fix the other end of the spring to a wall. Choose an origin as shown in the figure. Stretch the spring by an amount x_i and release the object. How much work does the spring do on the object when the spring is stretched by an amount x_f?

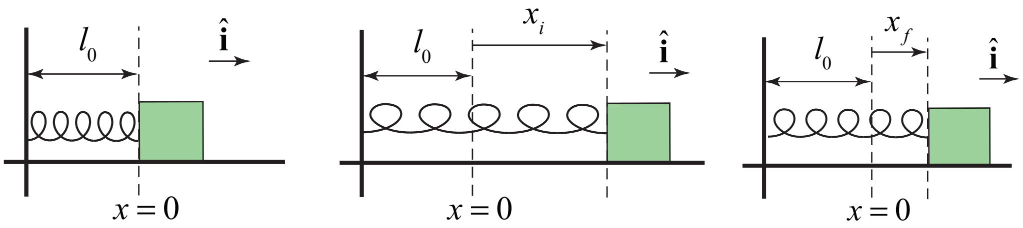


Figure 13.6 Equilibrium, initial and final states for a spring

Solution: We first begin by choosing a coordinate system with our origin located at the position of the object when the spring is unstretched (or uncompressed). We choose the $\hat{\mathbf{i}}$ unit vector to point in the direction the object moves when the spring is being stretched. We choose the coordinate function x to denote the position of the object with respect to the origin. We show the coordinate function and free-body force diagram in the figure below.

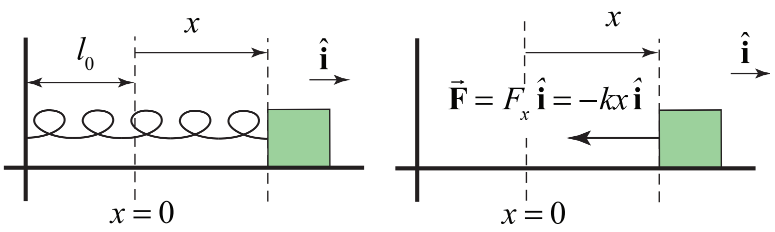


Figure 13.6a Spring force

The spring force on the object is given by (Figure 13.6a)

$$\vec{\mathbf{F}} = F_x \hat{\mathbf{i}} = -k x \hat{\mathbf{i}} \qquad (13.5.4)$$

In Figure 13.7 we show the graph of the x-component of the spring force, $F_x(x)$, as a function of x.

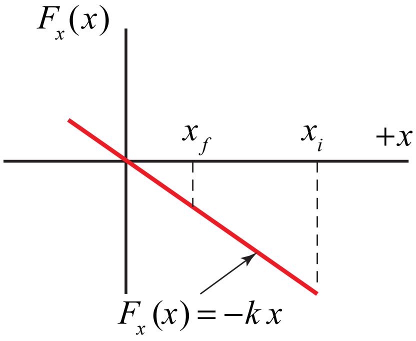


Figure 13.7 Plot of spring force $F_x(x)$ vs. displacement x

The work done is just the area under the curve for the interval x_i to x_f,

$$W = \int_{x'=x_i}^{x'=x_f} F_x(x')\,dx' = \int_{x'=x_i}^{x'=x_f} -kx'\,dx' = -\frac{1}{2}k(x_f^2 - x_i^2) \qquad (13.5.5)$$

This result is independent of the sign of x_i and x_f because both quantities appear as squares. If the spring is less stretched or compressed in the final state than in the initial state, then the absolute value, $\left|x_f\right| < \left|x_i\right|$, and the work done by the spring force is positive. The spring force does positive work on the body when the spring goes from a state of "greater tension" to a state of "lesser tension."

13.6 Work-Kinetic Energy Theorem

There is a direct connection between the work done on a point-like object and the change in kinetic energy the point-like object undergoes. If the work done on the object is non-zero, this implies that an unbalanced force has acted on the object, and the object will have undergone acceleration. For an object undergoing one-dimensional motion the left hand side of Equation (13.3.16) is the work done on the object by the component of the sum of the forces in the direction of displacement,

$$W = \int_{x=x_i}^{x=x_f} F_x\, dx = \frac{1}{2}mv_f^{\,2} - \frac{1}{2}mv_i^{\,2} = K_f - K_i = \Delta K \tag{13.6.1}$$

When the work done on an object is positive, the object will increase its speed, and negative work done on an object causes a decrease in speed. When the work done is zero, the object will maintain a constant speed. In fact, the work-energy relationship is quite precise; the work done by the applied force on an object is identically equal to the change in kinetic energy of the object.

Example 13.7 Gravity and the Work-Energy Theorem

Suppose a ball of mass $m = 0.2\text{ kg}$ starts from rest at a height $y_0 = 15\text{ m}$ above the surface of the earth and falls down to a height $y_f = 5.0\text{ m}$ above the surface of the earth. What is the change in the kinetic energy? Find the final velocity using the work-energy theorem.

Solution: As only one force acts on the ball, the change in kinetic energy is the work done by gravity,

$$\begin{aligned} W^g &= -mg(y_f - y_0) \\ &= (-2.0\times10^{-1}\text{ kg})(9.8\text{ m}\cdot\text{s}^{-2})(5\text{ m} - 15\text{ m}) = 2.0\times10^{1}\text{ J.} \end{aligned} \tag{13.6.2}$$

The ball started from rest, $v_{y,0} = 0$. So the change in kinetic energy is

$$\Delta K = \frac{1}{2}mv_{y,f}^{\;2} - \frac{1}{2}mv_{y,0}^{\;2} = \frac{1}{2}mv_{y,f}^{\;2}. \tag{13.6.3}$$

We can solve Equation (13.6.3) for the final velocity using Equation (13.6.2)

$$v_{y,f} = \sqrt{\frac{2\Delta K}{m}} = \sqrt{\frac{2W^g}{m}} = \sqrt{\frac{2(2.0\times10^{1}\text{ J})}{0.2\text{ kg}}} = 1.4\times10^{1}\text{ m}\cdot\text{s}^{-1}. \tag{13.6.4}$$

For the falling ball in a constant gravitation field, the positive work of the gravitation force on the body corresponds to an increasing kinetic energy and speed. For a rising body in the same field, the kinetic energy and hence the speed decrease since the work done is negative.

Example 13.7 Final Kinetic Energy of Moving Cup

A person pushes a cup of mass 0.2 kg along a horizontal table with a force of magnitude 2.0 N at an angle of 30° with respect to the horizontal for a distance of 0.5 m as in Example 13.4. The coefficient of friction between the table and the cup is $\mu_k = 0.1$. If the cup was initially at rest, what is the final kinetic energy of the cup after being pushed 0.5 m? What is the final speed of the cup?

Solution: The total work done on the cup is the sum of the work done by the pushing force and the work done by the friction force, as given in Equations (13.4.9) and (13.4.14),

$$\begin{aligned} W = W^a + W^f &= (F_x^a - \mu_k N)(x_f - x_i) \\ &= (1.7\text{ N} - 9.6\times 10^{-2}\text{ N})(0.5\text{ m}) = 8.0\times 10^{-1}\text{ J} \end{aligned} . \tag{13.6.5}$$

The initial velocity is zero so the change in kinetic energy is just

$$\Delta K = \frac{1}{2}mv_{y,f}{}^2 - \frac{1}{2}mv_{y,0}{}^2 = \frac{1}{2}mv_{y,f}{}^2 . \tag{13.6.6}$$

Thus the work-kinetic energy theorem, Eq.(13.6.1)), enables us to solve for the final kinetic energy,

$$K_f = \frac{1}{2}mv_f{}^2 = \Delta K = W = 8.0\times 10^{-1}\text{ J} . \tag{13.6.7}$$

We can solve for the final speed,

$$v_{y,f} = \sqrt{\frac{2K_f}{m}} = \sqrt{\frac{2W}{m}} = \sqrt{\frac{2(8.0\times 10^{-1}\text{ J})}{0.2\text{ kg}}} = 2.9\text{ m}\cdot\text{s}^{-1} . \tag{13.6.8}$$

13.7 Power Applied by a Constant Force

Suppose that an applied force $\vec{\mathbf{F}}^a$ acts on a body during a time interval Δt, and the displacement of the point of application of the force is in the x-direction by an amount Δx. The work done, ΔW^a, during this interval is

$$\Delta W^a = F_x^a \,\Delta x . \tag{13.7.1}$$

where F_x^a is the x-component of the applied force. (Equation (13.7.1) is the same as Equation (13.4.2).)

The ***average power*** of an applied force is defined to be the rate at which work is done,

$$P^a_{ave} = \frac{\Delta W^a}{\Delta t} = \frac{F^a_x \, \Delta x}{\Delta t} = F^a_x v_{\text{ave},x} \, . \qquad (13.7.2)$$

The average power delivered to the body is equal to the component of the force in the direction of motion times the component of the average velocity of the body. Power is a scalar quantity and can be positive, zero, or negative depending on the sign of work. The SI units of power are called watts $[\text{W}]$ and $[1\ \text{W}] = [1\ \text{J} \cdot \text{s}^{-1}]$.

The ***instantaneous power*** at time t is defined to be the limit of the average power as the time interval $[t, t + \Delta t]$ approaches zero,

$$P^a = \lim_{\Delta t \to 0} \frac{\Delta W^a}{\Delta t} = \lim_{\Delta t \to 0} \frac{F^a_x \, \Delta x}{\Delta t} = F^a_x \left(\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \right) = F^a_x v_x \, . \qquad (13.7.3)$$

The instantaneous power of a constant applied force is the product of the component of the force in the direction of motion and the instantaneous velocity of the moving object.

Example 13.8 Gravitational Power for a Falling Object

Suppose a ball of mass $m = 0.2\ \text{kg}$ starts from rest at a height $y_0 = 15\ \text{m}$ above the surface of the earth and falls down to a height $y_f = 5.0\ \text{m}$ above the surface of the earth. What is the average power exerted by the gravitation force? What is the instantaneous power when the ball is at a height $y_f = 5.0\ \text{m}$ above the surface of the Earth? Make a graph of power vs. time. You may ignore the effects of air resistance.

Solution: There are two ways to solve this problem. Both approaches require calculating the time interval Δt for the ball to fall. Set $t_0 = 0$ for the time the ball was released. We can solve for the time interval $\Delta t = t_f$ that it takes the ball to fall using the equation for a freely falling object that starts from rest,

$$y_f = y_0 - \frac{1}{2} g t_f^{\,2} \, . \qquad (13.7.4)$$

Thus the time interval for falling is

$$t_f = \sqrt{\frac{2}{g}(y_0 - y_f)} = \sqrt{\frac{2}{9.8\ \text{m} \cdot \text{s}^{-2}}(15\ \text{m} - 5\ \text{m})} = 1.4\ \text{s} \, . \qquad (13.7.5)$$

First approach: we can calculate the work done by gravity,

$$\begin{aligned} W^{g} &= -mg(y_{f} - y_{0}) \\ &= (-2.0\times10^{-1}\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2})(5\ \text{m} - 15\ \text{m}) = 2.0\times10^{1}\ \text{J}. \end{aligned} \tag{13.7.6}$$

Then the average power is

$$P^{g}_{\text{ave}} = \frac{\Delta W}{\Delta t} = \frac{2.0\times10^{1}\ \text{J}}{1.4\ \text{s}} = 1.4\times10^{1}\ \text{W}\,. \tag{13.7.7}$$

Second Approach. We calculate the gravitation force and the average velocity. The gravitation force is

$$F^{g}_{y} = -mg = -(2.0\times10^{-1}\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2}) = -2.0\ \text{N}\,. \tag{13.7.8}$$

The average velocity is

$$v_{\text{ave},y} = \frac{\Delta y}{\Delta t} = \frac{5\ \text{m} - 15\ \text{m}}{1.4\ \text{s}} = -7.0\ \text{m}\cdot\text{s}^{-1}\,. \tag{13.7.9}$$

The average power is therefore

$$\begin{aligned} P^{g}_{\text{ave}} &= F^{g}_{y}\, v_{\text{ave},y} = (-mg)v_{\text{ave},y} \\ &= (-2.0\ \text{N})(-7.0\ \text{m}\cdot\text{s}^{-1}) = 1.4\times10^{1}\ \text{W}. \end{aligned} \tag{13.7.10}$$

In order to find the instantaneous power at any time, we need to find the instantaneous velocity at that time. The ball takes a time $t_{f} = 1.4\ \text{s}$ to reach the height $y_{f} = 5.0\ \text{m}$. The velocity at that height is given by

$$v_{y} = -gt_{f} = -(9.8\ \text{m}\cdot\text{s}^{-2})(1.4\ \text{s}) = -1.4\times10^{1}\ \text{m}\cdot\text{s}^{-1}. \tag{13.7.11}$$

So the instantaneous power at time $t_{f} = 1.4\ \text{s}$ is

$$\begin{aligned} P^{g} &= F^{g}_{y}\, v_{y} = (-mg)(-gt_{f}) = mg^{2}t_{f} \\ &= (0.2\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2})^{2}(1.4\ \text{s}) = 2.7\times10^{1}\ \text{W} \end{aligned} \tag{13.7.12}$$

If this problem were done symbolically, the answers given in Equation (13.7.11) and Equation (13.7.12) would differ by a factor of two; the answers have been rounded to two significant figures.

The instantaneous power grows linearly with time. The graph of power vs. time is shown in Figure 13.8. From the figure, it should be seen that the instantaneous power at any time is twice the average power between $t = 0$ and that time.

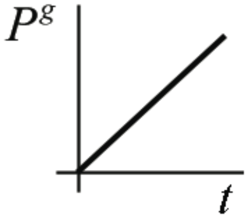


Figure 13.8 Graph of power vs. time

Example 13.9 Power Pushing a Cup

A person pushes a cup of mass 0.2 kg along a horizontal table with a force of magnitude 2.0 N at an angle of 30^o with respect to the horizontal for a distance of 0.5 m, as in Example 13.4. The coefficient of friction between the table and the cup is $\mu_k = 0.1$. What is the average power of the pushing force? What is the average power of the kinetic friction force?

Solution: We will use the results from Examples 13.4 and 13.7 but keeping extra significant figures in the intermediate calculations. The work done by the pushing force is

$$W^a = F_x^a(x_f - x_0) = (1.732\text{ N})(0.50\text{ m}) = 8.660\times10^{-1}\text{ J}. \tag{13.7.13}$$

The final speed of the cup is $v_{x,f} = 2.860\text{ m}\cdot\text{s}^{\text{-1}}$. Assuming constant acceleration, the time during which the cup was pushed is

$$t_f = \frac{2(x_f - x_0)}{v_{x,f}} = 0.3496\text{ s}. \tag{13.7.14}$$

The average power of the pushing force is then, with $\Delta t = t_f$,

$$P^a_{\text{ave}} = \frac{\Delta W^a}{\Delta t} = \frac{8.660\times10^{-1}\text{ J}}{0.3496\text{ s}} = 2.340\text{ W}, \tag{13.7.15}$$

or 2.3 W to two significant figures. The work done by the friction force is

$$\begin{aligned} W^f &= f_{\text{k}}(x_f - x_0) \\ &= -\mu_{\text{k}}N(x_f - x_0) = -(9.6\times10^{-2}\text{ N})(0.50\text{ m}) = -(4.8\times10^{-2}\text{ J}). \end{aligned} \tag{13.7.16}$$

The average power of kinetic friction is

$$P^f_{\text{ave}} = \frac{\Delta W^f}{\Delta t} = \frac{-4.8 \times 10^{-2}\ \text{J}}{0.3496\ \text{s}} = -1.4 \times 10^{-1}\ \text{W}. \qquad (13.7.17)$$

The time rate of change of the kinetic energy for a body of mass m moving in the x-direction is

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2} m v_x^{\ 2}\right) = m \frac{dv_x}{dt} v_x = m\, a_x v_x \,. \qquad (13.7.18)$$

By Newton's Second Law, $F_x = m\, a_x$, and so Equation (13.7.18) becomes

$$\frac{dK}{dt} = F_x v_x = P \,. \qquad (13.7.19)$$

The instantaneous power delivered to the body is equal to the time rate of change of the kinetic energy of the body.

13.8 Work and the Scalar Product

We shall introduce a vector operation, called the **scalar product** or "dot product" that takes any two vectors and generates a scalar quantity (a number). We shall see that the physical concept of work can be mathematically described by the scalar product between the force and the displacement vectors.

13.8.1 Scalar Product

Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors. Because any two non-collinear vectors form a plane, we define the angle θ to be the angle between the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ as shown in Figure 13.9. Note that θ can vary from 0 to π.

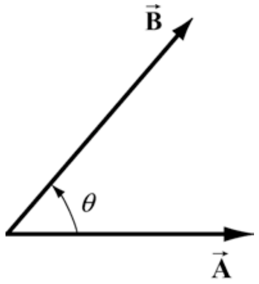


Figure 13.9 Scalar product geometry.

*The **scalar product** $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$ of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is defined to be product of the magnitude of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with the cosine of the angle θ between the two vectors:*

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos(\theta)\,, \tag{13.8.1}$$

where $A = |\vec{\mathbf{A}}|$ *and* $B = |\vec{\mathbf{B}}|$ *represent the magnitude of* $\vec{\mathbf{A}}$ *and* $\vec{\mathbf{B}}$ *respectively. The scalar product can be positive, zero, or negative, depending on the value of* $\cos\theta$ *. The scalar product is always a scalar quantity.*

The angle formed by two vectors is therefore

$$\theta = \cos^{-1}\left(\frac{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}{|\vec{\mathbf{A}}||\vec{\mathbf{B}}|}\right). \tag{13.8.2}$$

The magnitude of a vector $\vec{\mathbf{A}}$ is given by the square root of the scalar product of the vector $\vec{\mathbf{A}}$ with itself.

$$|\vec{\mathbf{A}}| = (\vec{\mathbf{A}} \cdot \vec{\mathbf{A}})^{1/2}. \tag{13.8.3}$$

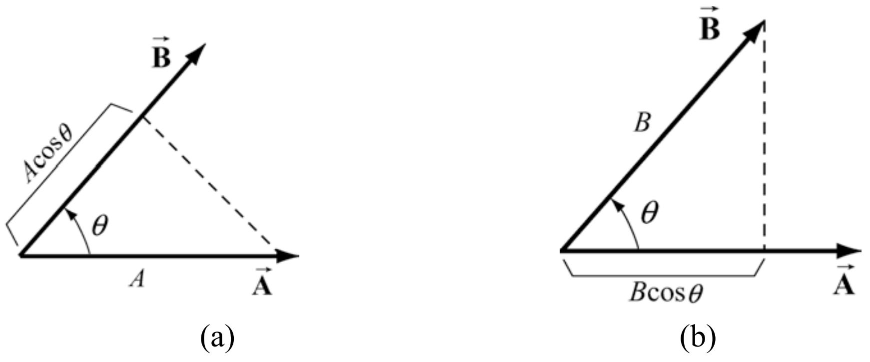
We can give a geometric interpretation to the scalar product by writing the definition as

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = (A\cos(\theta))\,B. \tag{13.8.4}$$

In this formulation, the term $A\cos\theta$ is the projection of the vector $\vec{\mathbf{B}}$ in the direction of the vector $\vec{\mathbf{B}}$. This projection is shown in Figure 13.10a. So the scalar product is the product of the projection of the length of $\vec{\mathbf{A}}$ in the direction of $\vec{\mathbf{B}}$ with the length of $\vec{\mathbf{B}}$. Note that we could also write the scalar product as

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A(B\cos(\theta)). \tag{13.8.5}$$

Now the term $B\cos(\theta)$ is the projection of the vector $\vec{\mathbf{B}}$ in the direction of the vector $\vec{\mathbf{A}}$ as shown in Figure 13.10b. From this perspective, the scalar product is the product of the projection of the length of $\vec{\mathbf{B}}$ in the direction of $\vec{\mathbf{A}}$ with the length of $\vec{\mathbf{A}}$.



(a) (b)

Figure 13.10 (a) and (b) Projection of vectors and the scalar product

From our definition of the scalar product we see that the scalar product of two vectors that are perpendicular to each other is zero since the angle between the vectors is $\pi/2$ and $\cos(\pi/2) = 0$.

We can calculate the scalar product between two vectors in a Cartesian coordinates system as follows. Consider two vectors $\vec{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}} + B_z\,\hat{\mathbf{k}}$. Recall that

$$\begin{aligned} \hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1 \\ \hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{i}}\cdot\hat{\mathbf{k}} = 0. \end{aligned} \tag{13.8.6}$$

The scalar product between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is then

$$\vec{\mathbf{A}}\cdot\vec{\mathbf{B}} = A_x B_x + A_y B_y + A_z B_z\,. \tag{13.8.7}$$

The time derivative of the scalar product of two vectors is given by

$$\begin{aligned} &\frac{d}{dt}(\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}) = \frac{d}{dt}(A_x B_x + A_y B_y + A_z B_z) \\ &= \frac{d}{dt}(A_x)B_x + \frac{d}{dt}(A_y)B_y + \frac{d}{dt}(A_z)B_z + A_x\frac{d}{dt}(B_x) + A_y\frac{d}{dt}(B_y) + A_z\frac{d}{dt}(B_z) \\ &= \left(\frac{d}{dt}\vec{\mathbf{A}}\right)\cdot\vec{\mathbf{B}} + \vec{\mathbf{A}}\cdot\left(\frac{d}{dt}\vec{\mathbf{B}}\right). \end{aligned} \tag{13.8.8}$$

In particular when $\vec{\mathbf{A}} = \vec{\mathbf{B}}$, then the time derivative of the square of the magnitude of the vector $\vec{\mathbf{A}}$ is given by

$$\frac{d}{dt}A^2 = \frac{d}{dt}(\vec{\mathbf{A}}\cdot\vec{\mathbf{A}}) = \left(\frac{d}{dt}\vec{\mathbf{A}}\right)\cdot\vec{\mathbf{A}} + \vec{\mathbf{A}}\cdot\left(\frac{d}{dt}\vec{\mathbf{A}}\right) = 2\left(\frac{d}{dt}\vec{\mathbf{A}}\right)\cdot\vec{\mathbf{A}}\,. \tag{13.8.9}$$

13.8.2 Kinetic Energy and the Scalar Product

For an object undergoing three-dimensional motion, the velocity of the object in Cartesian components is given by $\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$. Recall that the magnitude of a vector is given by the square root of the scalar product of the vector with itself,

$$A \equiv \left|\vec{\mathbf{A}}\right| \equiv (\vec{\mathbf{A}}\cdot\vec{\mathbf{A}})^{1/2} = (A_x^2 + A_y^2 + A_z^2)^{1/2}. \tag{13.8.10}$$

Therefore the square of the magnitude of the velocity is given by the expression

$$v^2 \equiv (\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}) = v_x^2 + v_y^2 + v_z^2 . \tag{13.8.11}$$

Hence the kinetic energy of the object is given by

$$K = \frac{1}{2} m(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}) = \frac{1}{2} m(v_x^2 + v_y^2 + v_z^2) . \tag{13.8.12}$$

13.8.2 Work and the Scalar Product

Work is an important physical example of the mathematical operation of taking the scalar product between two vectors. Recall that when a constant force acts on a body and the point of application of the force undergoes a displacement along the x-axis, only the component of the force along that direction contributes to the work,

$$W = F_x \Delta x . \tag{13.8.13}$$

Suppose we are pulling a body along a horizontal surface with a force $\vec{\mathbf{F}}$. Choose coordinates such that horizontal direction is the x-axis and the force $\vec{\mathbf{F}}$ forms an angle β with the positive x-direction. In Figure 13.11 we show the force vector $\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}}$ and the displacement vector of the point of application of the force $\Delta\vec{\mathbf{x}} = \Delta x\,\hat{\mathbf{i}}$. Note that $\Delta\vec{\mathbf{x}} = \Delta x\,\hat{\mathbf{i}}$ is the component of the displacement and hence can be greater, equal, or less than zero (but is shown as greater than zero in the figure for clarity). The scalar product between the force vector $\vec{\mathbf{F}}$ and the displacement vector $\Delta\vec{\mathbf{x}}$ is

$$\vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{x}} = (F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}}) \cdot (\Delta x\,\hat{\mathbf{i}}) = F_x\,\Delta x . \tag{13.8.14}$$

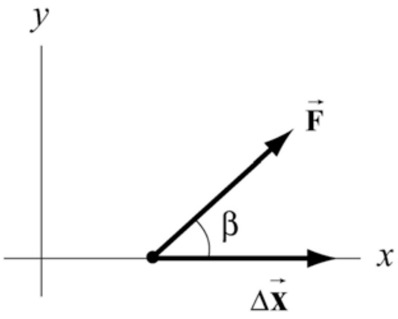


Figure 13.11 Force and displacement vectors

The work done by the force is then

$$\Delta W = \vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{x}} . \tag{13.8.15}$$

In general, the angle β takes values within the range $-\pi \le \beta \le \pi$ (in Figure 13.11, $0 \le \beta \le \pi / 2$). Because the x-component of the force is $F_x = F\cos(\beta)$ where $F = |\vec{\mathbf{F}}|$ denotes the magnitude of $\vec{\mathbf{F}}$, the work done by the force is

$$W = \vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{x}} = (F\cos(\beta))\Delta x\,. \tag{13.8.16}$$

Example 13.10 Object Sliding Down an Inclined Plane

An object of mass $m = 4.0\,kg$, starting from rest, slides down an inclined plane of length $l = 3.0\,m$. The plane is inclined by an angle of $\theta = 30^0$ to the ground. The coefficient of kinetic friction is $\mu_k = 0.2$. (a) What is the work done by each of the three forces while the object is sliding down the inclined plane? (b) For each force, is the work done by the force positive or negative? (c) What is the sum of the work done by the three forces? Is this positive or negative?

Solution: (a) and (b) Choose a coordinate system with the origin at the top of the inclined plane and the positive x-direction pointing down the inclined plane, and the positive y-direction pointing towards the upper right as shown in Figure 13.12. While the object is sliding down the inclined plane, three uniform forces act on the object, the gravitational force which points downward and has magnitude $F_g = mg$, the normal force N which is perpendicular to the surface of the inclined plane, and the friction force which opposes the motion and is equal in magnitude to $f_k = \mu_k N$. A force diagram on the object is shown in Figure 13.13.

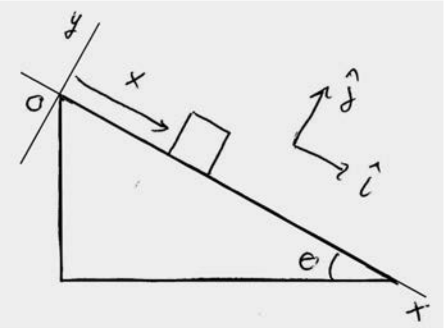


Figure 13.12 Coordinate system for object sliding down inclined plane

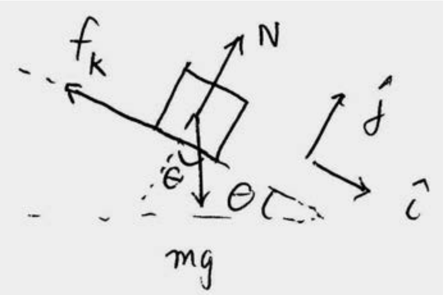


Figure 13.13 Free-body force diagram for object

In order to calculate the work we need to determine which forces have a component in the direction of the displacement. Only the component of the gravitational force along the positive x-direction $F_{gx} = mg\sin\theta$ and the friction force are directed along the displacement and therefore contribute to the work. We need to use Newton's Second Law to determine the magnitudes of the normal force. Because the object is constrained to move along the positive x-direction, $a_y = 0$, Newton's Second Law in the $\hat{\mathbf{j}}$-direction $N - mg\cos\theta = 0$. Therefore $N = mg\cos\theta$ and the magnitude of the friction force is $f_k = \mu_k mg\cos\theta$.

With our choice of coordinate system with the origin at the top of the inclined plane and the positive x-direction pointing down the inclined plane, the displacement of the object is given by the vector $\Delta\vec{\mathbf{r}} = \Delta x\,\hat{\mathbf{i}}$ (Figure 13.14).

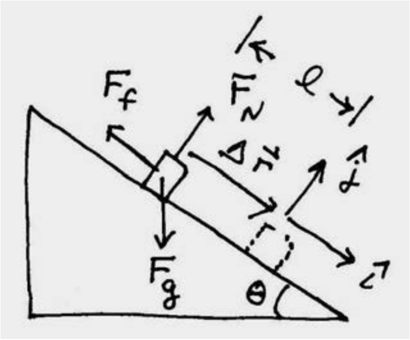


Figure 13.14 Force vectors and displacement vector for object

The vector decomposition of the three forces are $\vec{\mathbf{F}}^g = mg\sin\theta\,\hat{\mathbf{i}} - mg\cos\theta\hat{\mathbf{j}}$, $\vec{\mathbf{F}}^f = -\mu_k mg\cos\theta\hat{\mathbf{i}}$, and $\vec{\mathbf{F}}^N = mg\cos\theta\hat{\mathbf{j}}$. The work done by the normal force is zero because the normal force is perpendicular the displacement

$$W^N = \vec{\mathbf{F}}^N \cdot \Delta\vec{\mathbf{r}} = mg\cos\theta\hat{\mathbf{j}} \cdot l\,\hat{\mathbf{i}} = 0\,.$$

Then the work done by the friction force is negative and given by

$$W^f = \vec{\mathbf{F}}^f \cdot \Delta\vec{\mathbf{r}} = -\mu_k mg\cos\theta\hat{\mathbf{i}} \cdot l\,\hat{\mathbf{i}} = -\mu_k mg\cos\theta l < 0\,.$$

Substituting in the appropriate values yields

$$W^f = -\mu_k mg\cos\theta l = -(0.2)(4.0\,\mathrm{kg})(9.8\mathrm{m}\cdot\mathrm{s}^{-2})(3.0\,\mathrm{m})(\cos(30^o)(3.0\,\mathrm{m}) = -20.4\ \mathrm{J}\,.$$

The work done by the gravitational force is positive and given by

$$W^g = \vec{\mathbf{F}}^g \cdot \Delta\vec{\mathbf{r}} = (mg\sin\theta\,\hat{\mathbf{i}} - mg\cos\theta\hat{\mathbf{j}}) \cdot l\,\hat{\mathbf{i}} = mgl\sin\theta > 0\,.$$

Substituting in the appropriate values yields

$$W^g = mgl\sin\theta = (4.0\,\mathrm{kg})(9.8\,\mathrm{m}\cdot\mathrm{s}^{-2})(3.0\,\mathrm{m})(\sin(30^o) = 58.8\ \mathrm{J}\,.$$

(c) The scalar sum of the work done by the three forces is then

$$W = W^g + W^f = mgl(\sin\theta - \mu_k \cos\theta)$$
$$W = (4.0\,\text{kg})(9.8\,\text{m}\cdot\text{s}^{-2})(3.0\,\text{m})(\sin(30^o) - (0.2)(\cos(30^o)) = 38.4\ \text{J}.$$

13.9 Work done by a Non-Constant Force Along an Arbitrary Path

Suppose that a non-constant force $\vec{\mathbf{F}}$ acts on a point-like body of mass m while the body is moving on a three dimensional curved path. The position vector of the body at time t with respect to a choice of origin is $\vec{\mathbf{r}}(t)$. In Figure 13.15 we show the orbit of the body for a time interval $[t_i, t_f]$ moving from an initial position $\vec{\mathbf{r}}_i \equiv \vec{\mathbf{r}}(t = t_i)$ at time $t = t_i$ to a final position $\vec{\mathbf{r}}_f \equiv \vec{\mathbf{r}}(t = t_f)$ at time $t = t_f$.

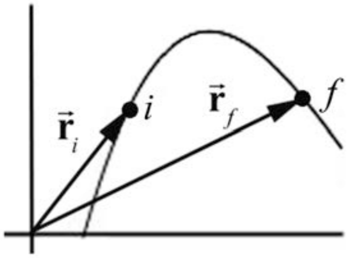


Figure 13.15 Path traced by the motion of a body.

We divide the time interval $[t_i, t_f]$ into N smaller intervals with $[t_{j-1}, t_j]$, $j = 1, \cdots, N$ with $t_N = t_f$. Consider two position vectors $\vec{\mathbf{r}}_j \equiv \vec{\mathbf{r}}(t = t_j)$ and $\vec{\mathbf{r}}_{j-1} \equiv \vec{\mathbf{r}}(t = t_{j-1})$ the displacement vector during the corresponding time interval as $\Delta\vec{\mathbf{r}}_j = \vec{\mathbf{r}}_j - \vec{\mathbf{r}}_{j-1}$. Let $\vec{\mathbf{F}}$ denote the force acting on the body during the interval $[t_{j-1}, t_j]$. The average force in this interval is $(\vec{\mathbf{F}}_j)_{\text{ave}}$ and the average work ΔW_j done by the force during the time interval $[t_{j-1}, t_j]$ is the scalar product between the average force vector and the displacement vector,

$$\Delta W_j = (\vec{\mathbf{F}}_j)_{\text{ave}} \cdot \Delta\vec{\mathbf{r}}_j. \tag{13.8.17}$$

The force and the displacement vectors for the time interval $[t_{j-1}, t_j]$ are shown in Figure 13.16 (note that the subscript "ave" on $(\vec{\mathbf{F}}_j)_{\text{ave}}$ has been suppressed).

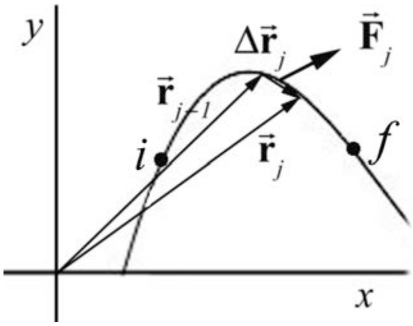


Figure 13.16 An infinitesimal work element.

We calculate the work by adding these scalar contributions to the work for each interval $[t_{j-1}, t_j]$, for $j = 1$ to N,

$$W_N = \sum_{j=1}^{j=N} \Delta W_j = \sum_{j=1}^{j=N} (\vec{\mathbf{F}}_j)_{\text{ave}} \cdot \Delta\vec{\mathbf{r}}_j \,. \qquad (13.8.18)$$

We would like to define work in a manner that is independent of the way we divide the interval, so we take the limit as $N \to \infty$ and $\left|\Delta\vec{\mathbf{r}}_j\right| \to 0$ for all j. In this limit, as the intervals become smaller and smaller, the distinction between the average force and the actual force vanishes. Thus if this limit exists and is well defined, then the work done by the force is

$$W = \lim_{\substack{N\to\infty \\ \left|\Delta\vec{\mathbf{r}}_j\right|\to 0}} \sum_{j=1}^{j=N} (\vec{\mathbf{F}}_j)_{\text{ave}} \cdot \Delta\vec{\mathbf{r}}_j = \int_i^f \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \,. \qquad (13.8.19)$$

Notice that this summation involves adding scalar quantities. This limit is called the ***line integral*** of the force $\vec{\mathbf{F}}$. The symbol $d\vec{\mathbf{r}}$ is called the ***infinitesimal vector line element***. At time t, $d\vec{\mathbf{r}}$ is tangent to the orbit of the body and is the limit of the displacement vector $\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)$ as Δt approaches zero. In this limit, the parameter t does not appear in the expression in Equation (13.8.19).

In general this line integral depends on the particular path the body takes between the initial position $\vec{\mathbf{r}}_i$ and the final position $\vec{\mathbf{r}}_f$, which matters when the force $\vec{\mathbf{F}}$ is non-constant in space, and when the contribution to the work can vary over different paths in space. We can represent the integral in Equation (13.8.19) explicitly in a coordinate system by specifying the infinitesimal vector line element $d\vec{\mathbf{r}}$ and then explicitly computing the scalar product.

13.9.1 Work Integral in Cartesian Coordinates

In Cartesian coordinates the line element is

$$d\vec{\mathbf{r}} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}\,, \tag{13.8.20}$$

where dx, dy, and dz represent arbitrary displacements in the $\hat{\mathbf{i}}$-, $\hat{\mathbf{j}}$-, and $\hat{\mathbf{k}}$-directions respectively as seen in Figure 13.17.

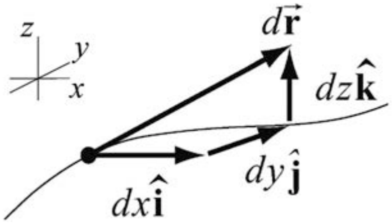


Figure 13.17 A line element in Cartesian coordinates.

The force vector can be represented in vector notation by

$$\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}} + F_z\,\hat{\mathbf{k}}\,. \tag{13.8.21}$$

The infinitesimal work is the sum of the work done by the component of the force times the component of the displacement in each direction,

$$dW = F_x dx + F_y dy + F_z dz\,. \tag{13.8.22}$$

Eq. (13.8.22) is just the scalar product

$$\begin{aligned} dW = \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} &= (F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}} + F_z\,\hat{\mathbf{k}})\cdot(dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}) \\ &= F_x dx + F_y dy + F_z dz \end{aligned}\,, \tag{13.8.23}$$

The work is

$$W = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} (F_x dx + F_y dy + F_z dz) = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_x dx + \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_y dy + \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_z dz\,. \tag{13.8.24}$$

13.9.2 Work Integral in Cylindrical Coordinates

In cylindrical coordinates the line element is

$$d\vec{\mathbf{r}} = dr\,\hat{\mathbf{r}} + rd\theta\,\hat{\boldsymbol{\theta}} + dz\,\hat{\mathbf{k}}\,, \tag{13.8.25}$$

where dr, $rd\theta$, and dz represent arbitrary displacements in the $\hat{\mathbf{r}}$-, $\hat{\boldsymbol{\theta}}$-, and $\hat{\mathbf{k}}$-directions respectively as seen in Figure 13.18.

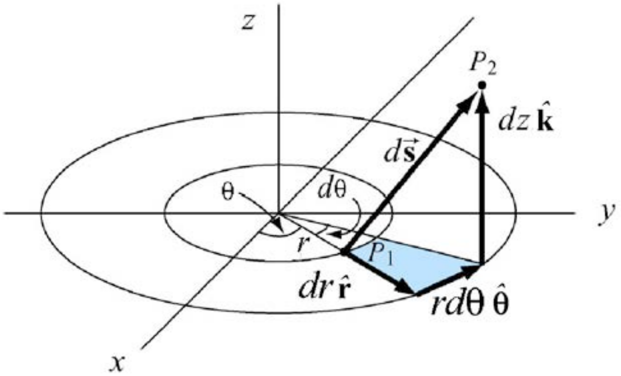


Figure 13.18 Displacement vector $d\,\vec{\mathbf{s}}$ between two points

The force vector can be represented in vector notation by

$$\vec{\mathbf{F}} = F_r\,\hat{\mathbf{r}} + F_\theta\,\hat{\mathbf{\theta}} + F_z\,\hat{\mathbf{k}}\,. \tag{13.8.26}$$

The infinitesimal work is the scalar product

$$\begin{aligned} dW &= \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = (F_r\,\hat{\mathbf{r}} + F_\theta\,\hat{\mathbf{\theta}} + F_z\,\hat{\mathbf{k}}) \cdot (dr\,\hat{\mathbf{r}} + rd\theta\,\hat{\mathbf{\theta}} + dz\,\hat{\mathbf{k}}) \\ &= F_r dr + F_\theta r d\theta + F_z dz. \end{aligned} \tag{13.8.27}$$

The work is

$$W = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} (F_r dr + F_\theta r d\theta + F_z dz) = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_r dr + \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_\theta r d\theta + \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_z dz\,. \tag{13.8.28}$$

13.10 Worked Examples

Example 13.11 Work Done in a Constant Gravitation Field

The work done in a uniform gravitation field is a fairly straightforward calculation when the body moves in the direction of the field. Suppose the body is moving under the influence of gravity, $\vec{\mathbf{F}} = -mg\,\hat{\mathbf{j}}$ along a parabolic curve. The body begins at the point (x_0, y_0) and ends at the point (x_f, y_f). What is the work done by the gravitation force on the body?

Solution: The infinitesimal line element $d\vec{\mathbf{r}}$ is therefore

$$d\vec{\mathbf{r}} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}}\,. \tag{13.9.1}$$

The scalar product that appears in the line integral can now be calculated,

$$\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -mg\,\hat{\mathbf{j}} \cdot [dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}}] = -mgdy\,. \tag{13.9.2}$$

This result is not surprising since the force is only in the y-direction. Therefore the only non-zero contribution to the work integral is in the y-direction, with the result that

$$W = \int_{\mathbf{r}_0}^{\mathbf{r}_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{y=y_0}^{y=y_f} F_y dy = \int_{y=y_0}^{y=y_f} -mgdy = -mg(y_f - y_0)\,. \tag{13.9.3}$$

In this case of a constant force, the work integral is independent of path.

Example 13.12 Hooke's Law Spring-Body System

Consider a spring-body system lying on a frictionless horizontal surface with one end of the spring fixed to a wall and the other end attached to a body of mass m (Figure 13.19). Calculate the work done by the spring force on body as the body moves from some initial position to some final position.

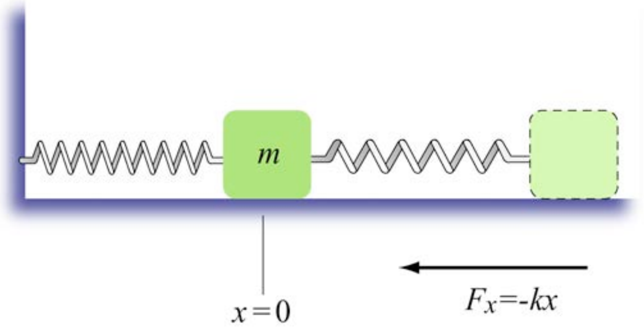


Figure 13.19 A spring-body system.

Solution: Choose the origin at the position of the center of the body when the spring is relaxed (the equilibrium position). Let x be the displacement of the body from the origin. We choose the $+\hat{\mathbf{i}}$ unit vector to point in the direction the body moves when the spring is being stretched (to the right of $x = 0$ in the figure). The spring force on the body is then given by

$$\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} = -kx\,\hat{\mathbf{i}}\,. \tag{13.9.4}$$

The work done by the spring force on the mass is

$$W_{\text{spring}} = \int_{x=x_0}^{x=x_f} (-kx)\,dx = -\frac{1}{2}k(x_f^{\,2} - x_0^{\,2})\,. \tag{13.9.5}$$

Example 13.13 Work done by the Inverse Square Gravitation Force

Consider a body of mass m in moving in a fixed orbital plane about the sun. The mass of the sun is m_s. How much work does the gravitation interaction between the sun and the body done on the body during this motion?

Solution: Let's assume that the sun is fixed and choose a polar coordinate system with the origin at the center of the sun. Initially the body is at a distance r_0 from the center of the sun. In the final configuration the body has moved to a distance $r_f < r_0$ from the center of the sun. The infinitesimal displacement of the body is given by $d\vec{\mathbf{r}} = dr\ \hat{\mathbf{r}} + rd\theta\ \hat{\boldsymbol{\theta}}$. The gravitation force between the sun and the body is given by

$$\vec{\mathbf{F}}_{grav} = F_{grav}\ \hat{\mathbf{r}} = -\frac{Gm_s m}{r^2}\hat{\mathbf{r}}. \qquad (13.9.6)$$

The infinitesimal work done work done by this gravitation force on the body is given by

$$dW = \vec{\mathbf{F}}_{grav} \cdot d\vec{\mathbf{r}} = (F_{grav,r}\ \hat{\mathbf{r}}) \cdot (dr\ \hat{\mathbf{r}} + rd\theta\ \hat{\boldsymbol{\theta}}) = F_{grav,r} dr\ . \qquad (13.9.7)$$

Therefore the work done on the object as the object moves from r_i to r_f is given by the integral

$$W = \int_{r_i}^{r_f} \vec{\mathbf{F}}_{grav} \cdot d\vec{\mathbf{r}} = \int_{r_i}^{r_f} F_{grav,r}\ dr = \int_{r_i}^{r_f} \left(-\frac{Gm_{\text{sun}} m}{r^2} \right) dr\,. \qquad (13.9.8)$$

Upon evaluation of this integral, we have for the work

$$W = \int_{r_i}^{r_f} \left(-\frac{Gm_{\text{sun}} m}{r^2} \right) dr = \left. \frac{Gm_{\text{sun}} m}{r} \right|_{r_i}^{r_f} = Gm_{\text{sun}} m \left(\frac{1}{r_f} - \frac{1}{r_i} \right). \qquad (13.9.9)$$

Because the body has moved closer to the sun, $r_f < r_i$, hence $1/r_f > 1/r_i$. Thus the work done by gravitation force between the sun and the body, on the body is positive,

$$W = Gm_{\text{sun}} m \left(\frac{1}{r_f} - \frac{1}{r_i} \right) > 0 \qquad (13.9.10)$$

We expect this result because the gravitation force points along the inward radial direction, so the scalar product and hence work of the force and the displacement is positive when the body moves closer to the sun. Also we expect that the sign of the work is the same for a body moving closer to the sun as a body falling towards the earth in a constant gravitation field, as seen in Example 4.7.1 above.

Example 13.14 Work Done by the Inverse Square Electrical Force

Let's consider two point-like bodies, body 1 and body 2, with charges q_1 and q_2 respectively interacting via the electric force alone. Body 1 is fixed in place while body 2 is free to move in an orbital plane. How much work does the electric force do on the body 2 during this motion?

Solution: The calculation in nearly identical to the calculation of work done by the gravitational inverse square force in Example 13.13. The most significant difference is that the electric force can be either attractive or repulsive while the gravitation force is always attractive. Once again we choose polar coordinates centered on body 2 in the plane of the orbit. Initially a distance r_0 separates the bodies and in the final state a distance r_f separates the bodies. The electric force between the bodies is given by

$$\vec{\mathbf{F}}_{\text{elec}} = F_{\text{elec}}\,\hat{\mathbf{r}} = F_{elec,r}\,\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\,\hat{\mathbf{r}}\,. \tag{13.9.11}$$

The work done by this electric force on the body 2 is given by the integral

$$W = \int_{r_i}^{r_f} \vec{\mathbf{F}}_{elec}\cdot d\vec{\mathbf{r}} = \int_{r_i}^{r_f} F_{elec,r}\,dr = \frac{1}{4\pi\varepsilon_0}\int_{r_i}^{r_f}\frac{q_1 q_2}{r^2}\,dr\,. \tag{13.9.12}$$

Evaluating this integral, we have for the work done by the electric force

$$W = \int_{r_i}^{r_f}\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\,dr = -\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\bigg|_{r_i}^{r_f} = -\frac{1}{4\pi\varepsilon_0}q_1 q_2\left(\frac{1}{r_f}-\frac{1}{r_i}\right). \tag{13.9.13}$$

If the charges have opposite signs, $q_1 q_2 < 0$, we expect that the body 2 will move closer to body 1 so $r_f < r_i$, and $1/r_f > 1/r_i$. From our result for the work, the work done by electrical force in moving body 2 is positive,

$$W = -\frac{1}{4\pi\varepsilon_0}q_1 q_2\left(\frac{1}{r_f}-\frac{1}{r_i}\right) > 0\,. \tag{13.9.14}$$

Once again we see that bodies under the influence of electric forces only will naturally move in the directions in which the force does positive work. If the charges have the same sign, then $q_1 q_2 > 0$. They will repel with $r_f > r_i$ and $1/r_f < 1/r_i$. Thus the work is once again positive:

$$W = -\frac{1}{4\pi\varepsilon_0} q_1 q_2 \left(\frac{1}{r_f} - \frac{1}{r_i} \right) > 0 . \tag{13.9.15}$$

13.11 Work-Kinetic Energy Theorem in Three Dimensions

Recall our mathematical result that for one-dimensional motion

$$m\int_i^f a_x\, dx = m\int_i^f \frac{dv_x}{dt} dx = m\int_i^f dv_x \frac{dx}{dt} = m\int_i^f v_x\, dv_x = \frac{1}{2} m v_{x,f}^2 - \frac{1}{2} m v_{x,i}^2 . \tag{13.11.1}$$

Using Newton's Second Law in the form $F_x = m\, a_x$, we concluded that

$$\int_i^f F_x\, dx = \frac{1}{2} m v_{x,f}^2 - \frac{1}{2} m v_{x,i}^2 . \tag{13.11.2}$$

Eq. (13.11.2) generalizes to the y- and z-directions:

$$\int_i^f F_y\, dy = \frac{1}{2} m v_{y,f}^2 - \frac{1}{2} m v_{y,i}^2 , \tag{13.11.3}$$

$$\int_i^f F_z\, dz = \frac{1}{2} m v_{z,f}^2 - \frac{1}{2} m v_{z,i}^2 . \tag{13.11.4}$$

Adding Eqs. (13.11.2), (13.11.3), and (13.11.4) yields

$$\int_i^f (F_x\, dx + F_y\, dy + F_z\, dz) = \frac{1}{2} m (v_{x,f}^2 + v_{y,f}^2 + v_{z,f}^2) - \frac{1}{2} m (v_{x,i}^2 + v_{y,i}^2 + v_{z,i}^2) . \tag{13.11.5}$$

Recall (Eq. (13.8.24)) that the left hand side of Eq. (13.11.5) is the work done by the force $\vec{\mathbf{F}}$ on the object

$$W = \int_i^f dW = \int_i^f (F_x\, dx + F_y\, dy + F_z\, dz) = \int_i^f \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{13.11.6}$$

The right hand side of Eq. (13.11.5) is the change in kinetic energy of the object

$$\Delta K \equiv K_f - K_i = \frac{1}{2} m v_f{}^2 - \frac{1}{2} m v_0{}^2 = \frac{1}{2} m (v_{x,f}^2 + v_{y,f}^2 + v_{z,f}^2) - \frac{1}{2} m (v_{x,i}^2 + v_{y,i}^2 + v_{z,i}^2) . \tag{13.11.7}$$

Therefore Eq. (13.11.5) is the three dimensional generalization of the work-kinetic energy theorem

$$\int_i^f \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = K_f - K_i. \tag{13.11.8}$$

When the work done on an object is positive, the object will increase its speed, and negative work done on an object causes a decrease in speed. When the work done is zero, the object will maintain a constant speed.

13.11.1 Instantaneous Power Applied by a Non-Constant Force for Three Dimensional Motion

Recall that for one-dimensional motion, the *instantaneous power* at time t is defined to be the limit of the average power as the time interval $[t, t + \Delta t]$ approaches zero,

$$P(t) = F_x^a(t) v_x(t) . \tag{13.11.9}$$

A more general result for the instantaneous power is found by using the expression for dW as given in Equation (13.8.23),

$$P = \frac{dW}{dt} = \frac{\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}}{dt} = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}} . \tag{13.11.10}$$

The time rate of change of the kinetic energy for a body of mass m is equal to the power,

$$\frac{dK}{dt} = \frac{1}{2} m \frac{d}{dt}\left(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \right) = m \frac{d\vec{\mathbf{v}}}{dt} \cdot \vec{\mathbf{v}} = m \vec{\mathbf{a}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}} = P . \tag{13.11.11}$$

where the we used Eq. (13.8.9), Newton’s Second Law and Eq. (13.11.10).

Appendix 13A Work Done on a System of Two Particles

We shall show that the work done by an internal force in changing a system of two particles of masses m_1 and m_2 respectively from an initial state A to a final state B is equal to

$$W_c = \frac{1}{2}\mu(v_B^2 - v_A^2) \tag{13.1.1}$$

where v_B^2 is the square of the relative velocity in state B, v_A^2 is the square of the relative velocity in state A, and $\mu = m_1 m_2 / (m_1 + m_2)$.

Consider two bodies 1 and 2 and an interaction pair of forces shown in Figure 13A.1.

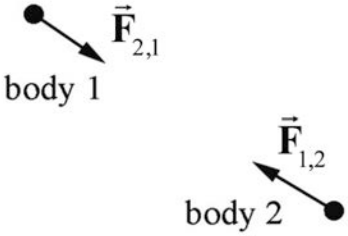


Figure 13A.1 System of two bodies interacting

We choose a coordinate system shown in Figure 13A.2.

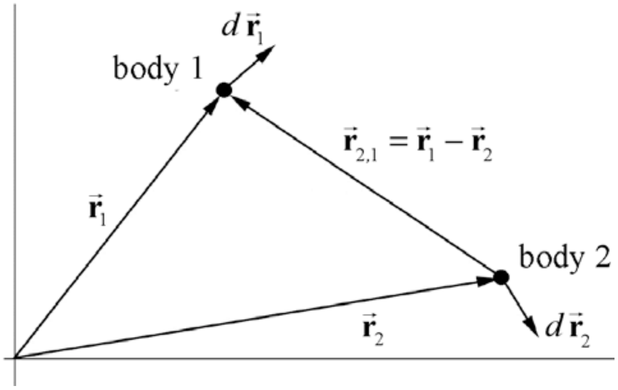


Figure 13A.2 Coordinate system for two-body interaction

Newton's Second Law applied to body 1 is

$$\vec{\mathbf{F}}_{2,1} = m_1 \frac{d^2\vec{\mathbf{r}}_1}{dt^2} \tag{13.1.2}$$

and applied to body 2 is

$$\vec{\mathbf{F}}_{1,2} = m_2 \frac{d^2\vec{\mathbf{r}}_2}{dt^2}. \qquad (13.1.3)$$

Divide each side of Equation (13.1.2) by m_1,

$$\frac{\vec{\mathbf{F}}_{2,1}}{m_1} = \frac{d^2\vec{\mathbf{r}}_1}{dt^2} \qquad (13.1.4)$$

and divide each side of Equation (13.1.3) by m_2,

$$\frac{\vec{\mathbf{F}}_{1,2}}{m_2} = \frac{d^2\vec{\mathbf{r}}_2}{dt^2}. \qquad (13.1.5)$$

Subtract Equation (13.1.5) from Equation (13.1.4) yielding

$$\frac{\vec{\mathbf{F}}_{2,1}}{m_1} - \frac{\vec{\mathbf{F}}_{1,2}}{m_2} = \frac{d^2\vec{\mathbf{r}}_1}{dt^2} - \frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \frac{d^2\vec{\mathbf{r}}_{2,1}}{dt^2}, \qquad (13.1.6)$$

where $\vec{\mathbf{r}}_{2,1} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$. Use Newton's Third Law, $\vec{\mathbf{F}}_{2,1} = -\vec{\mathbf{F}}_{1,2}$ on the left hand side of Equation (13.1.6) to obtain

$$\vec{\mathbf{F}}_{2,1}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{d^2\vec{\mathbf{r}}_1}{dt^2} - \frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \frac{d^2\vec{\mathbf{r}}_{2,1}}{dt^2}. \qquad (13.1.7)$$

The quantity $d^2\vec{\mathbf{r}}_{1,2} / dt^2$ is the *relative acceleration* of body 1 with respect to body 2. Define

$$\frac{1}{\mu} \equiv \frac{1}{m_1} + \frac{1}{m_2}. \qquad (13.1.8)$$

The quantity μ is known as the ***reduced mass*** of the system. Equation (13.1.7) now takes the form

$$\vec{\mathbf{F}}_{2,1} = \mu \frac{d^2\vec{\mathbf{r}}_{2,1}}{dt^2}. \qquad (13.1.9)$$

The work done in the system in displacing the two masses from an initial state A to a final state B is given by

$$W = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_1 + \int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\vec{\mathbf{r}}_2. \qquad (13.1.10)$$

Recall by the work energy theorem that the LHS is the work done on the system,

$$W = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_1 + \int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\vec{\mathbf{r}}_2 = \Delta K\,. \tag{13.1.11}$$

From Newton's Third Law, the sum in Equation (13.1.10) becomes

$$W = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_1 - \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_2 = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot (d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2) = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_{2,1}\,, \tag{13.1.12}$$

where $d\vec{\mathbf{r}}_{2,1}$ is the relative displacement of the two bodies. We can now substitute Newton's Second Law, Equation (13.1.9), for the relative acceleration into Equation (13.1.12),

$$W = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_{2,1} = \int_A^B \mu \frac{d^2\vec{\mathbf{r}}_{2,1}}{dt^2} \cdot d\vec{\mathbf{r}}_{2,1} = \mu \int_A^B \left(\frac{d^2\vec{\mathbf{r}}_{2,1}}{dt^2} \cdot \frac{d\vec{\mathbf{r}}_{2,1}}{dt} \right) dt\,, \tag{13.1.13}$$

where we have used the relation between the differential elements $d\vec{\mathbf{r}}_{2,1} = \dfrac{d\vec{\mathbf{r}}_{2,1}}{dt} dt$. The product rule for derivatives of the scalar product of a vector with itself is given for this case by

$$\frac{1}{2}\frac{d}{dt}\left(\frac{d\vec{\mathbf{r}}_{2,1}}{dt} \cdot \frac{d\vec{\mathbf{r}}_{2,1}}{dt} \right) = \frac{d^2\vec{\mathbf{r}}_{2,1}}{dt^2} \cdot \frac{d\vec{\mathbf{r}}_{2,1}}{dt}\,. \tag{13.1.14}$$

Substitute Equation (13.1.14) into Equation (13.1.13), which then becomes

$$W = \mu \int_A^B \frac{1}{2}\frac{d}{dt}\left(\frac{d\vec{\mathbf{r}}_{2,1}}{dt} \cdot \frac{d\vec{\mathbf{r}}_{2,1}}{dt} \right) dt\,. \tag{13.1.15}$$

Equation (13.1.15) is now the integral of an exact derivative, yielding

$$W = \frac{1}{2}\mu \left(\frac{d\vec{\mathbf{r}}_{2,1}}{dt} \cdot \frac{d\vec{\mathbf{r}}_{2,1}}{dt} \right)\Bigg|_A^B = \frac{1}{2}\mu (\vec{\mathbf{v}}_{2,1} \cdot \vec{\mathbf{v}}_{2,1})\Big|_A^B = \frac{1}{2}\mu (v_B^2 - v_A^2)\,, \tag{13.1.16}$$

where $\vec{\mathbf{v}}_{2,1}$ is the *relative velocity* between the two bodies. It's important to note that in the above derivation had we exchanged the roles of body 1 and 2 i.e. $1 \to 2$ and $2 \to 1$, we would have obtained the identical result because

$$\begin{aligned}
\vec{\mathbf{F}}_{1,2} &= -\vec{\mathbf{F}}_{2,1} \\
\vec{\mathbf{r}}_{1,2} &= \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1 = -\vec{\mathbf{r}}_{2,1} \\
d\vec{\mathbf{r}}_{1,2} &= d(\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1) = -d\vec{\mathbf{r}}_{2,1} \\
\vec{\mathbf{v}}_{1,2} &= -\vec{\mathbf{v}}_{2,1}.
\end{aligned} \tag{13.1.17}$$

Equation (13.1.16) implies that the work done is the change in the kinetic energy of the system, which we can write in terms of the reduced mass and the change in the square of relative speed of the two objects

$$\Delta K = \frac{1}{2}\mu(v_B^2 - v_A^2)\,. \tag{13.1.18}$$

Chapter 14 Potential Energy and Conservation of Energy

Equation Chapter 8 Section 1 Chapter 14 Potential Energy and Conservation of Energy

> *There is a fact, or if you wish, a law, governing all natural phenomena that are known to date. There is no exception to this law — it is exact as far as we know. The law is called the conservation of energy. It states that there is a certain quantity, which we call energy that does not change in the manifold changes which nature undergoes. That is a most abstract idea, because it is a mathematical principle; it says that there is a numerical quantity, which does not change when something happens. It is not a description of a mechanism, or anything concrete; it is just a strange fact that we can calculate some number and when we finish watching nature go through her tricks and calculate the number again, it is the same.* [1]
>
> *Richard Feynman*

So far we have analyzed the motion of point-like objects under the action of forces using Newton's Laws of Motion. We shall now introduce the Principle of Conservation of Energy to study the change in energy of a system between its initial and final states. In particular we shall introduce the concept of potential energy to describe the effect of conservative internal forces acting on the constituent components of a system.

14.1 Conservation of Energy

Recall from Chapter 13.1, the principle of conservation of energy. When a system and its surroundings undergo a transition from an initial state to a final state, the change in energy is zero,

$$\Delta E = \Delta E_{\text{system}} + \Delta E_{\text{surroundings}} = 0\,. \tag{14.1.1}$$

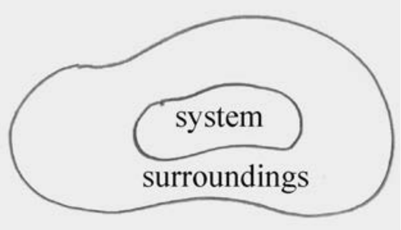


Figure 14.1 Diagram of a system and its surroundings

We shall study types of energy transformations due to interactions both inside and across the boundary of a system.

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[1] Richard P. Feynman, Robert B. Leighton, and Matthew Sands, *The Feynman Lectures on Physics,* Vol. 1, p. 4.1.

14.2 Conservative and Non-Conservative Forces

Our first type of "energy accounting" involves *mechanical energy*. There are two types of mechanical energy, *kinetic energy* and *potential energy*. Our first task is to define what we mean by the change of the potential energy of a system.

We defined the work done by a force $\vec{\mathbf{F}}$, on an object, which moves along a path from an initial position $\vec{\mathbf{r}}_i$ to a final position $\vec{\mathbf{r}}_f$, as the integral of the component of the force tangent to the path with respect to the displacement of the point of contact of the force and the object,

$$W = \int_{\text{path}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} . \qquad (14.2.1)$$

Does the work done on the object by the force depend on the path taken by the object?

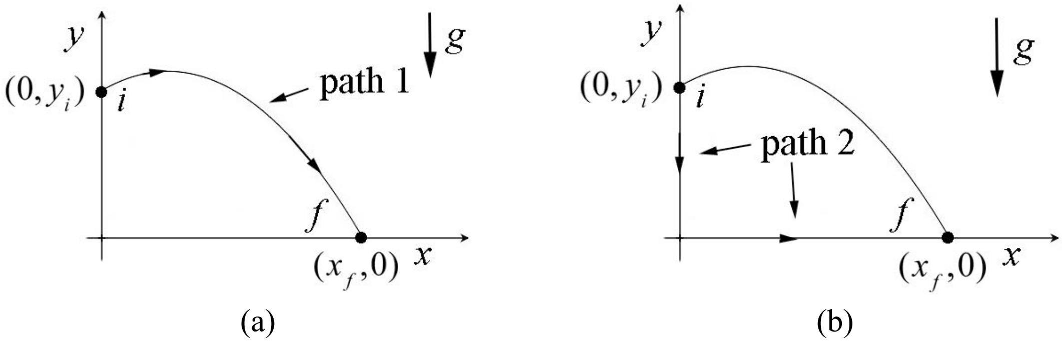

Figure 14.2 (a) and (b) Two different paths connecting the same initial and final points

First consider the motion of an object under the influence of a gravitational force near the surface of the earth. Let's consider two paths 1 and 2 shown in Figure 14.2. Both paths begin at the initial point $(x_i, y_i) = (0, y_i)$ and end at the final point $(x_f, y_f) = (x_f, 0)$. The gravitational force always points downward, so with our choice of coordinates, $\vec{\mathbf{F}} = -mg\,\hat{\mathbf{j}}$. The infinitesimal displacement along path 1 (Figure 14.2a) is given by $d\vec{\mathbf{r}}_1 = dx_1\,\hat{\mathbf{i}} + dy_1\,\hat{\mathbf{j}}$. The scalar product is then

$$\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}_1 = -mg\,\hat{\mathbf{j}} \cdot (dx_1\,\hat{\mathbf{i}} + dy_1\,\hat{\mathbf{j}}) = -mgdy_1 . \qquad (14.2.2)$$

The work done by gravity along path 1 is the integral

$$W_1 = \int_{\text{path 1}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{(0,y_i)}^{(x_f,0)} -mg\,dy_1 = -mg(0 - y_i) = \; mgy_i . \qquad (14.2.3)$$

Path 2 consists of two legs (Figure 14.2b), leg A goes from the initial point $(0, y_i)$ to the origin $(0,0)$, and leg B goes from the origin $(0,0)$ to the final point $(x_f, 0)$. We shall calculate the work done along the two legs and then sum them up. The infinitesimal displacement along leg A is given by $d\vec{\mathbf{r}}_A = dy_A\,\hat{\mathbf{j}}$. The scalar product is then

$$\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}_A = -mg\,\hat{\mathbf{j}} \cdot dy_A\,\hat{\mathbf{j}} = -mgdy_A\,. \tag{14.2.4}$$

The work done by gravity along leg A is the integral

$$W_A = \int\limits_{\text{leg } A} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}_{\text{A}} = \int\limits_{(0,y_i)}^{(0,0)} -mg\,dy_A = -mg(0 - y_i) = \; mgy_i\,. \tag{14.2.5}$$

The infinitesimal displacement along leg B is given by $d\vec{\mathbf{r}}_B = dx_B\,\hat{\mathbf{i}}$. The scalar product is then

$$\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}_B = -mg\,\hat{\mathbf{j}} \cdot dx_B\,\hat{\mathbf{i}} = 0\,. \tag{14.2.6}$$

Therefore the work done by gravity along leg B is zero, $W_B = 0$, which is no surprise because leg B is perpendicular to the direction of the gravitation force. Therefore the work done along path 2 is equal to the work along path 1,

$$W_2 = W_A + W_B = \; mgy_i = W_1\,. \tag{14.2.7}$$

Now consider the motion of an object on a surface with a kinetic frictional force between the object and the surface and denote the coefficient of kinetic friction by μ_{k}. Let's compare two paths from an initial point x_i to a final point x_f. The first path is a straight-line path. Along this path the work done is just

$$W^f = \int\limits_{\text{path 1}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int\limits_{\text{path 1}} F_x\;dx = -\mu_{\text{k}} N\,s_1 = -\mu_{\text{k}} N\,\Delta x < 0\,, \tag{14.2.8}$$

where the length of the path is equal to the displacement, $s_1 = \Delta x$. Note that the fact that the kinetic frictional force is directed opposite to the displacement, which is reflected in the minus sign in Equation (14.2.8). The second path goes past x_f some distance and them comes back to x_f (Figure 14.3). Because the force of friction always opposes the motion, the work done by friction is negative,

$$W^f = \int\limits_{\text{path 2}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int\limits_{\text{path 2}} F_x\;dx = -\mu_{\text{k}} N\,s_2 < 0\,. \tag{14.2.9}$$

The work depends on the total distance traveled s_2, and is greater than the displacement $s_2 > \Delta x$. The magnitude of the work done along the second path is greater than the magnitude of the work done along the first path.

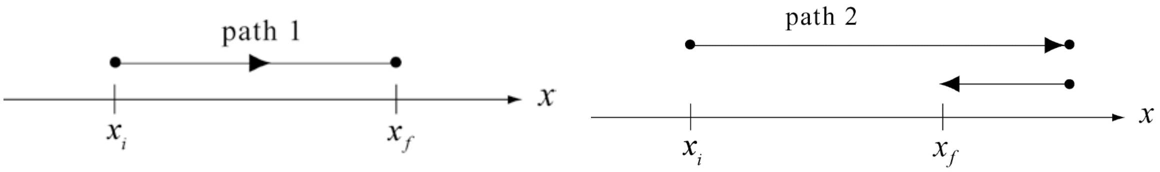

Figure 14.3 Two different paths from x_i to x_f.

These two examples typify two fundamentally different types of forces and their contribution to work. The work done by the gravitational force near the surface of the earth is independent of the path taken between the initial and final points. In the case of sliding friction, the work done depends on the path taken.

> *Whenever the work done by a force in moving an object from an initial point to a final point is independent of the path, the force is called a* ***conservative force****.*

The work done by a conservative force $\vec{\mathbf{F}}_c$ in going around a closed path is zero. Consider the two paths shown in Figure 14.4 that form a closed path starting and ending at the point A with Cartesian coordinates $(1,0)$.

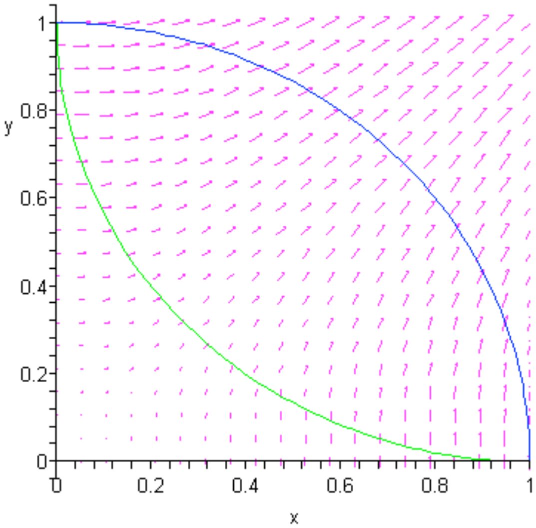

Figure 14.4 Two paths in the presence of a conservative force.

The work done along path 1 (the upper path in the figure, blue if viewed in color) from point A to point B with coordinates $(0,1)$ is given by

$$W_1 = \int_A^B \vec{\mathbf{F}}_{\mathrm{c}}(1) \cdot d\vec{\mathbf{r}}_1 \,. \tag{14.2.10}$$

The work done along path 2 (the lower path, green in color) from B to A is given by

$$W_2 = \int_B^A \vec{\mathbf{F}}_{\mathrm{c}}(2) \cdot d\vec{\mathbf{r}}_2 \,. \tag{14.2.11}$$

The work done around the closed path is just the sum of the work along paths 1 and 2,

$$W = W_1 + W_2 = \int_A^B \vec{\mathbf{F}}_{\mathrm{c}}(1) \cdot d\vec{\mathbf{r}}_1 + \int_B^A \vec{\mathbf{F}}_{\mathrm{c}}(2) \cdot d\vec{\mathbf{r}}_2 \,. \tag{14.2.12}$$

If we reverse the endpoints of path 2, then the integral changes sign,

$$W_2 = \int_B^A \vec{\mathbf{F}}_{\mathrm{c}}(2) \cdot d\vec{\mathbf{r}}_2 = -\int_A^B \vec{\mathbf{F}}_{\mathrm{c}}(2) \cdot d\vec{\mathbf{r}}_2 \,. \tag{14.2.13}$$

We can then substitute Equation (14.2.13) into Equation (14.2.12) to find that the work done around the closed path is

$$W = \int_A^B \vec{\mathbf{F}}_{\mathrm{c}}(1) \cdot d\vec{\mathbf{r}}_1 - \int_A^B \vec{\mathbf{F}}_{\mathrm{c}}(2) \cdot d\vec{\mathbf{r}}_2 \,. \tag{14.2.14}$$

Since the force is conservative, the work done between the points A to B is independent of the path, so

$$\int_A^B \vec{\mathbf{F}}_{\mathrm{c}}(1) \cdot d\vec{\mathbf{r}}_1 = \int_A^B \vec{\mathbf{F}}_{\mathrm{c}}(2) \cdot d\vec{\mathbf{r}}_2 \,. \tag{14.2.15}$$

We now use path independence of work for a conservative force (Equation (14.2.15) in Equation (14.2.14)) to conclude that the work done by a conservative force around a closed path is zero,

$$W = \oint_{\substack{\text{closed}\\\text{path}}} \vec{\mathbf{F}}_{\mathrm{c}} \cdot d\vec{\mathbf{r}} = 0 \,. \tag{14.2.16}$$

14.3 Changes in Potential Energies of a System

Consider an object near the surface of the earth as a system that is initially given a velocity directed upwards. Once the object is released, the gravitation force, acting as an external force, does a negative amount of work on the object, and the kinetic energy decreases until the object reaches its highest point, at which its kinetic energy is zero. The gravitational force then does positive work until the object returns to its initial starting point with a velocity directed downward. If we ignore any effects of air resistance, the descending object will then have the identical kinetic energy as when it was thrown. All the kinetic energy was completely recovered.

Now consider both the earth and the object as a system and assume that there are no other external forces acting on the system. Then the gravitational force is an internal conservative force, and does work on both the object and the earth during the motion. As the object moves upward, the kinetic energy of the system decreases, primarily because the object slows down, but there is also an imperceptible increase in the kinetic energy of the earth. The change in kinetic energy of the earth must also be included because the earth is part of the system. When the object returns to its original height (vertical distance from the surface of the earth), all the kinetic energy in the system is recovered, even though a very small amount has been transferred to the Earth.

If we included the air as part of the system, and the air resistance as a non-conservative internal force, then the kinetic energy lost due to the work done by the air resistance is not recoverable. This lost kinetic energy, which we have called thermal energy, is distributed as random kinetic energy in both the air molecules and the molecules that compose the object (and, to a smaller extent, the earth).

We shall define a new quantity, the change in the internal *potential energy* of the system, which measures the amount of lost kinetic energy that can be recovered during an interaction.

> *When only internal conservative forces act in a closed system, the sum of the changes of the kinetic and potential energies of the system is zero.*

Consider a closed system, $\Delta E_{sys} = 0$, that consists of two objects with masses m_1 and m_2 respectively. Assume that there is only one conservative force (internal force) that is the source of the interaction between two objects. We denote the force on object 1 due to the interaction with object 2 by $\vec{\mathbf{F}}_{2,1}$ and the force on object 2 due to the interaction with object 1 by $\vec{\mathbf{F}}_{1,2}$. From Newton's Third Law,

$$\vec{\mathbf{F}}_{2,1} = -\vec{\mathbf{F}}_{1,2} . \tag{14.3.1}$$

The forces acting on the objects are shown in Figure 14.5.

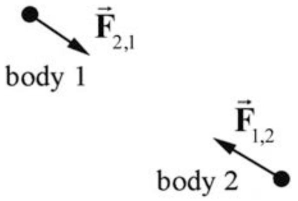

Figure 14.5 Internal forces acting on two objects

Choose a coordinate system (Figure 14.6) in which the position vector of object 1 is given by $\vec{\mathbf{r}}_1$ and the position vector of object 2 is given by $\vec{\mathbf{r}}_2$. The relative position of object 1 with respect to object 2 is given by $\vec{\mathbf{r}}_{2,1} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$. During the course of the interaction, object 1 is displaced by $d\vec{\mathbf{r}}_1$ and object 2 is displaced by $d\vec{\mathbf{r}}_2$, so the relative displacement of the two objects during the interaction is given by $d\vec{\mathbf{r}}_{2,1} = d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2$.

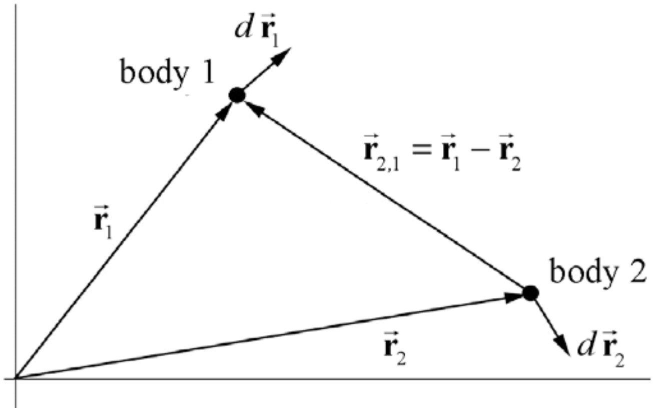

Figure 14.6 Coordinate system for two objects with relative position vector $\vec{\mathbf{r}}_{2,1} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$

Recall that the change in the kinetic energy of an object is equal to the work done by the forces in displacing the object. For two objects displaced from an initial state A to a final state B,

$$\Delta K_{\text{sys}} = \Delta K_1 + \Delta K_2 = W_{\text{c}} = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_1 + \int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\vec{\mathbf{r}}_2 . \tag{14.3.2}$$

(In Equation (14.3.2), the labels "A" and "B" refer to initial and final states, not paths.)

From Newton's Third Law, Equation (14.3.1), the sum in Equation (14.3.2) becomes

$$\Delta K_{\text{sys}} = W_c = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_1 - \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_2 = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot (d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2) = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_{2,1} \tag{14.3.3}$$

where $d\vec{\mathbf{r}}_{2,1} = d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2$ is the relative displacement of the two objects. Note that since $\vec{\mathbf{F}}_{2,1} = -\vec{\mathbf{F}}_{1,2}$ and $d\vec{\mathbf{r}}_{2,1} = -d\vec{\mathbf{r}}_{1,2}$, $\int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_{2,1} = \int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\vec{\mathbf{r}}_{1,2}$.

Consider a system consisting of two objects interacting through a conservative force. Let $\vec{\mathbf{F}}_{2,1}$ *denote the force on object 1 due to the interaction with object 2 and let* $d\vec{\mathbf{r}}_{2,1} = d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2$ *be the relative displacement of the two objects. The* ***change in internal potential energy of the system*** *is defined to be the negative of the work done by the conservative force when the objects undergo a relative displacement from the initial state* A *to the final state* B *along any displacement that changes the initial state* A *to the final state* B*,*

$$\Delta U_{\text{sys}} = -W_{\text{c}} = -\int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_{2,1} = -\int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\vec{\mathbf{r}}_{1,2} . \tag{14.3.4}$$

Our definition of potential energy only holds for conservative forces, because the work done by a conservative force does not depend on the path but only on the initial and final positions. Because the work done by the conservative force is equal to the change in kinetic energy, we have that

$$\Delta U_{\text{sys}} = -\Delta K_{\text{sys}}, \text{ (closed system with no non-conservative forces)} . \tag{14.3.5}$$

Recall that the work done by a conservative force in going around a closed path is zero (Equation (14.2.16)); therefore the change in kinetic energy when a system returns to its initial state is zero. This means that the kinetic energy is completely recoverable.

In the *Appendix 13A: Work Done on a System of Two Particles,* we showed that the work done by an internal force in changing a system of two particles of masses m_1 and m_2 respectively from an initial state A to a final state B is equal to

$$W = \frac{1}{2}\mu(v_B^2 - v_A^2) = \Delta K_{\text{sys}} , \tag{14.3.6}$$

where v_B^2 is the square of the relative velocity in state B, v_A^2 is the square of the relative velocity in state A, and $\mu = m_1 m_2 / (m_1 + m_2)$ is a quantity known as the *reduced mass* of the system.

14.3.1 Change in Potential Energy for Several Conservative Forces

When there are several internal conservative forces acting on the system we define a separate change in potential energy for the work done by each conservative force,

$$\Delta U_{\text{sys},\,i} = -W_{c,i} = -\int_A^B \vec{\mathbf{F}}_{\text{c},\,i} \cdot d\vec{\mathbf{r}}_i . \tag{14.3.7}$$

where $\vec{\mathbf{F}}_{c,i}$ is a conservative internal force and $d\vec{\mathbf{r}}_i$ a change in the relative positions of the objects on which $\vec{\mathbf{F}}_{c,i}$ when the system is changed from state A to state B. The work done is the sum of the work done by the individual conservative forces,

$$W_c = W_{c,1} + W_{c,2} + \cdots . \tag{14.3.8}$$

Hence, the sum of the changes in potential energies for the system is the sum

$$\Delta U_{sys} = \Delta U_{sys,1} + \Delta U_{sys,2} + \cdots . \tag{14.3.9}$$

Therefore the change in potential energy of the system is equal to the negative of the work done

$$\Delta U_{sys} = -W_c = -\sum_i \int_A^B \vec{\mathbf{F}}_{c,i} \cdot d\vec{\mathbf{r}}_i \,. \tag{14.3.10}$$

If the system is closed (external forces do no work), and there are no non-conservative internal forces then Eq. (14.3.5) holds.

14.4 Change in Potential Energy and Zero Point for Potential Energy

We already calculated the work done by different conservative forces: constant gravity near the surface of the earth, the spring force, and the universal gravitation force. We chose the system in each case so that the conservative force was an external force. In each case, there was no change of potential energy and the work done was equal to the change of kinetic energy,

$$W_{ext} = \Delta K_{sys} \,. \tag{14.4.1}$$

We now treat each of these conservative forces as internal forces and calculate the change in potential energy of the system according to our definition

$$\Delta U_{sys} = -W_c = -\int_A^B \vec{\mathbf{F}}_c \cdot d\vec{\mathbf{r}} \,. \tag{14.4.2}$$

We shall also choose a ***zero reference potential*** for the potential energy of the system, so that we can consider all changes in potential energy relative to this reference potential.

14.4.1 Change in Gravitational Potential Energy Near Surface of the Earth

Let's consider the example of an object falling near the surface of the earth. Choose our system to consist of the earth and the object. The gravitational force is now an internal conservative force acting inside the system. The distance separating the object and the center of mass of the earth, and the velocities of the earth and the object specifies the initial and final states.

Let's choose a coordinate system with the origin on the surface of the earth and the $+y$-direction pointing away from the center of the earth. Because the displacement of the earth is negligible, we need only consider the displacement of the object in order to calculate the change in potential energy of the system.

Suppose the object starts at an initial height y_i above the surface of the earth and ends at final height y_f. The gravitational force on the object is given by $\vec{\mathbf{F}}^g = -mg\,\hat{\mathbf{j}}$, the displacement is given by $d\vec{\mathbf{r}} = dy\,\hat{\mathbf{j}}$, and the scalar product is given by $\vec{\mathbf{F}}^g \cdot d\vec{\mathbf{r}} = -mg\,\hat{\mathbf{j}} \cdot dy\hat{\mathbf{j}} = -mgdy$. The work done by the gravitational force on the object is then

$$W^g = \int_{y_i)}^{y_f} \vec{\mathbf{F}}^g \cdot d\vec{\mathbf{r}} = \int_{y_i)}^{y_f} -mg\,dy = -mg(y_f - y_i) \ . \tag{14.4.3}$$

The change in potential energy is then given by

$$\Delta U^g = -W^g = mg\,\Delta y = mg\,y_f - mg\,y_i \,. \tag{14.4.4}$$

We introduce a potential energy function U so that

$$\Delta U^g \equiv U_f^g - U_i^g \,. \tag{14.4.5}$$

Only differences in the function U^g have a physical meaning. We can choose a zero reference point for the potential energy anywhere we like. We have some flexibility to adapt our choice of zero for the potential energy to best fit a particular problem. Because the change in potential energy only depended on the displacement, Δy. In the above expression for the change of potential energy (Eq. (14.4.4)), let $y_f = y$ be an arbitrary point and $y_i = 0$ denote the surface of the earth. Choose the zero reference potential for the potential energy to be at the surface of the earth corresponding to our origin $y = 0$, with $U^g(0) = 0$. Then

$$\Delta U^g = U^g(y) - U^g(0) = U^g(y) \,. \tag{14.4.6}$$

Substitute $y_i = 0$, $y_f = y$ and Eq. (14.4.6) into Eq. (14.4.4) yielding a potential energy as a function of the height y above the surface of the earth,

$$U^g(y) = mgy, \text{ with } U^g(y = 0) = 0 \,. \tag{14.4.7}$$

14.4.2 Hooke's Law Spring-Object System

Consider a spring-object system lying on a frictionless horizontal surface with one end of the spring fixed to a wall and the other end attached to an object of mass m (Figure 14.7). The spring force is an internal conservative force. The wall exerts an external force on the spring-object system but since the point of contact of the wall with the spring undergoes no displacement, this external force does no work.

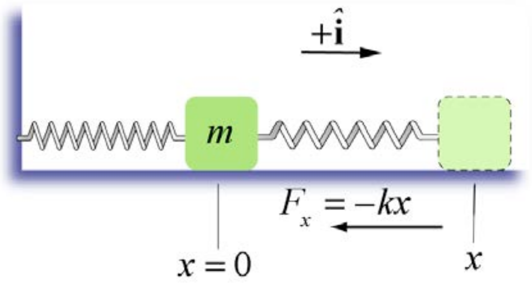

Figure 14.7 A spring-object system.

Choose the origin at the position of the center of the object when the spring is relaxed (the equilibrium position). Let x be the displacement of the object from the origin. We choose the $+\hat{\mathbf{i}}$ unit vector to point in the direction the object moves when the spring is being stretched (to the right of $x = 0$ in the figure). The spring force on a mass is then given by $\vec{\mathbf{F}}^s = F_x^s\,\hat{\mathbf{i}} = -kx\,\hat{\mathbf{i}}$. The displacement is $d\vec{\mathbf{r}} = dx\,\hat{\mathbf{i}}$. The scalar product is $\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -kx\,\hat{\mathbf{i}} \cdot dx\,\hat{\mathbf{i}} = -kx\,dx$. The work done by the spring force on the mass is

$$W^s = \int_{x=x_i}^{x=x_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\frac{1}{2}\int_{x=x_i}^{x=x_f} -\frac{1}{2}(-kx)\,dx = -\frac{1}{2}k({x_f}^2 - {x_i}^2). \qquad (14.4.8)$$

We then define the change in potential energy in the spring-object system in moving the object from an initial position x_i from equilibrium to a final position x_f from equilibrium by

$$\Delta U^{\mathrm{s}} \equiv U^{\mathrm{s}}(x_f) - U^{\mathrm{s}}(x_i) = -W^{\mathrm{s}} = \frac{1}{2}k(x_f^2 - x_i^2). \qquad (14.4.9)$$

Therefore an arbitrary stretch or compression of a spring-object system from equilibrium $x_i = 0$ to a final position $x_f = x$ changes the potential energy by

$$\Delta U^{\mathrm{s}} = U^{\mathrm{s}}(x_f) - U^{\mathrm{s}}(0) = \frac{1}{2}k\,x^2. \qquad (14.4.10)$$

For the spring-object system, there is an obvious choice of position where the potential energy is zero, the equilibrium position of the spring- object,

$$U^{s}(0) \equiv 0 . \tag{14.4.11}$$

Then with this choice of zero reference potential, the potential energy as a function of the displacement x from the equilibrium position is given by

$$U^{s}(x) = \frac{1}{2} k x^{2}, \text{ with } U^{s}(0) \equiv 0 . \tag{14.4.12}$$

14.4.3 Inverse Square Gravitation Force

Consider a system consisting of two objects of masses m_1 and m_2 that are separated by a center-to-center distance $r_{2,1}$. A coordinate system is shown in the Figure 14.8. The internal gravitational force on object 1 due to the interaction between the two objects is given by

$$\vec{\mathbf{F}}^{G}_{2,1} = -\frac{G m_1 m_2}{r^2_{2,1}} \hat{\mathbf{r}}_{2,1} . \tag{14.4.13}$$

The displacement vector is given by $d\vec{\mathbf{r}}_{2,1} = dr_{2,1}\, \hat{\mathbf{r}}_{2,1}$. So the scalar product is

$$\vec{\mathbf{F}}^{G}_{2,1} \cdot d\vec{\mathbf{r}}_{2,1} = -\frac{G m_1 m_2}{r^2_{2,1}} \hat{\mathbf{r}}_{2,1} \cdot dr_{2,1}\, \hat{\mathbf{r}}_{2,1} = -\frac{G m_1 m_2}{r^2_{2,1}} dr_{2,1} \quad . \tag{14.4.14}$$

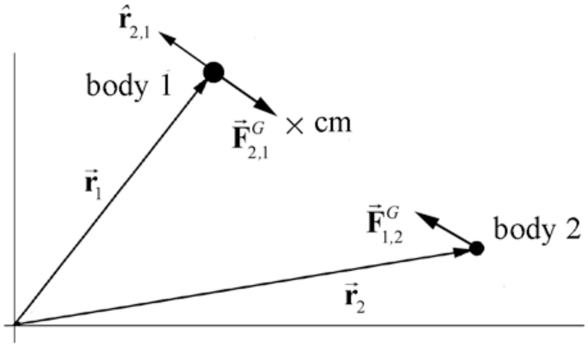

Figure 14.8 Gravitational interaction

Using our definition of potential energy (Eq. (14.3.4)), we have that the change in the gravitational potential energy of the system in moving the two objects from an initial position in which the center of mass of the two objects are a distance r_i apart to a final position in which the center of mass of the two objects are a distance r_f apart is given by

$$\Delta U^G = -\int_A^B \vec{\mathbf{F}}_{2,1}^G \cdot d\vec{\mathbf{r}}_{2,1} = -\int_{r_i}^{f} -\frac{G\,m_1\,m_2}{r_{2,1}^2}\,dr_{2,1} = -\frac{G\,m_1\,m_2}{r_{2,1}}\Bigg|_{r_i}^{r_f} = -\frac{G\,m_1\,m_2}{r_f} + \frac{G\,m_1\,m_2}{r_i}\,. \quad (14.4.15)$$

We now choose our reference point for the zero of the potential energy to be at infinity, $r_i = \infty$, with the choice that $U^G(\infty) \equiv 0$. By making this choice, the term $1/r$ in the expression for the change in potential energy vanishes when $r_i = \infty$. The gravitational potential energy as a function of the relative distance r between the two objects is given by

$$U^G(r) = -\frac{G\,m_1\,m_2}{r}, \text{ with } U^G(\infty) \equiv 0\,. \quad (14.4.16)$$

14.5 Mechanical Energy and Conservation of Mechanical Energy

> *The total change in the **mechanical energy** of the system is defined to be the sum of the changes of the kinetic and the potential energies,*

$$\Delta E_m = \Delta K_{\text{sys}} + \Delta U_{\text{sys}}\,. \quad (14.4.17)$$

For a closed system with only conservative internal forces, the total change in the mechanical energy is zero,

$$\Delta E_m = \Delta K_{\text{sys}} + \Delta U_{\text{sys}} = 0\,. \quad (14.4.18)$$

Equation (14.4.18) is the symbolic statement of what is called ***conservation of mechanical energy.*** Recall that the work done by a conservative force in going around a closed path is zero (Equation (14.2.16)), therefore both the changes in kinetic energy and potential energy are zero when a closed system with only conservative internal forces returns to its initial state. Throughout the process, the kinetic energy may change into internal potential energy but if the system returns to its initial state, the kinetic energy is completely recoverable. We shall refer to a closed system in which processes take place in which only conservative forces act as ***completely reversible processes***.

14.5.1 Change in Gravitational potential Energy Near Surface of the Earth

Let's consider the example of an object of mass m_o falling near the surface of the earth (mass m_e). Choose our system to consist of the earth and the object. The gravitational force is now an internal conservative force acting inside the system. The initial and final states are specified by the distance separating the object and the center of mass of the earth, and the velocities of the earth and the object. The change in kinetic energy between the initial and final states for the system is

$$\Delta K_{\text{sys}} = \Delta K_e + \Delta K_o\,, \quad (14.4.19)$$

$$\Delta K_{\text{sys}} = \left(\frac{1}{2}m_{\text{e}}(v_{e,f})^2 - \frac{1}{2}m_{\text{e}}(v_{\text{e,i}})^2\right) + \left(\frac{1}{2}m_o(v_{o,f})^2 - \frac{1}{2}m_o(v_{o,i})^2\right). \qquad (14.4.20)$$

The change of kinetic energy of the earth due to the gravitational interaction between the earth and the object is negligible. The change in kinetic energy of the system is approximately equal to the change in kinetic energy of the object,

$$\Delta K_{\text{sys}} \cong \Delta K_o = \frac{1}{2}m_o(v_{o,f})^2 - \frac{1}{2}m_o(v_{o,i})^2. \qquad (14.4.21)$$

We now define the mechanical energy function for the system

$$E_m = K + U^g = \frac{1}{2}m_o(v_b)^2 + m_o gy, \text{ with } U^g(0) = 0, \qquad (14.4.22)$$

where K is the kinetic energy and U^g is the potential energy. The change in mechanical energy is then

$$\Delta E_m \equiv E_{m,f} - E_{m,i} = (K_f + U_f^g) - (K_i + U_i^g). \qquad (14.4.23)$$

When the work done by the external forces is zero and there are no internal non-conservative forces, the total mechanical energy of the system is constant,

$$E_{m,f} = E_{m,i}, \qquad (14.4.24)$$

or equivalently

$$(K_f + U_f) = (K_i + U_i). \qquad (14.4.25)$$

14.6 Spring Force Energy Diagram

The spring force on an object is a restoring force $\vec{\mathbf{F}}^s = F_x^s\,\hat{\mathbf{i}} = -k\,x\,\hat{\mathbf{i}}$ where we choose a coordinate system with the equilibrium position at $x_i = 0$ and x is the amount the spring has been stretched $(x > 0)$ or compressed $(x < 0)$ from its equilibrium position. We calculate the potential energy difference Eq. (14.4.9) and found that

$$U^s(x) - U^s(x_i) = -\int_{x_i}^{x} F_x^s\,dx = \frac{1}{2}k(x^2 - x_i^2). \qquad (14.5.1)$$

The first fundamental theorem of calculus states that

$$U(x) - U(x_i) = \int_{x'=x_i}^{x'=x} \frac{dU}{dx'}\,dx'. \qquad (14.5.2)$$

Comparing Equation (14.5.1) with Equation (14.5.2) shows that the force is the negative derivative (with respect to position) of the potential energy,

$$F_x^s = -\frac{dU^s(x)}{dx}. \tag{14.5.3}$$

Choose the zero reference point for the potential energy to be at the equilibrium position, $U^s(0) \equiv 0$. Then the potential energy function becomes

$$U^s(x) = \frac{1}{2}k x^2. \tag{14.5.4}$$

From this, we obtain the spring force law as

$$F_x^s = -\frac{dU^s(x)}{dx} = -\frac{d}{dx}\left(\frac{1}{2}k x^2\right) = -k x. \tag{14.5.5}$$

In Figure 14.9 we plot the potential energy function $U^s(x)$ for the spring force as function of x with $U^s(0) \equiv 0$ (the units are arbitrary).

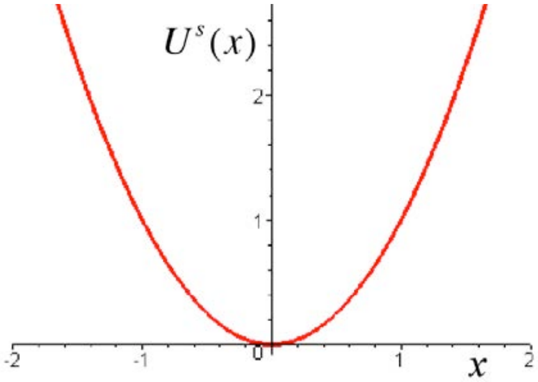

Figure 14.9 Graph of potential energy function as function of x for the spring.

The minimum of the potential energy function occurs at the point where the first derivative vanishes

$$\frac{dU^s(x)}{dx} = 0. \tag{14.5.6}$$

From Equation (14.5.4), the minimum occurs at $x = 0$,

$$0 = \frac{dU^s(x)}{dx} = k x. \tag{14.5.7}$$

Because the force is the negative derivative of the potential energy, and this derivative vanishes at the minimum, we have that the spring force is zero at the minimum $x = 0$ agreeing with our force law, $F_x^s\big|_{x=0} = -k\,x\big|_{x=0} = 0$.

The potential energy function has positive curvature in the neighborhood of a minimum equilibrium point. If the object is extended a small distance $x > 0$ away from equilibrium, the slope of the potential energy function is positive, $dU(x)/dx > 0$, hence the component of the force is negative because $F_x = -dU(x)/dx < 0$. Thus the object experiences a restoring force towards the minimum point of the potential. If the object is compresses with $x < 0$ then $dU(x)/dx < 0$, hence the component of the force is positive, $F_x = -dU(x)/dx > 0$, and the object again experiences a restoring force back towards the minimum of the potential energy as in Figure 14.10.

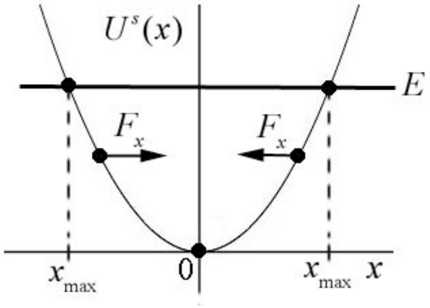

Figure 14.10 Stability diagram for the spring force.

The mechanical energy at any time is the sum of the kinetic energy $K(x)$ and the potential energy $U^s(x)$

$$E_m = K(x) + U^s(x). \tag{14.5.8}$$

Suppose our spring-object system has no loss of mechanical energy due to dissipative forces such as friction or air resistance. Both the kinetic energy and the potential energy are functions of the position of the object with respect to equilibrium. The energy is a constant of the motion and with our choice of $U^s(0) \equiv 0$, the energy can be either a positive value or zero. When the energy is zero, the object is at rest at the equilibrium position.

In Figure 14.10, we draw a straight horizontal line corresponding to a non-zero positive value for the energy E_m on the graph of potential energy as a function of x. The energy intersects the potential energy function at two points $\{-x_{\max}, x_{\max}\}$ with $x_{\max} > 0$. These points correspond to the maximum compression and maximum extension of the spring, which are called the ***turning points***. The kinetic energy is the difference between the energy and the potential energy,

$$K(x) = E_m - U^s(x). \tag{14.5.9}$$

At the turning points, where $E_m = U^s(x)$, the kinetic energy is zero. Regions where the kinetic energy is negative, $x < -x_{\max}$ or $x > x_{\max}$ are called the ***classically forbidden regions***, which the object can never reach if subject to the laws of classical mechanics. In quantum mechanics, with similar energy diagrams for quantum systems, there is a very small probability that the quantum object can be found in a classically forbidden region.

Example 14.1 Energy Diagram

The potential energy function for a particle of mass m, moving in the x-direction is given by

$$U(x) = -U_1\left(\left(\frac{x}{x_1}\right)^3 - \left(\frac{x}{x_1}\right)^2\right), \tag{14.5.10}$$

where U_1 and x_1 are positive constants and $U(0) = 0$. (a) Sketch $U(x)/U_1$ as a function of x/x_1. (b) Find the points where the force on the particle is zero. Classify them as stable or unstable. Calculate the value of $U(x)/U_1$ at these equilibrium points. (c) For energies E that lies in $0 < E < (4/27)U_1$ find an equation whose solution yields the turning points along the x-axis about which the particle will undergo periodic motion. (d) Suppose $E = (4/27)U_1$ and that the particle starts at $x = 0$ with speed v_0. Find v_0.

Solution: a) Figure 14.11 shows a graph of $U(x)$ vs. x, with the choice of values $x_1 = 1.5\text{ m}$, $U_1 = 27/4\text{ J}$, and $E = 0.2\text{ J}$.

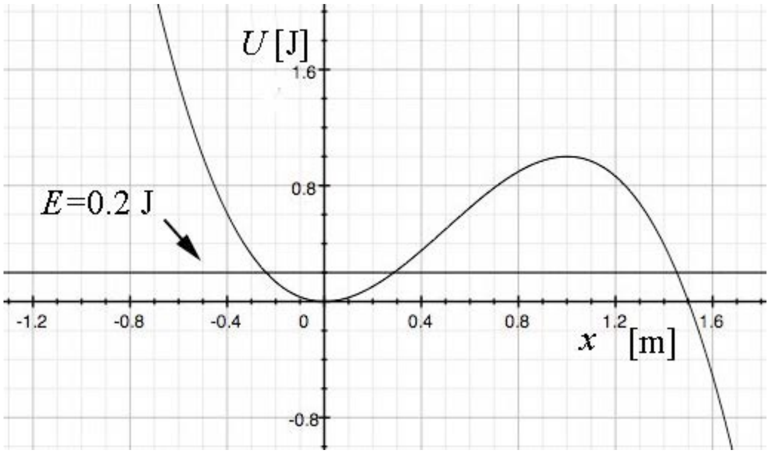

Figure 14.11 Energy diagram for Example 14.1

b) The force on the particle is zero at the minimum of the potential which occurs at

$$F_x(x) = -\frac{dU}{dx}(x) = U_1\left(\left(\frac{3}{x_1^{\,3}}\right)x^2 - \left(\frac{2}{x_1^{\,2}}\right)x\right) = 0 \tag{14.5.11}$$

which becomes

$$x^2 = (2x_1 / 3)x\,. \tag{14.5.12}$$

We can solve Eq. (14.5.12) for the extrema. This has two solutions

$$x = (2x_1 / 3) \quad \text{and} \quad x = 0\,. \tag{14.5.13}$$

The second derivative is given by

$$\frac{d^2U}{dx^2}(x) = -U_1\left(\left(\frac{6}{x_1^{\,3}}\right)x - \left(\frac{2}{x_1^{\,2}}\right)\right). \tag{14.5.14}$$

Evaluating the second derivative at $x = (2x_1 / 3)$ yields a negative quantity

$$\frac{d^2U}{dx^2}(x = (2x_1 / 3)) = -U_1\left(\left(\frac{6}{x_1^{\,3}}\right)\frac{2x_1}{3} - \left(\frac{2}{x_1^{\,2}}\right)\right) = -\frac{2U_1}{x_1^{\,2}} < 0\,, \tag{14.5.15}$$

indicating the solution $x = (2x_1 / 3)$ represents a local maximum and hence is an unstable point. At $x = (2x_1 / 3)$, the potential energy is given by the value $U((2x_1 / 3)) = (4 / 27)U_1$. Evaluating the second derivative at $x = 0$ yields a positive quantity

$$\frac{d^2U}{dx^2}(x = 0) = -U_1\left(\left(\frac{6}{x_1^{\,3}}\right)0 - \left(\frac{2}{x_1^{\,2}}\right)\right) = \frac{2U_1}{x_1^{\,2}} > 0\,, \tag{14.5.16}$$

indicating the solution $x = 0$ represents a local minimum and is a stable point. At the local minimum $x = 0$, the potential energy $U(0) = 0$.

c) Consider a fixed value of the energy of the particle within the range

$$U(0) = 0 < E < U(2x_1 / 3) = \frac{4U_1}{27}\ . \tag{14.5.17}$$

If the particle at any time is found in the region $x_a < x < x_b < 2x_1 / 3$, where x_a and x_b are the turning points and are solutions to the equation

$$E = U(x) = -U_1\left(\left(\frac{x}{x_1}\right)^3 - \left(\frac{x}{x_1}\right)^2\right). \qquad (14.5.18)$$

then the particle will undergo periodic motion between the values $x_a < x < x_b$. Within this region $x_a < x < x_b$, the kinetic energy is always positive because $K(x) = E - U(x)$. There is another solution x_c to Eq. (14.5.18) somewhere in the region $x_c > 2x_1 / 3$. If the particle at any time is in the region $x > x_c$ then it at any later time it is restricted to the region $x_c < x < +\infty$.

For $E > U(2x_1 / 3) = (4 / 27)U_1$, Eq. (14.5.18) has only one solution x_d. For all values of $x > x_d$, the kinetic energy is positive, which means that the particle can "escape" to infinity but can never enter the region $x < x_d$.

For $E < U(0) = 0$, the kinetic energy is negative for the range $-\infty < x < x_e$ where x_e satisfies Eq. (14.5.18) and therefore this region of space is forbidden.

(d) If the particle has speed v_0 at $x = 0$ where the potential energy is zero, $U(0) = 0$, the energy of the particle is constant and equal to kinetic energy

$$E = K(0) = \frac{1}{2} m v_0^2. \qquad (14.5.19)$$

Therefore

$$(4 / 27)U_1 = \frac{1}{2} m v_0^2, \qquad (14.5.20)$$

which we can solve for the speed

$$v_0 = \sqrt{8U_1 / 27m}. \qquad (14.5.21)$$

14.7 Change of Mechanical Energy for Closed System with Internal Non-conservative Forces

Consider a closed system (energy of the system is constant) that undergoes a transformation from an initial state to a final state by a prescribed set of changes.

> *Whenever the work done by a force in moving an object from an initial point to a final point depends on the path, the force is called a* ***non-conservative force****.*

Suppose the internal forces are both conservative and non-conservative. The work W done by the forces is a sum of the conservative work W_{c}, which is path-independent, and the non-conservative work W_{nc}, which is path-dependent,

$$W = W_{\text{c}} + W_{\text{nc}} . \tag{14.6.1}$$

The work done by the conservative forces is equal to the negative of the change in the potential energy

$$\Delta U = -W_{\text{c}} . \tag{14.6.2}$$

Substituting Equation (14.6.2) into Equation (14.6.1) yields

$$W = -\Delta U + W_{\text{nc}} . \tag{14.6.3}$$

The work done is equal to the change in the kinetic energy,

$$W = \Delta K . \tag{14.6.4}$$

Substituting Equation (14.6.4) into Equation (14.6.3) yields

$$\Delta K = -\Delta U + W_{\text{nc}} . \tag{14.6.5}$$

which we can rearrange as

$$W_{\text{nc}} = \Delta K + \Delta U . \tag{14.6.6}$$

We can now substitute Equation (14.6.4) into our expression for the change in the mechanical energy, Equation (14.4.17), with the result

$$W_{\text{nc}} = \Delta E_m . \tag{14.6.7}$$

The mechanical energy is no longer constant. The total change in energy of the system is zero,

$$\Delta E_{\text{system}} = \Delta E_m - W_{\text{nc}} = 0 . \tag{14.6.8}$$

Energy is conserved but some mechanical energy has been transferred into non-recoverable energy W_{nc}. We shall refer to processes in which there is non-zero non-recoverable energy as ***irreversible processes***.

14.7.1 Change of Mechanical Energy for a Non-closed System

When the system is no longer closed but in contact with its surroundings, the change in energy of the system is equal to the negative of the change in energy of the surroundings (Eq. (14.1.1)),

$$\Delta E_{\text{system}} = -\Delta E_{\text{surroundings}} \tag{14.6.9}$$

If the system is not isolated, the change in energy of the system can be the result of external work done by the surroundings on the system (which can be positive or negative)

$$W_{\text{ext}} = \int_A^B \vec{\mathbf{F}}_{\text{ext}} \cdot d\vec{\mathbf{r}}\,. \tag{14.6.10}$$

This work will result in the system undergoing ***coherent motion***. Note that $W_{\text{ext}} > 0$ if work is done on the system ($\Delta E_{\text{surroundings}} < 0$) and $W_{\text{ext}} < 0$ if the system does work on the surroundings ($\Delta E_{\text{surroundings}} > 0$). If the system is in thermal contact with the surroundings, then energy can flow into or out of the system. This energy flow due to thermal contact is often denoted by Q with the convention that $Q > 0$ if the energy flows into the system ($\Delta E_{\text{surroundings}} < 0$) and $Q < 0$ if the energy flows out of the system ($\Delta E_{\text{surroundings}} > 0$). Then Eq. (14.6.9) can be rewritten as

$$W^{\text{ext}} + Q = \Delta E_{\text{sys}} \tag{14.6.11}$$

Equation (14.6.11) is also called ***the first law of thermodynamics.***

This will result in either an increase or decrease in random thermal motion of the molecules inside the system, There may also be other forms of energy that enter the system, for example *radiative energy*.

Several questions naturally arise from this set of definitions and physical concepts. Is it possible to identify all the conservative forces and calculate the associated changes in potential energies? How do we account for non-conservative forces such as friction that act at the boundary of the system?

14.8 Dissipative Forces: Friction

Suppose we consider an object moving on a rough surface. As the object slides it slows down and stops. While the sliding occurs both the object and the surface increase in temperature. The increase in temperature is due to the molecules inside the materials increasing their kinetic energy. This random kinetic energy is called *thermal energy*. Kinetic energy associated with the coherent motion of the molecules of the object has been dissipated into kinetic energy associated with random motion of the molecules composing the object and surface.

If we define the system to be just the object, then the friction force acts as an external force on the system and results in the dissipation of energy into both the block and the surface. Without knowing further properties of the material we cannot determine the exact changes in the energy of the system.

Friction introduces a problem in that the point of contact is not well defined because the surface of contact is constantly deforming as the object moves along the surface. If we considered the object and the surface as the system, then the friction force is an internal force, and the decrease in the kinetic energy of the moving object ends up as an increase in the internal random kinetic energy of the constituent parts of the system. When there is dissipation at the boundary of the system, we need an additional model (thermal equation of state) for how the dissipated energy distributes itself among the constituent parts of the system.

14.8.1 Source Energy

Consider a person walking. The frictional force between the person and the ground does no work because the point of contact between the person's foot and the ground undergoes no displacement as the person applies a force against the ground, (there may be some slippage but that would be opposite the direction of motion of the person). However the kinetic energy of the object increases. Have we disproved the work-energy theorem? The answer is no! The chemical energy stored in the body tissue is converted to kinetic energy and thermal energy. Because the person-air-ground can be treated as a closed system, we have that

$$0 = \Delta E_{\text{sys}} = \Delta E_{\text{chemical}} + \Delta E_{\text{thermal}} + \Delta E_{\text{mechanical}}, \quad (\text{closed system}). \qquad (14.7.1)$$

If we assume that there is no change in the potential energy of the system, then $\Delta E_{\text{mechanical}} = \Delta K$. Therefore some of the internal chemical energy has been transformed into thermal energy and the rest has changed into the kinetic energy of the system,

$$-\Delta E_{\text{chemical}} = \Delta E_{\text{thermal}} + \Delta K \, . \qquad (14.7.2)$$

14.9 Worked Examples

Example 14.2 Escape Velocity of Toro

The asteroid Toro, discovered in 1964, has a radius of about $R = 5.0\,\text{km}$ and a mass of about $m_t = 2.0 \times 10^{15}\,\text{kg}$. Let's assume that Toro is a perfectly uniform sphere. What is the escape velocity for an object of mass m on the surface of Toro? Could a person reach this speed (on earth) by running?

Solution: The only potential energy in this problem is the gravitational potential energy. We choose the zero point for the potential energy to be when the object and Toro are an infinite distance apart, $U^G(\infty) \equiv 0$. With this choice, the potential energy when the object and Toro are a finite distance r apart is given by

$$U^G(r) = -\frac{Gm_t\, m}{r} \qquad (14.8.1)$$

with $U^G(\infty) \equiv 0$. The expression ***escape velocity*** refers to the minimum speed necessary for an object to escape the gravitational interaction of the asteroid and move off to an infinite distance away. If the object has a speed less than the escape velocity, it will be unable to escape the gravitational force and must return to Toro. If the object has a speed greater than the escape velocity, it will have a non-zero kinetic energy at infinity. The condition for the escape velocity is that the object will have exactly zero kinetic energy at infinity.

We choose our initial state, at time t_i, when the object is at the surface of the asteroid with speed equal to the escape velocity. We choose our final state, at time t_f, to occur when the separation distance between the asteroid and the object is infinite.

The initial kinetic energy is $K_i = (1/2)m{v_{\text{esc}}}^2$. The initial potential energy is $U_i = -Gm_t m / R$, and so the initial mechanical energy is

$$E_i = K_i + U_i = \frac{1}{2}m v_{\text{esc}}^2 - \frac{Gm_t m}{R}. \tag{14.8.2}$$

The final kinetic energy is $K_f = 0$, because this is the condition that defines the escape velocity. The final potential energy is zero, $U_f = 0$ because we chose the zero point for potential energy at infinity. The final mechanical energy is then

$$E_f = K_f + U_f = 0. \tag{14.8.3}$$

There is no non-conservative work, so the change in mechanical energy is zero

$$0 = W_{\text{nc}} = \Delta E_m = E_f - E_i. \tag{14.8.4}$$

Therefore

$$0 = -\left(\frac{1}{2}m v_{\text{esc}}^2 - \frac{Gm_t\, m}{R}\right). \tag{14.8.5}$$

This can be solved for the escape velocity,

$$\begin{aligned} v_{\text{esc}} &= \sqrt{\frac{2Gm_t}{R}} \\ &= \sqrt{\frac{2(6.67\times 10^{-11}\text{N}\cdot\text{m}^2\cdot\text{kg}^{-2})(2.0\times 10^{15}\,\text{kg})}{(5.0\times 10^3\,\text{m})}} = 7.3\ \text{m}\cdot\text{s}^{-1}. \end{aligned} \tag{14.8.6}$$

Considering that Olympic sprinters typically reach velocities of $12\ \text{m}\cdot\text{s}^{-1}$, this is an easy speed to attain by running on earth. It may be harder on Toro to generate the acceleration necessary to reach this speed by pushing off the ground, since any slight upward force will raise the runner's center of mass and it will take substantially more time than on earth to come back down for another push off the ground.

Example 14.3 Spring-Block-Loop-the-Loop

A small block of mass m is pushed against a spring with spring constant k and held in place with a catch. The spring is compressed an unknown distance x (Figure 14.12). When the catch is removed, the block leaves the spring and slides along a frictionless circular loop of radius r. When the block reaches the top of the loop, the force of the loop on the block (the normal force) is equal to twice the gravitational force on the mass. (a) Using conservation of energy, find the kinetic energy of the block at the top of the loop. (b) Using Newton's Second Law, derive the equation of motion for the block when it is at the top of the loop. Specifically, find the speed v_{top} in terms of the gravitation constant g and the loop radius r. (c) What distance was the spring compressed?

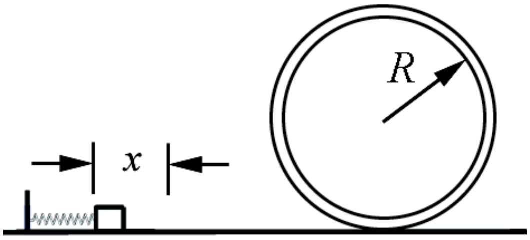

Figure 14.12 Initial state for spring-block-loop-the-loop system

Solution: a) Choose for the initial state the instant before the catch is released. The initial kinetic energy is $K_i = 0$. The initial potential energy is non-zero, $U_i = (1/2)k x^2$**.** The initial mechanical energy is then

$$E_i = K_i + U_i = \frac{1}{2}k x^2 . \tag{14.8.7}$$

Choose for the final state the instant the block is at the top of the loop. The final kinetic energy is $K_f = (1/2)m v_{\text{top}}^2$; the block is in motion with speed v_{top}. The final potential energy is non-zero, $U_f = (mg)(2R)$. The final mechanical energy is then

$$E_f = K_f + U_f = 2mgR + \frac{1}{2}m v_{\text{top}}^2 . \tag{14.8.8}$$

Because we are assuming the track is frictionless and neglecting air resistance, there is no non- conservative work. The change in mechanical energy is therefore zero,

$$0 = W_{\text{nc}} = \Delta E_m = E_f - E_i \,. \tag{14.8.9}$$

Mechanical energy is conserved, $E_f = E_i$, therefore

$$2mgR + \frac{1}{2} m v_{\text{top}}^2 = \frac{1}{2} k x^2 \,. \tag{14.8.10}$$

From Equation (14.8.10), the kinetic energy at the top of the loop is

$$\frac{1}{2} m v_{\text{top}}^2 = \frac{1}{2} k x^2 - 2mgR \,. \tag{14.8.11}$$

b) At the top of the loop, the forces on the block are the gravitational force of magnitude mg and the normal force of magnitude N, both directed down. Newton's Second Law in the radial direction, which is the downward direction, is

$$-mg - N = -\frac{m v_{\text{top}}^2}{R} \,. \tag{14.8.12}$$

In this problem, we are given that when the block reaches the top of the loop, the force of the loop on the block (the normal force, *downward* in this case) is equal to twice the weight of the block, $N = 2mg$. The Second Law, Eq. (14.8.12), then becomes

$$3mg = \frac{m v_{\text{top}}^2}{R} \,. \tag{14.8.13}$$

We can rewrite Equation (14.8.13) in terms of the kinetic energy as

$$\frac{3}{2} mg R = \frac{1}{2} m v_{\text{top}}^2 \,. \tag{14.8.14}$$

The speed at the top is therefore

$$v_{\text{top}} = \sqrt{3mg R} \,. \tag{14.8.15}$$

c) Combing Equations (14.8.11) and (14.8.14) yields

$$\frac{7}{2} mg R = \frac{1}{2} k x^2 . \tag{14.8.16}$$

Thus the initial displacement of the spring from equilibrium is

$$x = \sqrt{\frac{7mg\,R}{k}}\,. \tag{14.8.17}$$

Example 14.4 Mass-Spring on a Rough Surface

A block of mass m slides along a horizontal table with speed v_0. At $x = 0$ it hits a spring with spring constant k and begins to experience a friction force. The coefficient of friction is variable and is given by $\mu = bx$, where b is a positive constant. Find the loss in mechanical energy when the block first momentarily comes to rest.

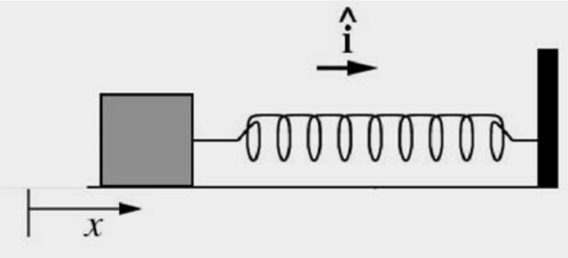

Figure 14.13 Spring-block system

Solution: From the model given for the frictional force, we could find the non-conservative work done, which is the same as the loss of mechanical energy, if we knew the position x_f where the block first comes to rest. The most direct (and easiest) way to find x_f is to use the work-energy theorem. The initial mechanical energy is $E_i = mv_i^2 / 2$ and the final mechanical energy is $E_f = k\,x_f^2 / 2$ (note that there is no potential energy term in E_i and no kinetic energy term in E_f). The difference between these two mechanical energies is the non-conservative work done by the frictional force,

$$\begin{aligned} W_{\text{nc}} &= \int_{x=0}^{x=x_f} F_{\text{nc}}\,dx = \int_{x=0}^{x=x_f} -F_{\text{friction}}\,dx = \int_{x=0}^{x=x_f} -\mu\,N\,dx \\ &= -\int_0^{x_f} b\,x\,mg\,dx = -\frac{1}{2} bmg\,x_f^2. \end{aligned} \tag{14.8.18}$$

We then have that

$$\begin{aligned} W_{\text{nc}} &= \Delta E_m \\ W_{\text{nc}} &= E_f - E_i \\ -\frac{1}{2} bmg\,x_f^2 &= \frac{1}{2} k\,x_f^2 - \frac{1}{2} mv_i^2. \end{aligned} \tag{14.8.19}$$

Solving the last of these equations for x_f^2 yields

$$x_f^2 = \frac{mv_0^2}{k + bmg} \, . \tag{14.8.20}$$

Substitute Eq. (14.8.20) into Eq. (14.8.18) gives the result that

$$W_{\text{nc}} = -\frac{bmg}{2}\frac{mv_0^2}{k + bmg} = -\frac{mv_0^2}{2}\left(1 + \frac{k}{bmg}\right)^{-1} . \tag{14.8.21}$$

It is worth checking that the above result is dimensionally correct. From the model, the parameter b must have dimensions of inverse length (the coefficient of friction μ must be dimensionless), and so the product bmg has dimensions of force per length, as does the spring constant k; the result is dimensionally consistent.

Example 14.5 Cart-Spring on an Inclined Plane

An object of mass m slides down a plane that is inclined at an angle θ from the horizontal (Figure 14.14). The object starts out at rest. The center of mass of the cart is a distance d from an unstretched spring that lies at the bottom of the plane. Assume the spring is massless, and has a spring constant k. Assume the inclined plane to be frictionless. (a) How far will the spring compress when the mass first comes to rest? (b) Now assume that the inclined plane has a coefficient of kinetic friction μ_{k}. How far will the spring compress when the mass first comes to rest? The friction is primarily between the wheels and the bearings, not between the cart and the plane, but the friction force may be modeled by a coefficient of friction μ_{k}. (c) In case (b), how much energy has been lost to friction?

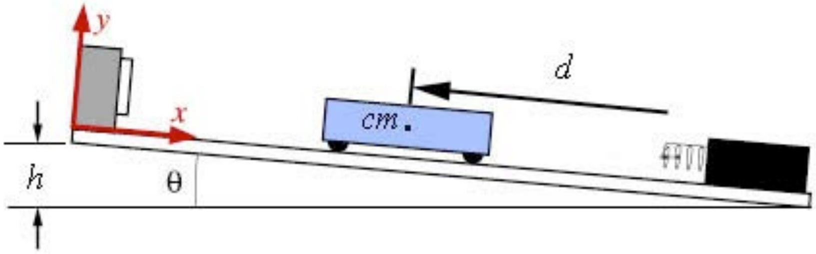

Figure 14.14 Cart on inclined plane

Solution: Let x denote the displacement of the spring from the equilibrium position. Choose the zero point for the gravitational potential energy $U^g(0) = 0$ not at the very bottom of the inclined plane, but at the location of the end of the unstretched spring. Choose the zero point for the spring potential energy where the spring is at its equilibrium position, $U^s(0) = 0$.

a) Choose for the initial state the instant the object is released (Figure 14.15). The initial kinetic energy is $K_i = 0$. The initial potential energy is non-zero, $U_i = mg\,d\sin\theta$. The initial mechanical energy is then

$$E_i = K_i + U_i = mg\,d\sin\theta \tag{14.8.22}$$

Choose for the final state the instant when the object first comes to rest and the spring is compressed a distance x at the bottom of the inclined plane (Figure 14.16). The final kinetic energy is $K_f = 0$ since the mass is not in motion. The final potential energy is non-zero, $U_f = k\,x^2/2 - x\,mg\sin\theta$**.** Notice that the gravitational potential energy is negative because the object has dropped below the height of the zero point of gravitational potential energy.

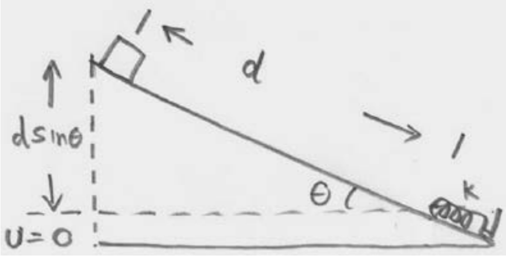

Figure 14.15 Initial state

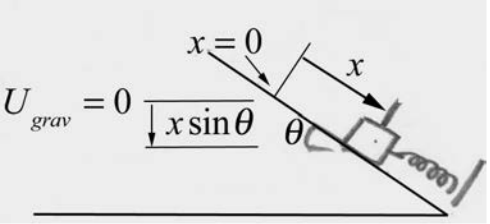

Figure 14.16 Final state

The final mechanical energy is then

$$E_f = K_f + U_f = \frac{1}{2}k\,x^2 - x\,mg\sin\theta\,. \tag{14.8.23}$$

Because we are assuming the track is frictionless and neglecting air resistance, there is no non- conservative work. The change in mechanical energy is therefore zero,

$$0 = W_{\text{nc}} = \Delta E_m = E_f - E_i\,. \tag{14.8.24}$$

Therefore

$$d\,mg\sin\theta = \frac{1}{2}k\,x^2 - x\,mg\sin\theta\,. \tag{14.8.25}$$

This is a quadratic equation in x,

$$x^2 - \frac{2mg\sin\theta}{k}x - \frac{2d\,mg\sin\theta}{k} = 0\,. \tag{14.8.26}$$

In the quadratic formula, we want the positive choice of square root for the solution to ensure a positive displacement of the spring from equilibrium,

$$x = \frac{mg\sin\theta}{k} + \left(\frac{m^2 g^2 \sin^2\theta}{k^2} + \frac{2d\, mg\sin\theta}{k} \right)^{1/2}$$
$$= \frac{mg}{k}(\sin\theta + \sqrt{1 + 2(k\,d\,/\,mg)\sin\theta}). \qquad (14.8.27)$$

(What would the solution with the negative root represent?)

b) The effect of kinetic friction is that there is now a non-zero non-conservative work done on the object, which has moved a distance, $d + x$, given by

$$W_{\text{nc}} = -f_{\text{k}}(d + x) = -\mu_{\text{k}} N(d + x) = -\mu_{\text{k}} mg\cos\theta(d + x)\,. \qquad (14.8.28)$$

Note the normal force is found by using Newton's Second Law in the perpendicular direction to the inclined plane,

$$N - mg\cos\theta = 0\,. \qquad (14.8.29)$$

The change in mechanical energy is therefore

$$W_{\text{nc}} = \Delta E_m = E_f - E_i\,, \qquad (14.8.30)$$

which becomes

$$-\mu_{\text{k}} mg\cos\theta(d + x) = \left(\frac{1}{2} k\,x^2 - x\,mg\sin\theta \right) - d\,mg\sin\theta\,. \qquad (14.8.31)$$

Equation (14.8.31) simplifies to

$$0 = \left(\frac{1}{2} k\,x^2 - x\,mg(\sin\theta - \mu_{\text{k}}\cos\theta) \right) - d\,mg(\sin\theta - \mu_{\text{k}}\cos\theta)\,. \qquad (14.8.32)$$

This is the same as Equation (14.8.25) above, but with $\sin\theta \to \sin\theta - \mu_k \cos\theta$. The maximum displacement of the spring is when there is friction is then

$$x = \frac{mg}{k}((\sin\theta - \mu_{\text{k}}\cos\theta) + \sqrt{1 + 2(k\,d\,/\,mg)(\sin\theta - \mu_{\text{k}}\cos\theta)})\,. \qquad (14.8.33)$$

.

c) The energy lost to friction is given by $W_{\text{nc}} = -\mu_{\text{k}} mg\cos\theta(d + x)$, where x is given in part b).

Example 14.6 Object Sliding on a Sphere

A small point like object of mass m rests on top of a sphere of radius R. The object is released from the top of the sphere with a negligible speed and it slowly starts to slide (Figure 14.17). Let g denote the gravitation constant. (a) Determine the angle θ_1 with respect to the vertical at which the object will lose contact with the surface of the sphere. (b) What is the speed v_1 of the object at the instant it loses contact with the surface of the sphere.

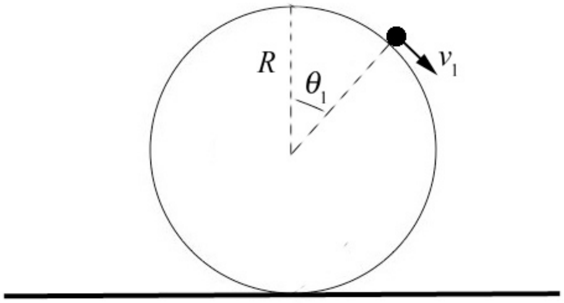

Figure 14.17 Object sliding on surface of sphere

Solution: We begin by identifying the forces acting on the object. There are two forces acting on the object, the gravitation and radial normal force that the sphere exerts on the particle that we denote by N. We draw a free-body force diagram for the object while it is sliding on the sphere. We choose polar coordinates as shown in Figure 14.18.

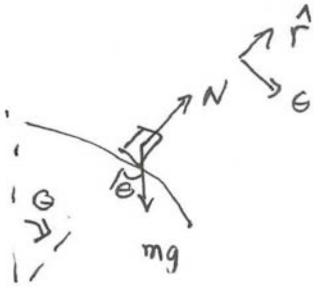

Figure 14.18 Free-body force diagram on object

The key constraint is that when the particle just leaves the surface the normal force is zero,

$$N(\theta_1) = 0, \tag{14.8.34}$$

where θ_1 denotes the angle with respect to the vertical at which the object will just lose contact with the surface of the sphere. Because the normal force is perpendicular to the displacement of the object, it does no work on the object and hence conservation of energy does not take into account the constraint on the motion imposed by the normal force. In order to analyze the effect of the normal force we must use the radial component of Newton's Second Law,

$$N - mg\cos\theta = -m\frac{v^2}{R}. \tag{14.8.35}$$

Then when the object just loses contact with the surface, Eqs. (14.8.34) and (14.8.35) require that

$$mg\cos\theta_1 = m\frac{v_1^{\ 2}}{R}. \tag{14.8.36}$$

where v_1 denotes the speed of the object at the instant it loses contact with the surface of the sphere. Note that the constrain condition Eq. (14.8.36) can be rewritten as

$$mgR\cos\theta_1 = mv_1^{\ 2}. \tag{14.8.37}$$

We can now apply conservation of energy. Choose the zero reference point $U = 0$ for potential energy to be the midpoint of the sphere.

Identify the initial state as the instant the object is released (Figure 14.19). We can neglect the very small initial kinetic energy needed to move the object away from the top of the sphere and so $K_i = 0$. The initial potential energy is non-zero, $U_i = mgR$**.** The initial mechanical energy is then

$$E_i = K_i + U_i = mgR. \tag{14.8.38}$$

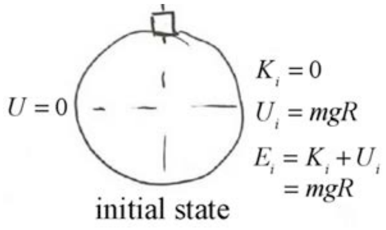

Figure 14.19 Initial state

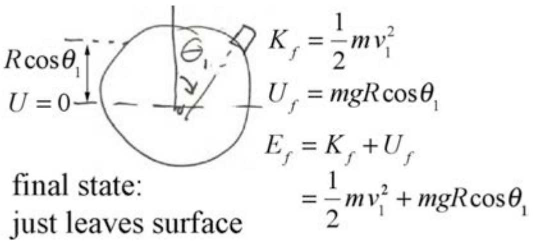

Figure 14.20 Final state

Choose for the final state the instant the object leaves the sphere (Figure 14.20). The final kinetic energy is $K_f = mv_1^2/2$; the object is in motion with speed v_1. The final potential energy is non-zero, $U_f = mgR\cos\theta_1$. The final mechanical energy is then

$$E_f = K_f + U_f = \frac{1}{2}mv_1^2 + mgR\cos\theta_1. \tag{14.8.39}$$

Because we are assuming the contact surface is frictionless and neglecting air resistance, there is no non-conservative work. The change in mechanical energy is therefore zero,

$$0 = W_{\text{nc}} = \Delta E_m = E_f - E_i. \tag{14.8.40}$$

Therefore

$$\frac{1}{2}mv_1^2 + mgR\cos\theta_1 = mgR. \tag{14.8.41}$$

We now solve the constraint condition Eq. (14.8.37) into Eq. (14.8.41) yielding

$$\frac{1}{2}mgR\cos\theta_1 + mgR\cos\theta_1 = mgR\,. \tag{14.8.42}$$

We can now solve for the angle at which the object just leaves the surface

$$\theta_1 = \cos^{-1}(2/3)\,. \tag{14.8.43}$$

We now substitute this result into Eq. (14.8.37) and solve for the speed

$$v_1 = \sqrt{2gR/3}\,. \tag{14.8.44}$$

Chapter 15 Collision Theory

Chapter 15 Collision Theory

> Despite my resistance to hyperbole, the LHC [Large Hadron Collider] belongs to a world that can only be described with superlatives. It is not merely large: the LHC is the biggest machine ever built. It is not merely cold: the 1.9 kelvin (1.9 degrees Celsius above absolute zero) temperature necessary for the LHC's supercomputing magnets to operate is the coldest extended region that we know of in the universe—even colder than outer space. The magnetic field is not merely big: the superconducting dipole magnets generating a magnetic field more than 100,000 times stronger than the Earth's are the strongest magnets in industrial production ever made.
>
> And the extremes don't end there. The vacuum inside the proton-containing tubes, a 10 trillionth of an atmosphere, is the most complete vacuum over the largest region ever produced. The energy of the collisions are the highest ever generated on Earth, allowing us to study the interactions that occurred in the early universe the furthest back in time.[1]
>
> *Lisa Randall*

15.1 Introduction

When discussing conservation of momentum, we considered examples in which two objects collide and stick together, and either there are no external forces acting in some direction (or the collision was nearly instantaneous) so the component of the momentum of the system along that direction is constant. We shall now study collisions between objects in more detail. In particular we shall consider cases in which the objects do not stick together. The momentum along a certain direction may still be constant but the mechanical energy of the system may change. We will begin our analysis by considering two-particle collision. We introduce the concept of the relative velocity between two particles and show that it is independent of the choice of reference frame. We then show that the change in kinetic energy only depends on the change of the square of the relative velocity and therefore is also independent of the choice of reference frame. We will then study one- and two-dimensional collisions with zero change in potential energy. In particular we will characterize the types of collisions by the change in kinetic energy and analyze the possible outcomes of the collisions.

15.2 Reference Frames and Relative Velocities

We shall recall our definition of relative inertial reference frames. Let $\vec{\mathbf{R}}$ be the vector from the origin of frame S to the origin of reference frame S'. Denote the

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[1] Randall, Lisa, *Knocking on Heaven's Door: How Physics and Scientific Thinking Illuminate the Universe and the Modern World*, Ecco, 2011.

position vector of the j^{th} particle with respect to the origin of reference frame S by $\vec{\mathbf{r}}_j$ and similarly, denote the position vector of the j^{th} particle with respect to the origin of reference frame S' by $\vec{\mathbf{r}}'_j$ (Figure 15.1).

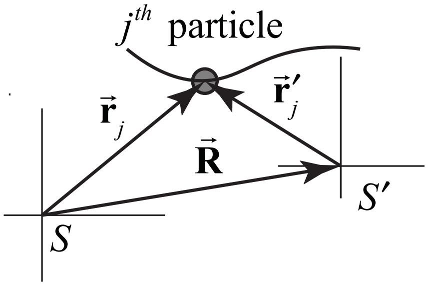


Figure 15.1 Position vector of j^{th} particle in two reference frames.

The position vectors are related by

$$\vec{\mathbf{r}}_j = \vec{\mathbf{r}}'_j + \vec{\mathbf{R}} \, . \tag{15.2.1}$$

The relative velocity (call this the *boost velocity*) between the two reference frames is given by

$$\vec{\mathbf{V}} = \frac{d\vec{\mathbf{R}}}{dt} \, . \tag{15.2.2}$$

Assume the boost velocity between the two reference frames is constant. Then, the relative acceleration between the two reference frames is zero,

$$\vec{\mathbf{A}} = \frac{d\vec{\mathbf{V}}}{dt} = \vec{\mathbf{0}} \, . \tag{15.2.3}$$

When Eq. (15.2.3) is satisfied, the reference frames S and S' are called *relatively inertial reference frames.*

Suppose the j^{th} particle in Figure 15.1 is moving; then observers in different reference frames will measure different velocities. Denote the velocity of j^{th} particle in frame S by $\vec{\mathbf{v}}_j = d\vec{\mathbf{r}}_j / dt$, and the velocity of the same particle in frame S' by $\vec{\mathbf{v}}'_j = d\vec{\mathbf{r}}'_j / dt$. Taking derivative, the velocities of the particles in two different reference frames are related according to

$$\vec{\mathbf{v}}_j = \vec{\mathbf{v}}'_j + \vec{\mathbf{V}} \, . \tag{15.2.4}$$

15.2.1 Relative Velocities

Consider two particles of masses m_1 and m_2 interacting via some force (Figure 15.2).

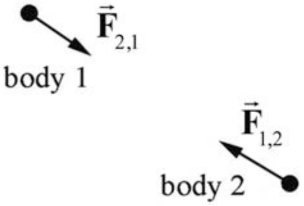


Figure 15.2 Two interacting particles

Choose a coordinate system (Figure 15.3) in which the position vector of body 1 is given by $\vec{\mathbf{r}}_1$ and the position vector of body 2 is given by $\vec{\mathbf{r}}_2$. The *relative position* of body 1 with respect to body 2 is given by $\vec{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$.

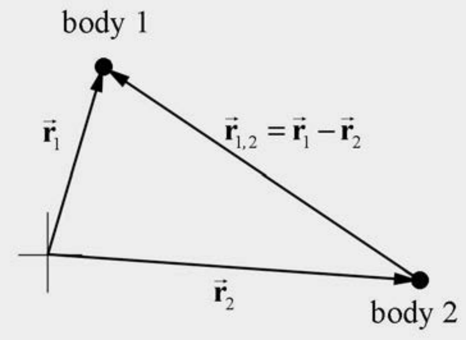


Figure 15.3 Coordinate system for two bodies.

During the course of the interaction, body 1 is displaced by $d\vec{\mathbf{r}}_1$ and body 2 is displaced by $d\vec{\mathbf{r}}_2$, so the ***relative displacement*** of the two bodies during the interaction is given by $d\vec{\mathbf{r}}_{1,2} = d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2$. The ***relative velocity*** between the particles is

$$\vec{\mathbf{v}}_{1,2} = \frac{d\vec{\mathbf{r}}_{1,2}}{dt} = \frac{d\vec{\mathbf{r}}_1}{dt} - \frac{d\vec{\mathbf{r}}_2}{dt} = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2 . \qquad (15.2.5)$$

We shall now show that the relative velocity between the two particles is independent of the choice of reference frame providing that the reference frames are relatively inertial. The relative velocity $\vec{\mathbf{v}}'_{12}$ in reference frame S' can be determined from using Eq. (15.2.4) to express Eq. (15.2.5) in terms of the velocities in the reference frame S',

$$\vec{\mathbf{v}}_{1,2} = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2 = (\vec{\mathbf{v}}'_1 + \vec{\mathbf{V}}) - (\vec{\mathbf{v}}'_2 + \vec{\mathbf{V}}) = \vec{\mathbf{v}}'_1 - \vec{\mathbf{v}}'_2 = \vec{\mathbf{v}}'_{1,2} \qquad (15.2.6)$$

and is equal to the relative velocity in frame S.

> *For a two-particle interaction, the relative velocity between the two vectors is independent of the choice of relatively inertial reference frames.*

15.2.2 Center-of-mass Reference Frame

Let $\vec{\mathbf{r}}_{cm}$ be the vector from the origin of frame S to the center-of-mass of the system of particles, a point that we will choose as the origin of reference frame S_{cm}, called the ***center-of-mass reference frame***. Denote the position vector of the j^{th} particle with respect to origin of reference frame S by $\vec{\mathbf{r}}_j$ and similarly, denote the position vector of the j^{th} particle with respect to origin of reference frame S_{cm} by $\vec{\mathbf{r}}'_j$ (Figure 15.4).

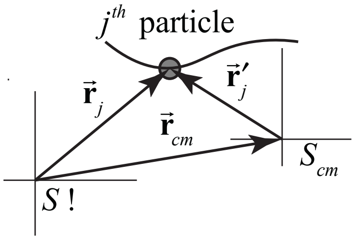


Figure 15.4 Position vector of j^{th} particle in the center-of-mass reference frame.

The position vector of the j^{th} particle in the center-of-mass frame is then given by

$$\vec{\mathbf{r}}'_j = \vec{\mathbf{r}}_j - \vec{\mathbf{r}}_{cm} . \tag{15.2.7}$$

The velocity of the j^{th} particle in the center-of-mass reference frame is then given by

$$\vec{\mathbf{v}}'_j = \vec{\mathbf{v}}_j - \vec{\mathbf{v}}_{cm} . \tag{15.2.8}$$

There are many collision problems in which the center-of-mass reference frame is the most convenient reference frame to analyze the collision.

Consider a system consisting of two particles, which we shall refer to as particle 1 and particle 2. We can use Eq. (15.2.8) to determine the velocities of particles 1 and 2 in the center-of-mass,

$$\vec{\mathbf{v}}'_1 = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_{cm} = \vec{\mathbf{v}}_1 - \frac{m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2}{m_1 + m_2} = \frac{m_2}{m_1 + m_2}(\vec{\mathbf{v}}_{1,} - \vec{\mathbf{v}}_2) = \frac{\mu}{m_1}\vec{\mathbf{v}}_{1,2} . \tag{15.2.9}$$

where $\vec{\mathbf{v}}_{12} = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2$ is the relative velocity of particle 1 with respect to particle 2. A similar result holds for particle 2:

$$\vec{\mathbf{v}}'_2 = \vec{\mathbf{v}}_2 - \vec{\mathbf{v}}_{cm} = \vec{\mathbf{v}}_2 - \frac{m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2}{m_1 + m_2} = -\frac{m_1}{m_1 + m_2}(\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2) = -\frac{\mu}{m_2}\vec{\mathbf{v}}_{1,2}\,. \quad (15.2.10)$$

The momentum of the system the center-of-mass reference frame is zero as we expect,

$$m_1\vec{\mathbf{v}}'_1 + m_2\vec{\mathbf{v}}'_2 = \mu\vec{\mathbf{v}}_{12} - \mu\vec{\mathbf{v}}_{12} = \vec{\mathbf{0}}\,. \quad (15.2.11)$$

15.2.3 Kinetic Energy in the Center-of-Mass Reference Frame

The kinetic energy in the center of mass reference frame is given by

$$K_{cm} = \frac{1}{2}m_1\vec{\mathbf{v}}'_1 \cdot \vec{\mathbf{v}}'_1 + \frac{1}{2}m_2\vec{\mathbf{v}}'_2 \cdot \vec{\mathbf{v}}'_2\,. \quad (15.2.12)$$

We now use Eqs. (15.2.9) and (15.2.10) to rewrite the kinetic energy in terms of the relative velocity $\vec{\mathbf{v}}'_{12} = \vec{\mathbf{v}}'_1 - \vec{\mathbf{v}}'_2$,

$$\begin{aligned} K_{cm} &= \frac{1}{2}m_1\left(\frac{\mu}{m_1}\vec{\mathbf{v}}_{1,2}\right)\cdot\left(\frac{\mu}{m_1}\vec{\mathbf{v}}_{1,2}\right) + \frac{1}{2}m_2\left(-\frac{\mu}{m_2}\vec{\mathbf{v}}_{1,2}\right)\cdot\left(-\frac{\mu}{m_2}\vec{\mathbf{v}}_{1,2}\right) \\ &= \frac{1}{2}\mu^2\vec{\mathbf{v}}_{1,2}\cdot\vec{\mathbf{v}}_{1,2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{1}{2}\mu v_{1,2}{}^2 \end{aligned}\,. \quad (15.2.13)$$

where we used the fact that we defined the reduced mass by

$$\frac{1}{\mu} \equiv \frac{1}{m_1} + \frac{1}{m_2}\,. \quad (15.2.14)$$

15.2.4 Change of Kinetic Energy and Relatively Inertial Reference Frames

The kinetic energy of the two particles in reference frame S is given by

$$K_S = \frac{1}{2}m_1 v_1{}^2 + \frac{1}{2}m_2 v_2{}^2\,. \quad (15.2.15)$$

We can take the scalar product of Eq. (15.2.8) to rewrite Eq. (15.2.15) as

$$
\begin{aligned}
K_S &= \frac{1}{2}m_1(\vec{\mathbf{v}}_1' + \vec{\mathbf{v}}_{cm})\cdot(\vec{\mathbf{v}}_1' + \vec{\mathbf{v}}_{cm}) + \frac{1}{2}m_2(\vec{\mathbf{v}}_2' + \vec{\mathbf{v}}_{cm})\cdot(\vec{\mathbf{v}}_2' + \vec{\mathbf{v}}_{cm}) \\
&= \frac{1}{2}m_1 v_1'^{\,2} + \frac{1}{2}m_2 v_2'^{\,2} + \frac{1}{2}(m_1+m_2)v_{cm}^{\;2} + (m_1\vec{\mathbf{v}}_1' + m_2\vec{\mathbf{v}}_2')\cdot\vec{\mathbf{v}}_{cm}
\end{aligned}
\quad . \tag{15.2.16}
$$

The last term is zero due to the fact that the momentum of the system in the center of mass reference frame is zero (Eq. (15.2.11)). Therefore Eq. (15.2.16) becomes

$$K_S = \frac{1}{2}m_1 v_1'^{\,2} + \frac{1}{2}m_2 v_2'^{\,2} + \frac{1}{2}(m_1+m_2)v_{cm}^{\;2} \,. \tag{15.2.17}$$

The first two terms correspond to the kinetic energy in the center of mass frame, thus the kinetic energies in the two reference frames are related by

$$K_S = K_{cm} + \frac{1}{2}(m_1+m_2)v_{cm}^{\;2} \ . \tag{15.2.18}$$

We now use Eq. (15.2.13) to rewrite Eq. (15.2.18) as

$$K_S = \frac{1}{2}\mu v_{1,2}^{\;2} + \frac{1}{2}(m_1+m_2)v_{cm}^{\;2} \tag{15.2.19}$$

Even though kinetic energy is a reference frame dependent quantity, because the second term in Eq. (15.2.19) is a constant, the change in kinetic energy in either reference frame is equal to

$$\Delta K = \frac{1}{2}\mu\left(\left(v_{1,2}^{\;2}\right)_f - \left(v_{1,2}^{\;2}\right)_i\right) \ . \tag{15.2.20}$$

This generalizes to any two relatively inertial reference frames because the relative velocity is a reference frame independent quantity,

> *the change in kinetic energy is independent of the choice of relatively inertial reference frames.*

We showed in Appendix 13A that when two particles of masses m_1 and m_2 interact, the work done by the interaction force is equal to

$$W = \frac{1}{2}\mu\left(\left(v_{1,2}^{\;2}\right)_f - \left(v_{1,2}^{\;2}\right)_i\right). \tag{15.2.21}$$

Hence we explicitly verified that for our two-particle system

$$W = \Delta K_{sys} \,. \tag{15.2.22}$$

15.3 Characterizing Collisions

In a collision, the ratio of the magnitudes of the initial and final relative velocities is called the coefficient of restitution and denoted by the symbol e,

$$e = \frac{v_B}{v_A}. \tag{15.2.23}$$

If the magnitude of the relative velocity does not change during a collision, $e = 1$, then the change in kinetic energy is zero, (Eq. (15.2.21)). Collisions in which there is no change in kinetic energy are called ***elastic collisions***,

$$\Delta K = 0, \quad \text{elastic collision}. \tag{15.2.24}$$

If the magnitude of the final relative velocity is less than the magnitude of the initial relative velocity, $e < 1$, then the change in kinetic energy is negative. Collisions in which the kinetic energy decreases are called ***inelastic collisions***,

$$\Delta K < 0, \ \text{inelastic collision}. \tag{15.2.25}$$

If the two objects stick together after the collision, then the relative final velocity is zero, $e = 0$. Such collisions are called ***totally inelastic***. The change in kinetic energy can be found from Eq. (15.2.21),

$$\Delta K = -\frac{1}{2}\mu v_A^2 = -\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2} v_A^2, \ \text{totally inelastic collision}. \tag{15.2.26}$$

If the magnitude of the final relative velocity is greater than the magnitude of the initial relative velocity, $e > 1$, then the change in kinetic energy is positive. Collisions in which the kinetic energy increases are called ***superelastic collisions***,

$$\Delta K > 0, \ \text{superelastic collision}. \tag{15.2.27}$$

15.4 One-Dimensional Collisions Between Two Objects

15.4.1 One Dimensional Elastic Collision in Laboratory Reference Frame

Consider a one-dimensional elastic collision between two objects moving in the x-direction. One object, with mass m_1 and initial x-component of the velocity $v_{1x,i}$, collides with an object of mass m_2 and initial x-component of the velocity $v_{2x,i}$. The scalar components $v_{1x,i}$ and $v_{1x,i}$ can be positive, negative or zero. No forces other than the interaction force between the objects act during the collision. After the collision, the final x-component of the velocities are $v_{1x,f}$ and $v_{2x,f}$. We call this reference frame the "laboratory reference frame".

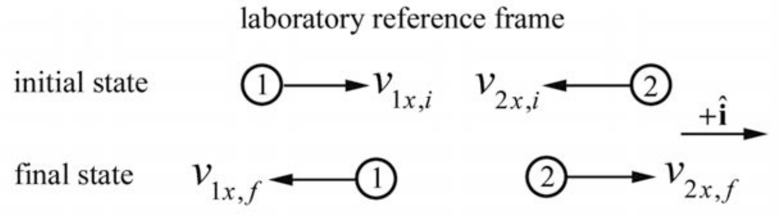


Figure 15.5 One-dimensional elastic collision, laboratory reference frame

For the collision depicted in Figure 15.5, $v_{1x,i} > 0$, $v_{2x,i} < 0$, $v_{1x,f} < 0$, and $v_{2x,f} > 0$. Because there are no external forces in the x-direction, momentum is constant in the x-direction. Equating the momentum components before and after the collision gives the relation

$$m_1 v_{1x,i} + m_2 v_{2x,i} = m_1 v_{1x,f} + m_2 v_{2x,f} . \tag{15.3.1}$$

Because the collision is elastic, kinetic energy is constant. Equating the kinetic energy before and after the collision gives the relation

$$\frac{1}{2} m_1 v_{1x,i}^2 + \frac{1}{2} m_2 v_{2x,i}^2 = \frac{1}{2} m_1 v_{1x,f}^2 + \frac{1}{2} m_2 v_{2x,f}^2 \tag{15.3.2}$$

Rewrite these Eqs. (15.3.1) and (15.3.2) as

$$m_1 (v_{1x,i} - v_{1x,f}) = m_2 (v_{2x,f} - v_{2x,i}) \tag{15.3.3}$$

$$m_1 (v_{1x,i}^2 - v_{1x,f}^2) = m_2 (v_{2x,f}^2 - v_{2x,i}^2) . \tag{15.3.4}$$

Eq. (15.3.4) can be written as

$$m_1 (v_{1x,i} - v_{1x,f})(v_{1x,i} + v_{1x,f}) = m_2 (v_{2x,f} - v_{2x,i})(v_{2x,f} + v_{2x,i}) . \tag{15.3.5}$$

Divide Eq. (15.3.4) by Eq. (15.3.3), yielding

$$v_{1x,i} + v_{1x,f} = v_{2x,i} + v_{2x,f} . \tag{15.3.6}$$

Eq. (15.3.6) may be rewritten as

$$v_{1x,i} - v_{2x,i} = v_{2x,f} - v_{1x,f} . \tag{15.3.7}$$

Recall that the relative velocity between the two objects is defined to be

$$\vec{\mathbf{v}}^{\text{rel}} \equiv \vec{\mathbf{v}}_{1,2} \equiv \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2 . \tag{15.3.8}$$

where we used the superscript “rel” to remind ourselves that the velocity is a relative velocity (and to simplify our notation). Thus $v_{x,i}^{\text{rel}} = v_{1x,i} - v_{2x,i}$ is the initial x-component of the relative velocity, and $v_{x,f}^{\text{rel}} = v_{1x,f} - v_{2x,f}$ is the final x-component of the relative velocity. Therefore Eq. (15.3.7) states that during the interaction the initial relative velocity is equal to the negative of the final relative velocity

$$\vec{\mathbf{v}}_i^{\text{rel}} = -\vec{\mathbf{v}}_f^{\text{rel}}, \quad (1-\text{dimensional energy-momentum prinicple}). \qquad (15.3.9)$$

Consequently the initial and final relative speeds are equal. We shall call this relationship between the relative initial and final velocities the **one-dimensional energy-momentum principle** because we have combined these two principles to realize this result. The energy-momentum principle is independent of the masses of the colliding particles.

Although we derived this result explicitly, we have already shown that the change in kinetic energy for a two-particle interaction (Eq. (15.2.20)), in our simplified notation is given by

$$\Delta K = \frac{1}{2}\mu((v^{\text{rel}})_f^2 - (v^{\text{rel}})_i^2) \qquad (15.3.10)$$

Therefore for an elastic collision where $\Delta K = 0$, the square of the relative speed remains constant

$$(v^{\text{rel}})_f^2 = (v^{\text{rel}})_i^2 \ . \qquad (15.3.11)$$

For a one-dimensional collision, the magnitude of the relative speed remains constant but the direction changes by 180°.

We can now solve for the final x-component of the velocities, $v_{1x,f}$ and $v_{2x,f}$, as follows. Eq. (15.3.7) may be rewritten as

$$v_{2x,f} = v_{1x,f} + v_{1x,i} - v_{2x,i} \, . \qquad (15.3.12)$$

Now substitute Eq. (15.3.12) into Eq. (15.3.1) yielding

$$m_1 v_{1x,i} + m_2 v_{2x,i} = m_1 v_{1x,f} + m_2 (v_{1x,f} + v_{1x,i} - v_{2x,i}) \, . \qquad (15.3.13)$$

Solving Eq. (15.3.13) for $v_{1x,f}$ involves some algebra and yields

$$v_{1x,f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1x,i} + \frac{2 m_2}{m_1 + m_2} v_{2x,i} \, . \qquad (15.3.14)$$

To find $v_{2x,f}$, rewrite Eq. (15.3.7) as

$$v_{1x,f} = v_{2x,f} - v_{1x,i} + v_{2x,i}\,. \tag{15.3.15}$$

Now substitute Eq. (15.3.15) into Eq. (15.3.1) yielding

$$m_1 v_{1x,i} + m_2 v_{2x,i} = m_1(v_{2x,f} - v_{1x,i} + v_{2x,i})v_{1x,f} + m_2 v_{2x,f}\,. \tag{15.3.16}$$

We can solve Eq. (15.3.16) for $v_{2x,f}$ and determine that

$$v_{2x,f} = v_{2x,i}\frac{m_2 - m_1}{m_2 + m_1} + v_{1x,i}\frac{2\,m_1}{m_2 + m_1}\,. \tag{15.3.17}$$

Consider what happens in the limits $m_1 >> m_2$ in Eq. (15.3.14). Then

$$v_{1x,f} \to v_{1x,i} + \frac{2}{m_1} m_2 v_{2x,i}\,; \tag{15.3.18}$$

the more massive object's velocity component is only slightly changed by an amount proportional to the less massive object's x-component of momentum. Similarly, the less massive object's final velocity approaches

$$v_{2x,f} \to -v_{2x,i} + 2v_{1x,i} = v_{1x,i} + v_{1x,i} - v_{2x,i}\,. \tag{15.3.19}$$

We can rewrite this as

$$v_{2x,f} - v_{1x,i} = v_{1x,i} - v_{2x,i} = v_{x,i}^{\text{rel}}\,. \tag{15.3.20}$$

i.e. the less massive object "rebounds" with the same speed relative to the more massive object which barely changed its speed.

If the objects are identical, or have the same mass, Eqs. (15.3.14) and (15.3.17) become

$$v_{1x,f} = v_{2x,i},\quad v_{2x,f} = v_{1x,i}\,; \tag{15.3.21}$$

the objects have exchanged x-components of velocities, and unless we could somehow distinguish the objects, we might not be able to tell if there was a collision at all.

15.4.2 One-Dimensional Collision Between Two Objects – Center-of-Mass Reference Frame

We analyzed the one-dimensional elastic collision (Figure 15.5) in Section 15.4.1 in the laboratory reference frame. Now let's view the collision from the center-of-mass (CM) frame. The x-component of velocity of the center-of-mass is

$$v_{x,\text{cm}} = \frac{m_1 v_{1x,i} + m_2 v_{2x,i}}{m_1 + m_2}. \tag{15.3.22}$$

With respect to the center-of-mass, the x-components of the velocities of the objects are

$$\begin{aligned} v'_{1x,i} &= v_{1x,i} - v_{x,\text{cm}} = (v_{1x,i} - v_{2x,i})\frac{m_2}{m_1 + m_2} \\ v'_{2x,i} &= v_{2x,i} - v_{x,\text{cm}} = (v_{2x,i} - v_{1x,i})\frac{m_1}{m_1 + m_2}. \end{aligned} \tag{15.3.23}$$

In the CM frame the momentum of the system is zero before the collision and hence the momentum of the system is zero after the collision. For an elastic collision, the only way for both momentum and kinetic energy to be the same before and after the collision is either the objects have the same velocity (a miss) or to reverse the direction of the velocities as shown in Figure 15.6.

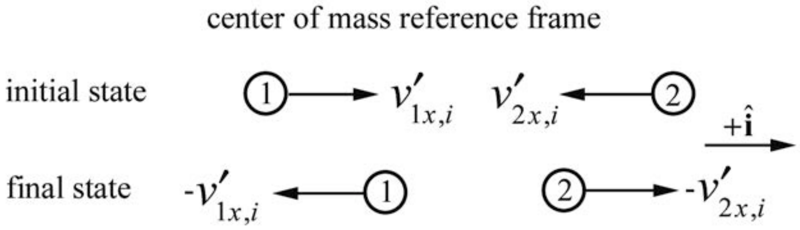


Figure 15.6 One-dimensional elastic collision in center-of-mass reference frame

In the CM frame, the final x-components of the velocities are

$$\begin{aligned} v'_{1x,f} &= -v'_{1x,i} = (v_{2x,i} - v_{1x,i})\frac{m_2}{m_1 + m_2} \\ v'_{2x,f} &= -v'_{2x,i} = (v_{2x,i} - v_{1x,i})\frac{m_1}{m_1 + m_2}. \end{aligned} \tag{15.3.24}$$

The final x-components of the velocities in the "laboratory frame" are then given by

$$
\begin{aligned}
v_{1x,f} &= v'_{1x,f} + v_{x,\text{cm}} \\
&= (v_{2x,i} - v_{1x,i})\frac{m_2}{m_1+m_2} + \frac{m_1 v_{1x,i} + m_2 v_{2x,i}}{m_1+m_2} \\
&= v_{1x,i}\frac{m_1-m_2}{m_1+m_2} + v_{2x,i}\frac{2m_2}{m_1+m_2}
\end{aligned}
\qquad (15.3.25)
$$

as in Eq. (15.3.14) and a similar calculation reproduces Eq. (15.3.17).

15.5 Worked Examples

Example 15.1 Elastic One-Dimensional Collision Between Two Objects

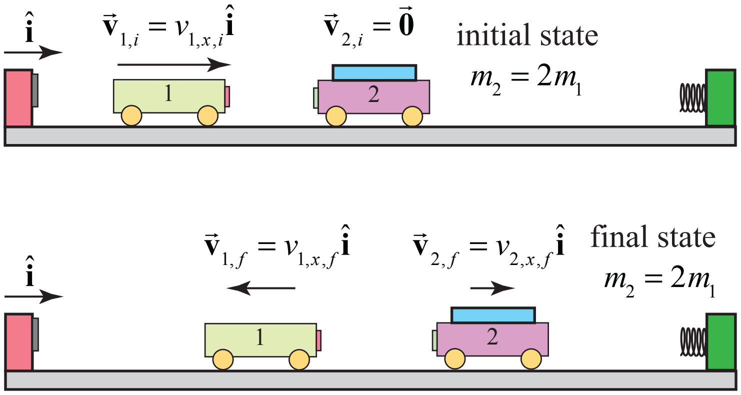


Figure 15.7 Elastic collision between two non-identical carts

Consider the elastic collision of two carts along a track; the incident cart 1 has mass m_1 and moves with initial speed $v_{1,i}$. The target cart has mass $m_2 = 2m_1$ and is initially at rest, $v_{2,i} = 0$, (Figure 15.7). Immediately after the collision, the incident cart has final speed $v_{1,f}$ and the target cart has final speed $v_{2,f}$. Calculate the final x-component of the velocities of the carts as a function of the initial speed $v_{1,i}$.

Solution The momentum flow diagram for the objects before (initial state) and after (final state) the collision are shown in Figure 15.7. We can immediately use our results above with $m_2 = 2m_1$ and $v_{2,i} = 0$. The final x-component of velocity of cart 1 is given by Eq. (15.3.14), where we use $v_{1x,i} = v_{1,i}$

$$v_{1x,f} = -\frac{1}{3}v_{1,i}\,. \tag{15.4.1}$$

The final x-component of velocity of cart 2 is given by Eq. (15.3.17)

$$v_{2x,f} = \frac{2}{3}v_{1,i}\,. \tag{15.4.2}$$

Example 15.2 The Dissipation of Kinetic Energy in a Completely Inelastic Collision Between Two Objects

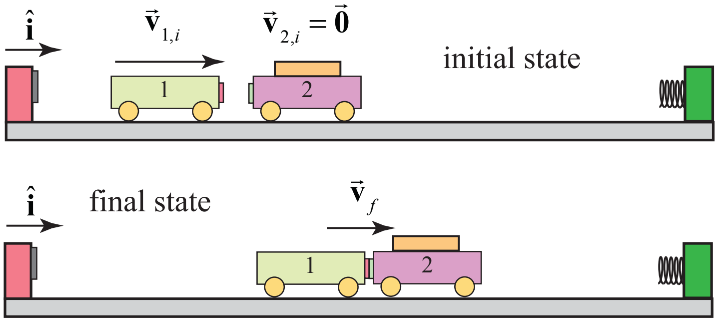


Figure 15.7b Inelastic collision between two non-identical carts

An incident cart of mass m_1 and initial speed $v_{1,i}$ collides completely inelastically with a cart of mass m_2 that is initially at rest (Figure 15.7b). There are no external forces acting on the objects in the direction of the collision. Find $\Delta K / K_{\text{initial}} = (K_{\text{final}} - K_{\text{initial}}) / K_{\text{initial}}$.

Solution: In the absence of any net force on the system consisting of the two carts, the momentum after the collision will be the same as before the collision. After the collision the carts will move in the direction of the initial velocity of the incident cart with a common speed v_f found from applying the momentum condition

$$\begin{aligned} &m_1 v_{1,i} = (m_1 + m_2) v_f \Rightarrow \\ &v_f = \frac{m_1}{m_1 + m_2} v_{1,i}\,. \end{aligned} \tag{15.4.3}$$

The initial relative speed is $v_i^{\text{rel}} = v_{1,i}$. The final relative velocity is zero because the carts stick together so using Eq. (15.2.26), the change in kinetic energy is

$$\Delta K = -\frac{1}{2}\mu (v_i^{rel})^2 = -\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2} v_{1,i}^2\,. \tag{15.4.4}$$

The ratio of the change in kinetic energy to the initial kinetic energy is then

$$\Delta K / K_{\text{initial}} = -\frac{m_2}{m_1 + m_2}. \tag{15.4.5}$$

As a check, we can calculate the change in kinetic energy via

$$\begin{aligned}\Delta K = (K_f - K_i) &= \frac{1}{2}(m_1 + m_2)v_f^2 - \frac{1}{2}v_{1,i}^2 \\ &= \frac{1}{2}(m_1 + m_2)\left(\frac{m_1}{m_1 + m_2}\right)^2 v_{1,i}^2 - \frac{1}{2}v_{1,i}^2 \\ &= \left(\frac{m_1}{m_1 + m_2} - 1\right)\left(\frac{1}{2}m_1 v_{1,i}^2\right) = -\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}v_{1,i}^2.\end{aligned} \tag{15.4.6}$$

in agreement with Eq. (15.4.4).

Example 15.3 Bouncing Superballs

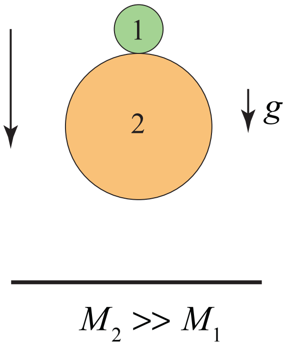
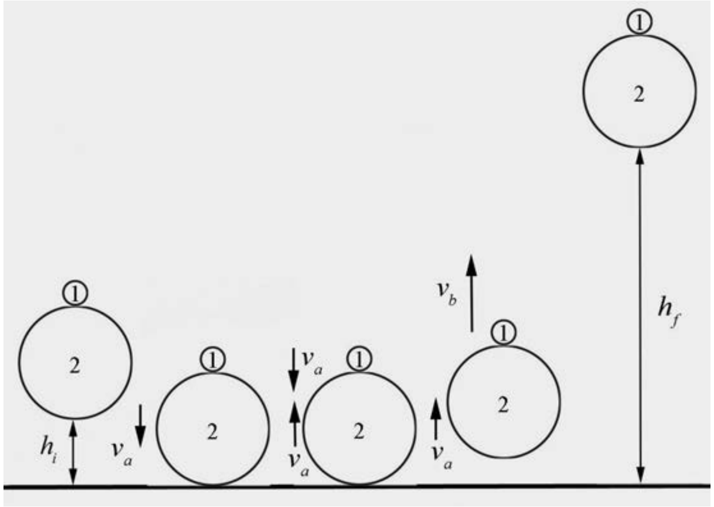


Figure 15.8b Two superballs dropping

Consider two balls that are dropped from a height h_i above the ground, one on top of the other (Figure 15.8). Ball 1 is on top and has mass M_1, and ball 2 is underneath and has mass M_2 with $M_2 >> M_1$. Assume that there is no loss of kinetic energy during all collisions. Ball 2 first collides with the ground and rebounds. Then, as ball 2`starts to move upward, it collides with the ball 1 which is still moving downwards (figure below left). How high will ball 1 rebound in the air? Hint: consider this collision as seen by an observer moving upward with the same speed as the ball 2 has after it collides with ground. What speed does ball 1 have in this reference frame after it collides with the ball 2?

Solution

The system consists of the two balls and the earth. There are five special states for this motion shown in the figure below.



part a)

Initial State: the balls are released from rest at a height h_i above the ground.

State A: the balls just reach the ground with speed $v_a = \sqrt{2gh_i}$. This follows from $\Delta E_{mech} = 0 \Rightarrow \Delta K = -\Delta U$. Thus $(1/2)mv_a^2 - 0 = -mg\Delta h = mgh_i \Rightarrow v_a = \sqrt{2gh_i}$.

State B: immediately before the collision of the balls. Ball 2 has collided with the ground and reversed direction with the same speed, v_a, but ball 1 is still moving downward with speed v_a.

State C: immediately after the collision of the balls. Because we are assuming that $m_2 >> m_1$, ball 2 does not change its speed as a result of the collision so it is still moving upward with speed v_a. As a result of the collision, ball 1 moves upward with speed v_b.

Final State: ball 1 reaches a maximum height $h_f = v_b{}^2 / 2g$ above the ground. This again follows from $\Delta K = -\Delta U \Rightarrow 0 - (1/2)mv_b^2 = -mg\Delta h = -mgh_f \Rightarrow h_f = v_b^2 / 2g$.

Choice of Reference Frame:

As indicated in the hint above, this collision is best analyzed from the reference frame of an observer moving upward with speed v_a, the speed of ball 2 just after it rebounded with the ground. In this frame immediately, before the collision, ball 1 is moving downward with a speed v'_b that is twice the speed seen by an observer at rest on the ground (lab reference frame).

$$v'_a = 2v_a \tag{15.4.7}$$

The mass of ball 2 is much larger than the mass of ball 1, $m_2 >> m_1$. This enables us to consider the collision (between States B and C) to be equivalent to ball 1 bouncing off a hard wall, while ball 2 experiences virtually no recoil. Hence ball 2 remains at rest in the reference frame moving upwards with speed v_a with respect to observer at rest on ground. Before the collision, ball 1 has speed $v'_a = 2v_a$. Since there is no loss of kinetic energy during the collision, the result of the collision is that ball 1 changes direction but maintains the same speed,

$$v'_b = 2v_a\,. \tag{15.4.8}$$

However, according to an observer at rest on the ground, after the collision ball 1 is moving upwards with speed

$$v_b = 2v_a + v_a = 3v_a\,. \tag{15.4.9}$$

While rebounding, the mechanical energy of the smaller superball is constant (we consider the smaller superball and the Earth as a system) hence between State C and the Final State,

$$\Delta K + \Delta U = 0\,. \tag{15.4.10}$$

The change in kinetic energy is

$$\Delta K = -\frac{1}{2}m_1(3v_a)^2\,. \tag{15.4.11}$$

The change in potential energy is

$$\Delta U = m_1 g h_f\,. \tag{15.4.12}$$

So the condition that mechanical energy is constant (Equation (15.4.10)) is now

$$-\frac{1}{2}m_1(3v_{1a})^2 + m_1 g h_f = 0\,. \tag{15.4.13}$$

We can rewrite Equation (15.4.13) as

$$m_1 g h_f = 9\frac{1}{2}m_1(v_a)^2\,. \tag{15.4.14}$$

Recall that we can also use the fact that the mechanical energy doesn't change between the Initial State and State A yielding an equation similar to Eq. (15.4.14),

$$m_1 g h_i = \frac{1}{2} m_1 (v_a)^2 . \qquad (15.4.15)$$

Now substitute the expression for the kinetic energy in Eq. (15.4.15) into Eq. (15.4.14) yielding

$$m_1 g h_f = 9 m_1 g h_i . \qquad (15.4.16)$$

Thus ball 1 reaches a maximum height

$$h_f = 9 h_i . \qquad (15.4.17)$$

15.6 Two Dimensional Elastic Collisions

15.6.1 Two-dimensional Elastic Collision in Laboratory Reference Frame

Consider the elastic collision between two particles in which we neglect any external forces on the system consisting of the two particles. Particle 1 of mass m_1 is initially moving with velocity $\vec{\mathbf{v}}_{1,i}$ and collides elastically with a particle 2 of mass m_2 that is initially at rest. We shall refer to the reference frame in which one particle is at rest, 'the target', as the **laboratory reference frame**. After the collision particle 1 moves with velocity $\vec{\mathbf{v}}_{1,f}$ and particle 2 moves with velocity $\vec{\mathbf{v}}_{2,f}$, (Figure 15.9). The angles $\theta_{1,f}$ and $\theta_{2,f}$ that the particles make with the positive forward direction of particle 1 are called the **laboratory scattering angles**.

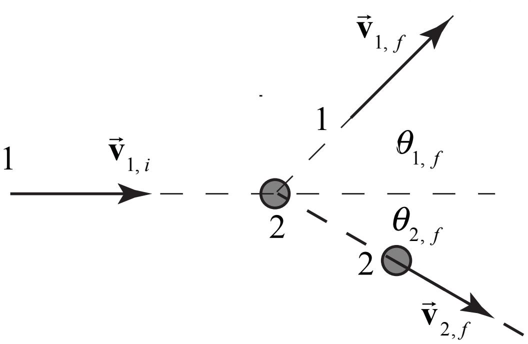


Figure 15.9 Two-dimensional collision in laboratory reference frame

Generally the initial velocity $\vec{\mathbf{v}}_{1,i}$ of particle 1 is known and we would like to determine the final velocities $\vec{\mathbf{v}}_{1,f}$ and $\vec{\mathbf{v}}_{2,f}$, which requires finding the magnitudes and directions of each of these vectors, $v_{1,f}$, $v_{2,f}$, $\theta_{1,f}$, and $\theta_{2,f}$. These quantities are related by the two equations describing the constancy of momentum, and the one equation describing constancy of the kinetic energy. Therefore there is one degree of freedom that we must specify in order to determine the outcome of the collision. In what follows we shall express our results for $v_{1,f}$, $v_{2,f}$, and $\theta_{2,f}$ in terms of $v_{1,i}$ and $\theta_{1,f}$.

The components of the total momentum $\vec{\mathbf{p}}_i^{\text{sys}} = m_1\vec{\mathbf{v}}_{1,i} + m_2\vec{\mathbf{v}}_{2,i}$ in the initial state are given by

$$\begin{aligned} p_{x,i}^{\text{sys}} &= m_1 v_{1,i} \\ p_{y,i}^{\text{sys}} &= 0. \end{aligned} \tag{15.5.1}$$

The components of the momentum $\vec{\mathbf{p}}_f^{\text{sys}} = m_1\vec{\mathbf{v}}_{1,f} + m_2\vec{\mathbf{v}}_{2,f}$ in the final state are given by

$$\begin{aligned} p_{x,f}^{\text{sys}} &= m_1\, v_{1,f} \cos\theta_{1,f} + m_2\, v_{2,f} \cos\theta_{2,f} \\ p_{y,f}^{\text{sys}} &= m_1\, v_{1,f} \sin\theta_{1,f} - m_2\, v_{2,f} \sin\theta_{2,f}. \end{aligned} \tag{15.5.2}$$

There are no any external forces acting on the system, so each component of the total momentum remains constant during the collision,

$$p_{x,i}^{\text{sys}} = p_{x,f}^{\text{sys}} \tag{15.5.3}$$

$$p_{y,i}^{\text{sys}} = p_{y,f}^{\text{sys}}. \tag{15.5.4}$$

Eqs. (15.5.3) and (15.5.4) become

$$m_1\, v_{1,i} = m_1\, v_{1,f} \cos\theta_{1,f} + m_2\, v_{2,f} \cos\theta_{2,f}, \tag{15.5.5}$$

$$0 = m_1\, v_{1,f} \sin\theta_{1,f} - m_2\, v_{2,f} \sin\theta_{2,f}\ . \tag{15.5.6}$$

The collision is elastic and therefore the system kinetic energy of is constant

$$K_i^{\text{sys}} = K_f^{\text{sys}}. \tag{15.5.7}$$

Using the given information, Eq. (15.5.7) becomes

$$\frac{1}{2} m_1 v_{1,i}^2 = \frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_2 v_{2,f}^2. \tag{15.5.8}$$

Rewrite the expressions in Eqs. (15.5.5) and (15.5.6) as

$$m_2 v_{2,f} \cos\theta_{2,f} = m_1(v_{1,i} - v_{1,f} \cos\theta_{1,f}), \tag{15.5.9}$$

$$m_2 v_{2,f} \sin\theta_{2,f} = m_1 v_{1,f} \sin\theta_{1,f} \ . \tag{15.5.10}$$

Square each of the expressions in Eqs. (15.5.9) and (15.5.10), add them together and use the identity $\cos^2\theta + \sin^2\theta = 1$ yielding

$$v_{2,f}^2 = \frac{m_1^2}{m_2^2}(v_{1,i}^2 - 2v_{1,i}v_{1,f}\cos\theta_{1,f} + v_{1,f}^2) \, . \tag{15.5.11}$$

Substituting Eq. (15.5.11) into Eq. (15.5.8) yields

$$\frac{1}{2}m_1 v_{1,i}^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}\frac{m_1^2}{m_2}(v_{1,i}^2 - 2v_{1,i}\, v_{1,f}\cos\theta_{1,f} + v_{1,f}^2) \, . \tag{15.5.12}$$

Eq. (15.5.12) simplifies to

$$0 = \left(1 + \frac{m_1}{m_2}\right)v_{1,f}^2 - \frac{m_1}{m_2}2v_{1,i}\, v_{1,f}\cos\theta_{1,f} - \left(1 - \frac{m_1}{m_2}\right)v_{1,i}^2 \, , \tag{15.5.13}$$

Let $\alpha = m_1 / m_2$ then Eq. (15.5.13) can be written as

$$0 = (1+\alpha)v_{1,f}^2 - 2\alpha v_{1,i}\, v_{1,f}\cos\theta_{1,f} - (1-\alpha)v_{1,i}^2 \, , \tag{15.5.14}$$

The solution to this quadratic equation is given by

$$v_{1,f} = \frac{\alpha v_{1,i}\cos\theta_{1,f} \pm \left(\alpha^2 v_{1,i}^2\cos^2\theta_{1,f} + (1-\alpha)v_{1,i}^2\right)^{1/2}}{(1+\alpha)} \, . \tag{15.5.15}$$

Divide the expressions in Eq. (15.5.9), yielding

$$\frac{v_{2,f}\sin\theta_{2,f}}{v_{2,f}\cos\theta_{2,f}} = \frac{v_{1,f}\sin\theta_{1,f}}{v_{1,i} - v_{1,f}\cos\theta_{1,f}} \, . \tag{15.5.16}$$

Eq. (15.5.16) simplifies to

$$\tan\theta_{2,f} = \frac{v_{1,f}\sin\theta_{1,f}}{v_{1,i} - v_{1,f}\cos\theta_{1,f}} \, . \tag{15.5.17}$$

The relationship between the scattering angles in Eq. (15.5.17) is independent of the masses of the colliding particles. Thus the scattering angle for particle 2 is

$$\theta_{2,f} = \tan^{-1}\left(\frac{v_{1,f}\sin\theta_{1,f}}{v_{1,i} - v_{1,f}\cos\theta_{1,f}}\right) \tag{15.5.18}$$

We can now use Eq. (15.5.10) to find an expression for the final velocity of particle 1

$$v_{2,f} = \frac{v_{1,f}\sin\theta_{1,f}}{\alpha\sin\theta_{2,f}} \ . \tag{15.5.19}$$

Example 15.5 Elastic Two-dimensional collision of identical particles

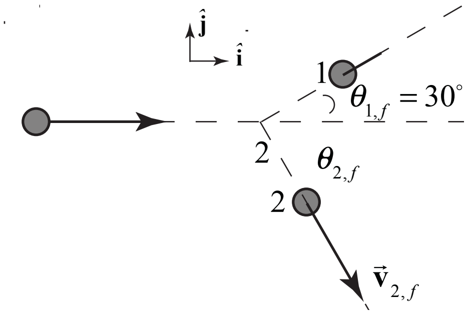


Figure 15.10 Momentum flow diagram for two-dimensional elastic collision

Object 1 with mass m_1 is initially moving with a speed $v_{1,i} = 3.0\,\text{m}\cdot\text{s}^{-1}$ and collides elastically with object 2 that has the same mass, $m_2 = m_1$, and is initially at rest. After the collision, object 1 moves with an unknown speed $v_{1,f}$ at an angle $\theta_{1,f}$ with respect to its initial direction of motion and object 2 moves with an unknown speed $v_{2,f}$, at an unknown angle $\theta_{2,f}$ (as shown in the Figure 15.10). Find the final speeds of each of the objects and the angle $\theta_{2,f}$.

Solution: Because the masses are equal, $\alpha = 1$. We are given that $v_{1,i} = 3.0\ \text{m}\cdot\text{s}^{-1}$ and $\theta_{1,f} = 30^o$. Hence Eq. (15.5.14) reduces to

$$v_{1,f} = v_{1,i}\cos\theta_{1,f} = (3.0\ \text{m}\cdot\text{s}^{-1})\cos 30^\circ = 2.6\ \text{m}\cdot\text{s}^{-1} \ . \tag{15.5.20}$$

Substituting Eq. (15.5.20) in Eq. (15.5.17) yields

$$\theta_{2,f} = \tan^{-1}\left(\frac{v_{1,f}\sin\theta_{1,f}}{v_{1,i} - v_{1,f}\cos\theta_{1,f}}\right)$$

$$\theta_{2,f} = \tan^{-1}\left(\frac{(2.6\,\mathrm{m}\cdot\mathrm{s}^{-1})\sin(30^\circ)}{3.0\,\mathrm{m}\cdot\mathrm{s}^{-1} - (2.6\,\mathrm{m}\cdot\mathrm{s}^{-1})\cos(30^\circ)}\right) \qquad (15.5.21)$$

$$= 60^\circ.$$

The above results for $v_{1,f}$ and $\theta_{2,f}$ may be substituted into either of the expressions in Eq. (15.5.9), or Eq. (15.5.11), to find $v_{2,f} = 1.5\mathrm{m}\cdot\mathrm{s}^{-1}$. Eq. (15.5.11) also has the solution $v_{2,f} = 0$, which would correspond to the incident particle missing the target completely.

Before going on, the fact that $\theta_{1,f} + \theta_{2,f} = 90^\circ$, that is, the objects move away from the collision point at right angles, is not a coincidence. A vector derivation is presented in Example 15.6. We can see this result algebraically from the above result. Substituting Eq. (15.5.20) $v_{1,f} = v_{1,i}\cos\theta_{1,f}$ in Eq. (15.5.17) yields

$$\tan\theta_{2,f} = \frac{\cos\theta_{1,f}\sin\theta_{1,f}}{1 - \cos\theta_{1,f}^2} = \cot\theta_{1,f} = \tan(90^\circ - \theta_{1,f}); \qquad (15.5.22)$$

showing that $\theta_{1,f} + \theta_{2,f} = 90^\circ$, the angles $\theta_{1,f}$ and $\theta_{2,f}$ are complements.

Example 15.6 Two-dimensional elastic collision between particles of equal mass

Show that the equal mass particles emerge from a two-dimensional elastic collision at right angles by making explicit use of the fact that momentum is a vector quantity.

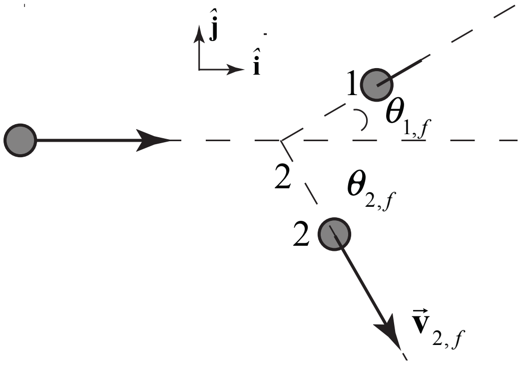


Figure 15.11 Elastic scattering of identical particles

Solution: Choose a reference frame in which particle 2 is initially at rest (Figure 15.11). There are no external forces acting on the two objects during the collision (the collision forces are all internal), therefore momentum is constant

$$\vec{\mathbf{p}}_i^{\text{sys}} = \vec{\mathbf{p}}_f^{\text{sys}} , \tag{15.5.23}$$

which becomes

$$m_1\vec{\mathbf{v}}_{1,i} = m_1\vec{\mathbf{v}}_{1,f} + m_1\vec{\mathbf{v}}_{2,f} \ . \tag{15.5.24}$$

Eq. (15.5.24) simplifies to

$$\vec{\mathbf{v}}_{1,i} = \vec{\mathbf{v}}_{1,f} + \vec{\mathbf{v}}_{2,f} \ . \tag{15.5.25}$$

Recall the vector identity that the square of the speed is given by the dot product $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = v^2$. With this identity in mind, we take the dot product of each side of Eq. (15.5.25) with itself,

$$\begin{aligned}\vec{\mathbf{v}}_{1,i} \cdot \vec{\mathbf{v}}_{1,i} &= (\vec{\mathbf{v}}_{1,f} + \vec{\mathbf{v}}_{2,f}) \cdot (\vec{\mathbf{v}}_{1,f} + \vec{\mathbf{v}}_{2,f}) \\ &= \vec{\mathbf{v}}_{1,f} \cdot \vec{\mathbf{v}}_{1,f} + 2\vec{\mathbf{v}}_{1,f} \cdot \vec{\mathbf{v}}_{2,f} + \vec{\mathbf{v}}_{2,f} \cdot \vec{\mathbf{v}}_{2,f} .\end{aligned} \tag{15.5.26}$$

This becomes

$$v_{1,i}^2 = v_{1,f}^2 + 2\vec{\mathbf{v}}_{1,f} \cdot \vec{\mathbf{v}}_{2,f} + v_{2,f}^2 . \tag{15.5.27}$$

Recall that kinetic energy is the same before and after an elastic collision, and the masses of the two objects are equal, so constancy of energy, (Eq. (15.3.2)) simplifies to

$$v_{1,i}^2 = v_{1,f}^2 + v_{2,f}^2 . \tag{15.5.28}$$

Comparing Eq. (15.5.27) to Eq. (15.5.28), we see that

$$\vec{\mathbf{v}}_{1,f} \cdot \vec{\mathbf{v}}_{2,f} = 0 \ . \tag{15.5.29}$$

The dot product of two nonzero vectors is zero when the two vectors are at right angles to each other justifying our claim that the collision particles emerge at right angles to each other.

Example 15.7 Two dimensional collision between particles of unequal mass

Particle 1 of mass m_1, initially moving in the positive x-direction (to the right in the figure below) with speed $v_{1,i}$, collides with particle 2 of mass $m_2 = m_1/3$, which is initially moving in the opposite direction (Figure 15.12) with an unknown speed $v_{2,i}$. Assume that the total external force acting on the particles is zero. Do not assume the collision is elastic. After the collision, particle 1 moves with speed $v_{1,f} = v_{1,i}/2$ in the negative y-direction. After the collision, particle 2 moves with an unknown speed $v_{2,f}$, at an angle $\theta_{2,f} = 45^o$ with respect to the positive x-direction. (i) Determine the initial speed $v_{2,i}$ of particle 2 and the final speed $v_{2,f}$ of particle 2 in terms of $v_{1,i}$. (ii) Is the collision elastic?

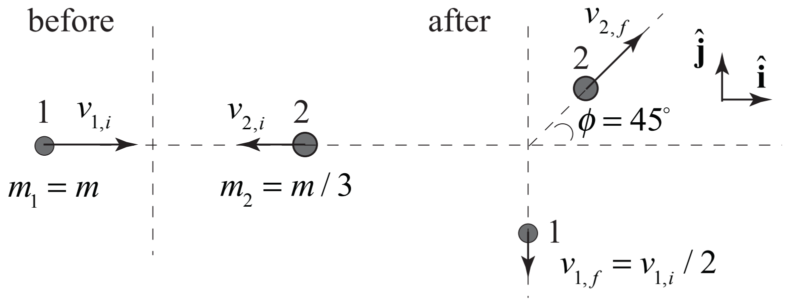


Figure 15.12 Two-dimensional collision between particles of unequal mass

Solution: We choose as our system the two particles. We are given that $v_{1,f} = v_{1,i}/2$. We apply the two momentum conditions,

$$m_1 v_{1,i} - (m_1/3) v_{2,i} = (m_1/3)\, v_{2,f} (\sqrt{2}/2) \tag{15.5.30}$$

$$0 = m_1 v_{1,f} - (m_1/3)\, v_{2,f} (\sqrt{2}/2)\ . \tag{15.5.31}$$

Solve Eq. (15.5.31) for $v_{2,f}$

$$v_{2,f} = 3\sqrt{2} v_{1,f} = \frac{3\sqrt{2}}{2} v_{1,i} \tag{15.5.32}$$

Substitute Eq. (15.5.32) into Eq. (15.5.30) and solve for $v_{2,i}$

$$v_{2,i} = (3/2) v_{1,i}\ . \tag{15.5.33}$$

The initial kinetic energy is then

$$K_i = \frac{1}{2} m_1 v_{1,i}^2 + \frac{1}{2}(m_1/3) v_{2,i}^2 = \frac{7}{8} m_1 v_{1,i}^2 . \tag{15.5.34}$$

The final kinetic energy is

$$K_f = \frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_2 v_{2,f}^2 = \frac{1}{8} m_1 v_{1,i}^2 + \frac{3}{4} m_1 v_{1,i}^2 = \frac{7}{8} m_1 v_{1,i}^2\ . \tag{15.5.35}$$

Comparing our results, we see that kinetic energy is constant so the collision is elastic.

15.7 Two-Dimensional Collisions in Center-of-Mass Reference Frame

15.7.1 Two-Dimensional Collision in Center-of-Mass Reference Frame

Consider the elastic collision between two particles in the laboratory reference frame (Figure 15.9). Particle 1 of mass m_1 is initially moving with velocity $\vec{\mathbf{v}}_{1,i}$ and collides elastically with a particle 2 of mass m_2 that is initially at rest. After the collision the particle 1 moves with velocity $\vec{\mathbf{v}}_{1,f}$ and particle 2 moves with velocity $\vec{\mathbf{v}}_{2,f}$. In section 15.7.1 we determined how to find $v_{1,f}$, $v_{2,f}$, and $\theta_{2,f}$ in terms of $v_{1,i}$ and $\theta_{2,f}$. We shall now analyze the collision in the center-of-mass reference frame, which is boosted form the laboratory frame by the velocity of center-of-mass given by

$$\vec{\mathbf{v}}_{cm} = \frac{m_1 \vec{\mathbf{v}}_{1,i}}{m_1 + m_2}. \qquad (15.5.36)$$

Because we assumed that there are no external forces acting on the system, the center-of-mass velocity remains constant during the interaction.

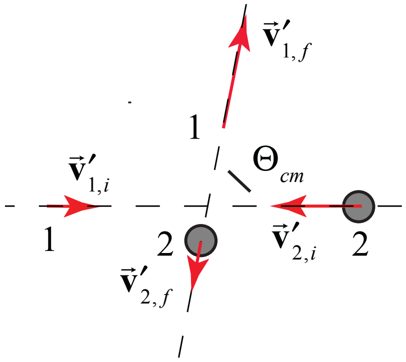


Figure 15.13 Two-dimensional elastic collision in center-of-mass reference frame

Recall the velocities of particles 1 and 2 in the center-of-mass frame are given by (Eq.,(15.2.9) and (15.2.10)). In the center-of-mass reference frame the velocities of the two incoming particles are in opposite directions, as are the velocities of the two outgoing particles after the collision (Figure 15.13). The angle Θ_{cm} between the incoming and outgoing velocities is called the **center-of-mass scattering angle**.

15.7.2 Scattering in the Center-of-Mass Reference Frame

Consider a collision between particle 1 of mass m_1 and velocity $\vec{\mathbf{v}}_{1,i}$ and particle 2 of mass m_2 at rest in the laboratory frame. Particle 1 is scattered elastically through a scattering angle Θ in the center-of-mass frame. The center-of-mass velocity is given by

$$\vec{\mathbf{v}}_{cm} = \frac{m_1 \vec{\mathbf{v}}_{1,i}}{m_1 + m_2}. \tag{15.5.37}$$

In the center-of-mass frame, the momentum of the system of two particles is zero

$$\vec{\mathbf{0}} = m_1 \vec{\mathbf{v}}'_{1,i} + m_2 \vec{\mathbf{v}}'_{2,i} = m_1 \vec{\mathbf{v}}'_{1,f} + m_2 \vec{\mathbf{v}}'_{2,f} \ . \tag{15.5.38}$$

Therefore

$$\vec{\mathbf{v}}'_{1,i} = -\frac{m_2}{m_1} \vec{\mathbf{v}}'_{2,i} \ . \tag{15.5.39}$$

$$\vec{\mathbf{v}}'_{1,f} = -\frac{m_2}{m_1} \vec{\mathbf{v}}'_{2,f} \tag{15.5.40}$$

The energy condition in the center-of-mass frame is

$$\frac{1}{2} m_1 v'^2_{1,i} + \frac{1}{2} m_2 v'^2_{2,i} = \frac{1}{2} m_1 v'^2_{1,f} + \frac{1}{2} m_2 v'^2_{2,f} . \tag{15.5.41}$$

Substituting Eqs. (15.5.39) and (15.5.40) into Eq. (15.5.41) yields

$$v'_{1,i} = v'_{1,f} . \tag{15.5.42}$$

(we are only considering magnitudes). Therefore

$$v'_{2,i} = v'_{2,f} . \tag{15.5.43}$$

Because the magnitude of the velocity of a particle in the center-of-mass reference frame is proportional to the relative velocity of the two particles, Eqs. (15.5.42) and (15.5.43) imply that the magnitude of the relative velocity also does not change

$$\left| \vec{\mathbf{v}}'_{1,2,i} \right| = \left| \vec{\mathbf{v}}'_{1,2,f} \right| , \tag{15.5.44}$$

verifying our earlier result that for an elastic collision the relative speed remains the same, (Eq. (15.2.20)). However the direction of the relative velocity is rotated by the center-of-mass scattering angle Θ_{cm}. This generalizes the energy-momentum principle to two dimensions. Recall that the relative velocity is independent of the reference frame,

$$\vec{\mathbf{v}}_{1,i} - \vec{\mathbf{v}}_{2,i} = \vec{\mathbf{v}}'_{1,i} - \vec{\mathbf{v}}'_{2,i} \tag{15.5.45}$$

In the laboratory reference frame $\vec{\mathbf{v}}_{2,i} = \vec{\mathbf{0}}$, hence the initial relative velocity is $\vec{\mathbf{v}}'_{1,2,i} = \vec{\mathbf{v}}_{1,2,i} = \vec{\mathbf{v}}_{1,i}$, and the velocities in the center-of-mass frame of the particles are then

$$\vec{\mathbf{v}}'_{1,i} = \frac{\mu}{m_1}\vec{\mathbf{v}}_{1,i} \tag{15.5.46}$$

$$\vec{\mathbf{v}}'_{2,i} = -\frac{\mu}{m_2}\vec{\mathbf{v}}_{1,i} \quad . \tag{15.5.47}$$

Therefore the magnitudes of the final velocities in the center-of-mass frame are

$$v'_{1,f} = v'_{1,i} = \frac{\mu}{m_1} v'_{1,2,i} = \frac{\mu}{m_1} v_{1,2,i} = \frac{\mu}{m_1} v_{1,i} \,. \tag{15.5.48}$$

$$v'_{2,f} = v'_{2,i} = \frac{\mu}{m_2} v'_{1,2,i} = \frac{\mu}{m_2} v_{1,2,i} = \frac{\mu}{m_2} v_{1,i} \,. \tag{15.5.49}$$

Example 15.8 Scattering in the Lab and CM Frames

Particle 1 of mass m_1 and velocity $\vec{\mathbf{v}}_{1,i}$ by a particle of mass m_2 at rest in the laboratory frame is scattered elastically through a scattering angle Θ in the center of mass frame, (Figure 15.14). Find (i) the scattering angle of the incoming particle in the laboratory frame, (ii) the magnitude of the final velocity of the incoming particle in the laboratory reference frame, and (iii) the fractional loss of kinetic energy of the incoming particle.

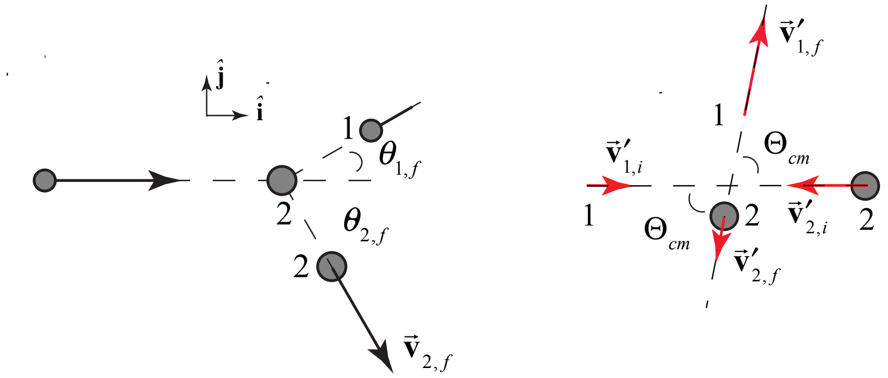


Figure 15.14 Scattering in the laboratory and center-of-mass reference frames

Solution:

i) In order to determine the center-of-mass scattering angle we use the transformation law for velocities

$$\vec{\mathbf{v}}'_{1,f} = \vec{\mathbf{v}}_{1,f} - \vec{\mathbf{v}}_{cm} \ . \tag{15.5.50}$$

In Figure 15.15 we show the collision in the center-of-mass frame along with the laboratory frame final velocities and scattering angles.

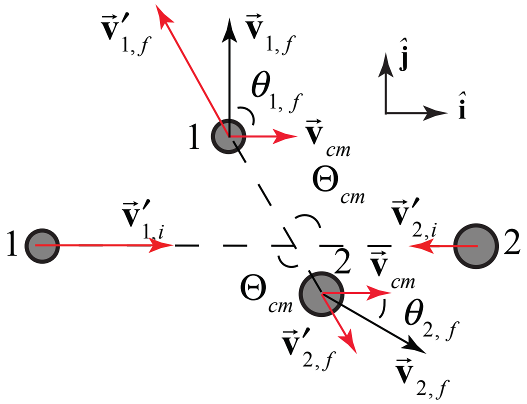


Figure 15.15 Final velocities of colliding particles

Vector decomposition of Eq. (15.5.50) yields

$$v_{1,f}\cos\theta_{1,i} = v'_{1,f}\cos\Theta_{cm} - v_{cm}\,, \tag{15.5.51}$$

$$v_{1,f}\sin\theta_{1,i} = v'_{1,f}\sin\Theta_{cm} \ . \tag{15.5.52}$$

where we choose as our directions the horizontal and vertical Divide Eq. (15.5.52) by (15.5.51) yields

$$\tan\theta_{1,i} = \frac{v_{1,f}\sin\theta_{1,i}}{v_{1,f}\cos\theta_{1,i}} = \frac{v'_{1,f}\sin\Theta_{cm}}{v'_{1,f}\cos\Theta_{cm} - v_{cm}} \tag{15.5.53}$$

Because $v'_{1,i} = v'_{1,f}$, we can rewrite Eq. (15.5.53) as

$$\tan\theta_{1,i} = \frac{v'_{1,i}\sin\Theta_{cm}}{v'_{1,i}\cos\Theta_{cm} - v_{cm}} \tag{15.5.54}$$

We now substitute Eqs. (15.5.48) and $v_{cm} = m_1 v_{1,i} / (m_1 + m_2)$ into Eq. (15.5.54) yielding

$$\tan\theta_{1,i} = \frac{m_2 \sin\Theta_{cm}}{\cos\Theta_{cm} - m_1 / m_2} \ . \tag{15.5.55}$$

Thus in the laboratory frame particle 1 scatters by an angle

$$\theta_{1,i} = \tan^{-1}\left(\frac{m_2 \sin\Theta_{cm}}{\cos\Theta_{cm} - m_1 / m_2}\right) \ . \tag{15.5.56}$$

ii) We can calculate the square of the final velocity in the laboratory frame

$$\vec{\mathbf{v}}_{1,f} \cdot \vec{\mathbf{v}}_{1,f} = (\vec{\mathbf{v}}'_{1,f} + \vec{\mathbf{v}}_{cm}) \cdot (\vec{\mathbf{v}}'_{1,f} + \vec{\mathbf{v}}_{cm}) \ . \tag{15.5.57}$$

which becomes

$$v_{1,f}{}^2 = v'_{1,f}{}^2 + 2\vec{\mathbf{v}}'_{1,f} \cdot \vec{\mathbf{v}}_{cm} + v_{cm}{}^2 = v'_{1,f}{}^2 + 2v'_{1,f} v_{cm} \cos\Theta_{cm} + v_{cm}{}^2 \ . \tag{15.5.58}$$

We use the fact that $v'_{1,f} = v'_{1,i} = (\mu / m_1) v_{1,2,i} = (\mu / m_1) v_{1,i} = (m_2 / m_1 + m_2) v_{1,i}$ to rewrite Eq. (15.5.58) as

$$v_{1,f}{}^2 = \left(\frac{m_2}{m_1 + m_2}\right)^2 v_{1,i}{}^2 + 2\frac{m_2 m_1}{(m_1 + m_2)^2} v_{1,i} \cos\Theta_{cm} + \frac{m_1{}^2}{(m_1 + m_2)^2} v_{1,i}{}^2 \ . \tag{15.5.59}$$

Thus

$$v_{1,f} = \frac{\left(m_2{}^2 + 2m_2 m_1 \cos\Theta_{cm} + m_1{}^2\right)^{1/2}}{m_1 + m_2} v_{1,i} \ . \tag{15.5.60}$$

(iii) The fractional change in the kinetic energy of particle 1 in the laboratory frame is given by

$$\frac{K_{1,f} - K_{1,i}}{K_{1,i}} = \frac{v_{1,f}{}^2 - v_{1,i}{}^2}{v_{1,i}{}^2} = \frac{m_2{}^2 + 2m_2 m_1 \cos\Theta_{cm} + m_1{}^2}{(m_1 + m_2)^2} - 1 = \frac{2m_2 m_1 (\cos\Theta_{cm} - 1)}{(m_1 + m_2)^2} \ . \tag{15.5.61}$$

We can also determine the scattering angle Θ_{cm} in the center-of-mass reference frame from the scattering angle $\theta_{1,i}$ of particle 1 in the laboratory. We now rewrite the momentum relations as

$$v_{1,f} \cos\theta_{1,i} + v_{cm} = v'_{1,f} \cos\Theta_{cm} , \tag{15.5.62}$$

$$v_{1,f} \sin\theta_{1,i} = v'_{1,f} \sin\Theta_{cm} \ . \tag{15.5.63}$$

In a similar fashion to the above argument, we have that

$$\tan\Theta_{cm} = \frac{v_{1,f}\sin\theta_{1,f}}{v_{1,f}\cos\theta_{1,f} + v_{cm}}. \qquad (15.5.64)$$

Recall from our analysis of the collision in the laboratory frame that if we specify one of the four parameters $v_{1,f}$, $v_{2,f}$, $\theta_{1,f}$, or $v_{1,f}$, then we can solve for the other three in terms of the initial parameters $v_{1,i}$ and $v_{2,i}$. With that caveat, we can use Eq. (15.5.64) to determine Θ_{cm}.

Chapter 16 Two Dimensional Rotational Kinematics

Chapter 16 Two Dimensional Rotational Kinematics

Most galaxies exhibit rising rotational velocities at the largest measured velocity; only for the very largest galaxies are the rotation curves flat. Thus the smallest SC's (i.e. lowest luminosity) exhibit the same lack of Keplerian velocity decrease at large R as do the high-luminosity spirals. The form for the rotation curves implies that the mass is not centrally condensed, but that significant mass is located at large R. The integral mass is increasing at least as fast as R. The mass is not converging to a limiting mass at the edge of the optical image. The conclusion is inescapable than non-luminous matter exists beyond the optical galaxy.[1]

Vera Rubin

16.1 Introduction

The physical objects that we encounter in the world consist of collections of atoms that are bound together to form systems of particles. When forces are applied, the shape of the body may be stretched or compressed like a spring, or sheared like jello. In some systems the constituent particles are very loosely bound to each other as in fluids and gasses, and the distances between the constituent particles will vary. We shall begin by restricting ourselves to an ideal category of objects, rigid bodies, which do not stretch, compress, or shear.

A body is called a ***rigid body*** if the distance between any two points in the body does not change in time. Rigid bodies, unlike point masses, can have forces applied at different points in the body. Let's start by considering the simplest example of rigid body motion, rotation about a fixed axis.

16.2 Fixed Axis Rotation: Rotational Kinematics

16.2.1 Fixed Axis Rotation

A simple example of rotation about a fixed axis is the motion of a compact disc in a CD player, which is driven by a motor inside the player. In a simplified model of this motion, the motor produces angular acceleration, causing the disc to spin. As the disc is set in motion, resistive forces oppose the motion until the disc no longer has any angular acceleration, and the disc now spins at a constant angular velocity. Throughout this process, the CD rotates about an axis passing through the center of the disc, and is perpendicular to the plane of the disc (see Figure 16.1). This type of motion is called ***fixed-axis rotation***.

---

[1] V.C. Rubin, W.K. Jr. Ford, N Thonnard, *Rotational properties of 21 SC galaxies with a large range of luminosities and radii, from NGC 4605 /R = 4kpc/ to UGC 2885 /R = 122 kpc/*, Astrophysical Journal, Part 1, vol. 238, June 1, 1980, p. 471-487.

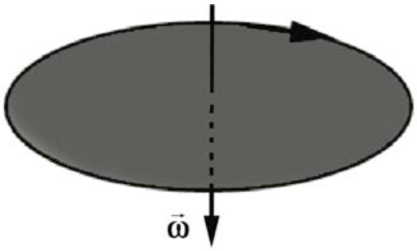


Figure 16.1 Rotation of a compact disc about a fixed axis.

When we ride a bicycle forward, the wheels rotate about an axis passing through the center of each wheel and perpendicular to the plane of the wheel (Figure 16.2). As long as the bicycle does not turn, this axis keeps pointing in the same direction. This motion is more complicated than our spinning CD because the wheel is both moving (translating) with some center of mass velocity, $\vec{\mathbf{v}}_{\text{cm}}$, and rotating with an angular speed ω.

Figure 16.2 Fixed axis rotation and center of mass translation for a bicycle wheel.

When we turn the bicycle's handlebars, we change the bike's trajectory and the axis of rotation of each wheel changes direction. Other examples of non-fixed axis rotation are the motion of a spinning top, or a gyroscope, or even the change in the direction of the earth's rotation axis. This type of motion is much harder to analyze, so we will restrict ourselves in this chapter to considering fixed axis rotation, with or without translation.

16.2.2 Angular Velocity and Angular Acceleration

For a rigid body undergoing fixed-axis rotation, we can divide the body up into small volume elements with mass Δm_i. Each of these volume elements is moving in a circle of radius r_i about the axis of rotation (Figure 16.3).

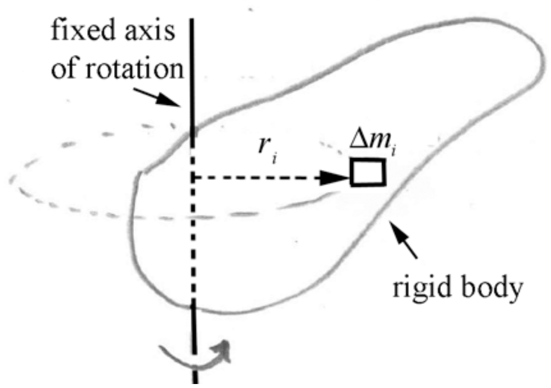


Figure 16.3 Coordinate system for fixed-axis rotation.

We will adopt the notation implied in Figure 16.3, and denote the vector from the axis to the point where the mass element is located as $\vec{\mathbf{r}}_i$, with magnitude $r_i = |\vec{\mathbf{r}}_i|$. Suppose the fixed axis of rotation is the z-axis. Introduce a right-handed coordinate system for an angle θ in the plane of rotation and the choice of the positive z-direction perpendicular to that plane of rotation. Recall our definition of the angular velocity vector. The angular velocity vector is directed along the z-axis with z-component equal to the time derivative of the angle θ,

$$\vec{\boldsymbol{\omega}} = \frac{d\theta}{dt}\hat{\mathbf{k}} = \omega_z \hat{\mathbf{k}} . \tag{16.1.1}$$

The angular velocity vector for the mass element undergoing fixed axis rotation with $\omega_z > 0$ is shown in Figure 16.4. Because the body is rigid, all the mass elements will have the same angular velocity $\vec{\boldsymbol{\omega}}$ and hence the same angular acceleration $\vec{\alpha}$. If the bodies did not have the same angular velocity, the mass elements would "catch up to" or "pass" each other, precluded by the rigid-body assumption.

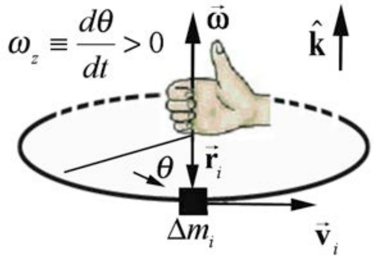


Figure 16.4 Angular velocity vector for a mass element for fixed axis rotation

In a similar fashion, all points in the rigid body have the same angular acceleration,

$$\vec{\boldsymbol{\alpha}} = \frac{d^2\theta}{dt^2}\hat{\mathbf{k}} = \alpha_z \hat{\mathbf{k}} . \tag{16.1.2}$$

The angular acceleration vector is shown in Figure 16.5.

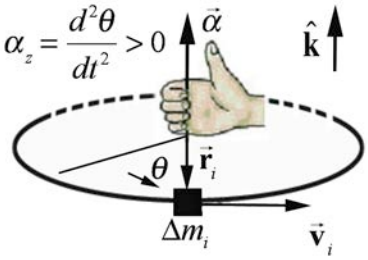


Figure 16.5 Angular acceleration vector for a rigid body rotating about the z-axis

16.2.3 Sign Convention: Angular Velocity and Angular Acceleration

For rotational problems we shall always choose a right-handed cylindrical coordinate system. If the positive z-axis points up, then we choose θ to be increasing in the counterclockwise direction as shown in Figures 16.4 and 16.5. If the rigid body rotates in the counterclockwise direction, then the z-component of the angular velocity is positive, $\omega_z = d\theta / dt > 0$. The angular velocity vector then points in the $+\hat{\mathbf{k}}$-direction as shown in Figure 16.4. If the rigid body rotates in the clockwise direction, then the z-component of the angular velocity angular velocity is negative, $\omega_z = d\theta / dt < 0$. The angular velocity vector then points in the $-\hat{\mathbf{k}}$-direction.

If the rigid body *increases* its rate of rotation in the counterclockwise (positive) direction then the z-component of the angular acceleration is positive, $\alpha_z \equiv d^2\theta/dt^2 = d\omega_z / dt > 0$. The angular acceleration vector then points in the $+\hat{\mathbf{k}}$-direction as shown in Figure 16.5. If the rigid body *decreases* its rate of rotation in the counterclockwise (positive) direction then the z-component of the angular acceleration is negative, $\alpha_z = d^2\theta / dt^2 = d\omega_z / dt < 0$. The angular acceleration vector then points in the $-\hat{\mathbf{k}}$-direction. To phrase this more generally, if $\vec{\alpha}$ and $\vec{\boldsymbol{\omega}}$ point in the same direction, the body is speeding up, if in opposite directions, the body is slowing down. This general result is independent of the choice of positive direction of rotation. Note that in Figure 16.1, the CD has the angular velocity vector points downward (in the $-\hat{\mathbf{k}}$-direction).

16.2.4 Tangential Velocity and Tangential Acceleration

Because the small element of mass, Δm_i, is moving in a circle of radius r_i with angular velocity $\vec{\boldsymbol{\omega}} = \omega_z \hat{\mathbf{k}}$, the element has a tangential velocity component

$$v_{\theta,\,i} = r_i\,\omega_z\,. \qquad (16.1.3)$$

If the magnitude of the tangential velocity is changing, the mass element undergoes a tangential acceleration given by

$$a_{\theta,\,i} = r_i \alpha_z \,. \tag{16.1.4}$$

Recall that the mass element is always accelerating inward with radial component given by

$$a_{r,\,i} = -\frac{v_{\theta,\,i}^2}{r_i} = -r_i \omega_z^2 \,. \tag{16.1.5}$$

Example 16.1 Turntable

A turntable is a uniform disc of mass $1.2\ \text{kg}$ and a radius $1.3\times 10^1\ \text{cm}$. The turntable is spinning initially in a counterclockwise direction when seen from above at a constant rate of $f_0 = 33\ \text{cycles}\cdot\text{min}^{-1}$ ($33\ \text{rpm}$). The motor is turned off and the turntable slows to a stop in $8.0\ \text{s}$. Assume that the angular acceleration is constant. (a) What is the initial angular velocity of the turntable? (b) What is the angular acceleration of the turntable?

Solution: (a) Choose a coordinate system shown in Figure 16.6.

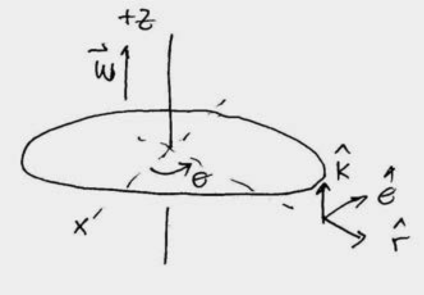


Figure 16.6 Coordinate system for turntable

Initially, the disc is spinning with a frequency

$$f_0 = \left(33\,\frac{\text{cycles}}{\text{min}}\right)\left(\frac{1\,\text{min}}{60\ \text{s}}\right) = 0.55\ \text{cycles}\cdot\text{s}^{-1} = 0.55\ \text{Hz}\,, \tag{16.1.6}$$

so the initial angular velocity has magnitude

$$\omega_0 = 2\pi\, f_0 = \left(2\pi\,\frac{\text{radian}}{\text{cycle}}\right)\left(0.55\,\frac{\text{cycles}}{\text{s}}\right) = 3.5\ \text{rad}\cdot\text{s}^{-1}\,. \tag{16.1.7}$$

The angular velocity vector points in the $+\hat{\mathbf{k}}$-direction as shown above.

(b) The final angular velocity is zero, so the component of the angular acceleration is

$$\alpha_z = \frac{\Delta\omega_z}{\Delta t} = \frac{\omega_f - \omega_0}{t_f - t_0} = \frac{-3.5\ \text{rad}\cdot\text{s}^{-1}}{8.0\ \text{s}} = -4.3\times10^{-1}\ \text{rad}\cdot\text{s}^{-2}\,. \qquad (16.1.8)$$

The z-component of the angular acceleration is negative, the disc is slowing down and so the angular acceleration vector then points in the $-\hat{\mathbf{k}}$-direction as shown in Figure 16.7.

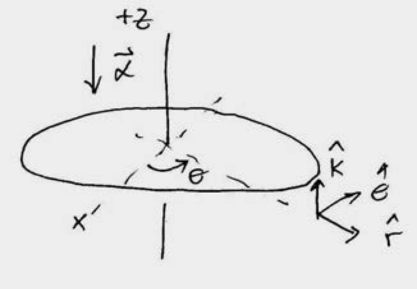


Figure 16.7 Angular acceleration vector for turntable

16.3 Rotational Kinetic Energy and Moment of Inertia

16.3.1 Rotational Kinetic Energy and Moment of Inertia

We have already defined translational kinetic energy for a point object as $K = (1/2)mv^2$; we now define the rotational kinetic energy for a rigid body about its center of mass.

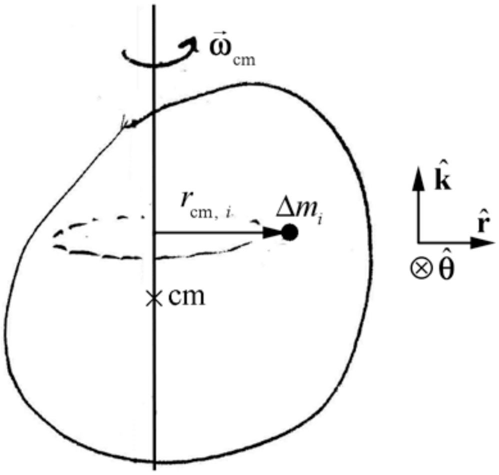


Figure 16.8 Volume element undergoing fixed-axis rotation about the z-axis that passes through the center of mass.

Choose the z-axis to lie along the axis of rotation passing through the center of mass. As in Section 16.2.2, divide the body into volume elements of mass Δm_i (Figure 16.8). Each individual mass element Δm_i undergoes circular motion about the center of mass with z-

component of angular velocity ω_{cm} in a circle of radius $r_{\mathrm{cm},\,i}$. Therefore the velocity of each element is given by $\vec{\mathbf{v}}_{\mathrm{cm},i} = r_{\mathrm{cm},i}\omega_{\mathrm{cm}}\,\hat{\boldsymbol{\theta}}$. The rotational kinetic energy is then

$$K_{\mathrm{cm},\,i} = \frac{1}{2}\Delta m_i\, v_{\mathrm{cm},\,i}^2 = \frac{1}{2}\Delta m_i r_{\mathrm{cm},\,i}^2 \omega_{\mathrm{cm}}^2 . \tag{16.2.1}$$

We now add up the kinetic energy for all the mass elements,

$$\begin{aligned} K_{\mathrm{cm}} &= \lim_{\substack{i\to\infty \\ \Delta m_i\to 0}} \sum_{i=1}^{i=N} K_{\mathrm{cm},\,i} = \lim_{\substack{i\to\infty \\ \Delta m_i\to 0}} \sum_{i=1}^{i=N} \left(\sum_i \frac{1}{2}\Delta m_i r_{\mathrm{cm},\,i}^2 \right) \omega_{\mathrm{cm}}^2 \\ &= \left(\frac{1}{2} \int_{\mathrm{body}} dm\, r_{\mathrm{dm}}^2 \right) \omega_{\mathrm{cm}}^2 , \end{aligned} \tag{16.2.2}$$

where dm is an infinitesimal mass element undergoing a circular orbit of radius r_{dm} about the axis passing through the center of mass.

> *The quantity*
>
> $$I_{cm} = \int_{body} dm\, r_{dm}^2 . \tag{16.2.3}$$
>
> *is called the* ***moment of inertia*** *of the rigid body about a fixed axis passing through the center of mass, and is a physical property of the body. The SI units for moment of inertia are* $\left[\mathrm{kg}\cdot\mathrm{m}^2\right]$.

Thus

$$K_{\mathrm{cm}} = \left(\frac{1}{2} \int_{\mathrm{body}} dm\, r_{\mathrm{dm}}^2 \right) \omega_{\mathrm{cm}}^2 \equiv \frac{1}{2} I_{cm} \omega_{\mathrm{cm}}^2 . \tag{16.2.4}$$

16.3.2 Moment of Inertia of a Rod of Uniform Mass Density

Consider a thin uniform rod of length L and mass m. In this problem, we will calculate the moment of inertia about an axis perpendicular to the rod that passes through the center of mass of the rod. A sketch of the rod, volume element, and axis is shown in Figure 16.9. Choose Cartesian coordinates, with the origin at the center of mass of the rod, which is midway between the endpoints since the rod is uniform. Choose the x-axis to lie along the length of the rod, with the positive x-direction to the right, as in the figure.

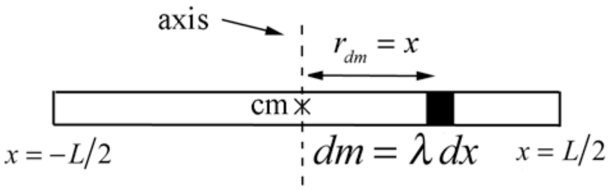


Figure 16.9 Moment of inertia of a uniform rod about center of mass.

Identify an infinitesimal mass element $dm = \lambda\, dx$, located at a displacement x from the center of the rod, where the mass per unit length $\lambda = m/L$ is a constant, as we have assumed the rod to be uniform. When the rod rotates about an axis perpendicular to the rod that passes through the center of mass of the rod, the element traces out a circle of radius $r_{dm} = x$. We add together the contributions from each infinitesimal element as we go from $x=-L/2$ to $x = L/2$. The integral is then

$$\begin{aligned} I_{\text{cm}} &= \int_{\text{body}} r_{\text{dm}}^2\, dm = \lambda \int_{-L/2}^{L/2} (x^2)\, dx = \lambda \left. \frac{x^3}{3} \right|_{-L/2}^{L/2} \\ &= \frac{m}{L}\frac{(L/2)^3}{3} - \frac{m}{L}\frac{(-L/2)^3}{3} = \frac{1}{12} m L^2. \end{aligned} \quad (16.2.5)$$

By using a constant mass per unit length along the rod, we need not consider variations in the mass density in any direction other than the x- axis. We also assume that the width is the rod is negligible. (Technically we should treat the rod as a cylinder or a rectangle in the x-y plane if the axis is along the z - axis. The calculation of the moment of inertia in these cases would be more complicated.)

Example 16.2 Moment of Inertia of a Uniform Disc

A thin uniform disc of mass M and radius R is mounted on an axle passing through the center of the disc, perpendicular to the plane of the disc. Calculate the moment of inertia about an axis that passes perpendicular to the disc through the center of mass of the disc

Solution: As a starting point, consider the contribution to the moment of inertia from the mass element dm show in Figure 16.10. Let r denote the distance form the center of mass of the disc to the mass element.

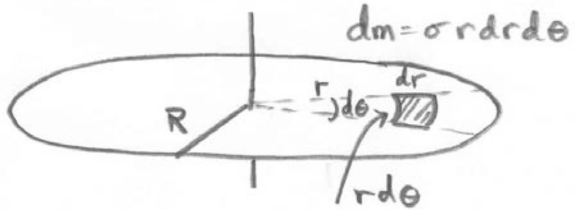


Figure 16.10 Infinitesimal mass element and coordinate system for disc.

Choose cylindrical coordinates with the coordinates (r,θ) in the plane and the z-axis perpendicular to the plane. The area element

$$da = r\,dr\,d\theta \tag{16.2.6}$$

may be thought of as the product of arc length $r\,d\theta$ and the radial width dr. Since the disc is uniform, the mass per unit area is a constant,

$$\sigma = \frac{dm}{da} = \frac{m_{\text{total}}}{\text{Area}} = \frac{M}{\pi R^2}\,. \tag{16.2.7}$$

Therefore the mass in the infinitesimal area element as given in Equation (16.2.6), a distance r from the axis of rotation, is given by

$$dm = \sigma\, r\,dr\,d\theta = \frac{M}{\pi R^2} r\,dr\,d\theta\,. \tag{16.2.8}$$

When the disc rotates, the mass element traces out a circle of radius $r_{dm} = r$; that is, the distance from the center is the perpendicular distance from the axis of rotation. The moment of inertia integral is now an integral in two dimensions; the angle θ varies from $\theta = 0$ to $\theta = 2\pi$, and the radial coordinate r varies from $r = 0$ to $r = R$. Thus the limits of the integral are

$$I_{\text{cm}} = \int\limits_{\text{body}} r_{dm}^2\, dm = \frac{M}{\pi R^2}\int_{r=0}^{r=R}\int_{\theta=0}^{\theta=2\pi} r^3\, d\theta\, dr\,. \tag{16.2.9}$$

The integral can now be explicitly calculated by first integrating the θ-coordinate

$$I_{\text{cm}} = \frac{M}{\pi R^2}\int_{r=0}^{r=R}\left(\int_{\theta=0}^{\theta=2\pi} d\theta\right) r^3 dr = \frac{M}{\pi R^2}\int_{r=0}^{r=R} 2\pi r^3 dr = \frac{2M}{R^2}\int_{r=0}^{r=R} r^3 dr \tag{16.2.10}$$

and then integrating the r-coordinate,

$$I_{\text{cm}} = \frac{2M}{R^2}\int_{r=0}^{r=R} r^3 dr = \frac{2M}{R^2}\frac{r^4}{4}\Bigg|_{r=0}^{r=R} = \frac{2M}{R^2}\frac{R^4}{4} = \frac{1}{2}MR^2\,. \tag{16.2.11}$$

Remark: Instead of taking the area element as a small patch $da = r\,dr\,d\theta$, choose a ring of radius r and width dr. Then the area of this ring is given by

$$da_{\text{ring}} = \pi(r + dr)^2 - \pi r^2 = \pi r^2 + 2\pi r\,dr + \pi(dr)^2 - \pi r^2 = 2\pi r\,dr + \pi(dr)^2\,. \tag{16.2.12}$$

In the limit that $dr \to 0$, the term proportional to $(dr)^2$ can be ignored and the area is $da = 2\pi r dr$. This equivalent to first integrating the $d\theta$ variable

$$da_{\text{ring}} = r\,dr\left(\int_{\theta=0}^{\theta=2\pi} d\theta\right) = 2\pi r\,dr\,. \tag{16.2.13}$$

Then the mass element is

$$dm_{\text{ring}} = \sigma da_{\text{ring}} = \frac{M}{\pi R^2} 2\pi r\,dr\,. \tag{16.2.14}$$

The moment of inertia integral is just an integral in the variable r,

$$I_{\text{cm}} = \int_{\text{body}} (r_\perp)^2\,dm = \frac{2\pi M}{\pi R^2}\int_{r=0}^{r=R} r^3 dr = \frac{1}{2} MR^2\,. \tag{16.2.15}$$

16.3.3 Parallel Axis Theorem

Consider a rigid body of mass m undergoing fixed-axis rotation. Consider two parallel axes. The first axis passes through the center of mass of the body, and the moment of inertia about this first axis is I_{cm}. The second axis passes through some other point S in the body. Let $d_{S,\text{cm}}$ denote the perpendicular distance between the two parallel axes (Figure 16.11).

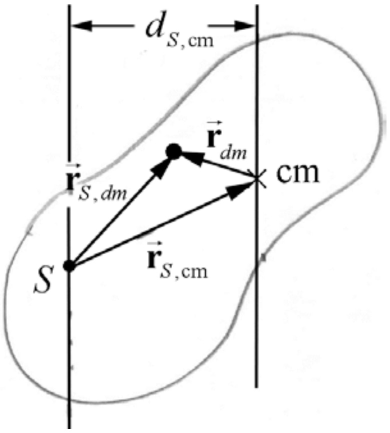


Figure 16.11 Geometry of the parallel axis theorem.

Then the moment of inertia I_S about an axis passing through a point S is related to I_{cm} by

$$I_S = I_{\text{cm}} + m\,d_{S,\text{cm}}^2\,. \tag{16.2.16}$$

16.3.4 Parallel Axis Theorem Applied to a Uniform Rod

Let point S be the left end of the rod of Figure 16.9. Then the distance from the center of mass to the end of the rod is $d_{S,\text{cm}} = L/2$. The moment of inertia $I_S = I_{\text{end}}$ about an axis passing through the endpoint is related to the moment of inertia about an axis passing through the center of mass, $I_{\text{cm}} = (1/12)\,m\,L^2$, according to Equation (16.2.16),

$$I_S = \frac{1}{12}m\,L^2 + \frac{1}{4}m\,L^2 = \frac{1}{3}m\,L^2 . \qquad (16.2.17)$$

In this case it's easy and useful to check by direct calculation. Use Equation (16.2.5) but with the limits changed to $x' = 0$ and $x' = L$, where $x' = x + L/2$,

$$\begin{aligned} I_{\text{end}} &= \int\limits_{\text{body}} r_{\perp}{}^2\, dm = \lambda \int_0^L x'^2\, dx' \\ &= \lambda \frac{x'^3}{3}\bigg|_0^L = \frac{m}{L}\frac{(L)^3}{3} - \frac{m}{L}\frac{(0)^3}{3} = \frac{1}{3}m\,L^2. \end{aligned} \qquad (16.2.18)$$

Example 16.3 Rotational Kinetic Energy of Disk

A disk with mass M and radius R is spinning with angular speed ω about an axis that passes through the rim of the disk perpendicular to its plane. The moment of inertia about cm is $I_{cm} = (1/2)mR^2$. What is the kinetic energy of the disk?

Solution: The parallel axis theorem states the moment of inertia about an axis passing perpendicular to the plane of the disc and passing through a point on the edge of the disc is equal to

$$I_{edge} = I_{cm} + mR^2 . \qquad (16.2.19)$$

The moment of inertia about an axis passing perpendicular to the plane of the disc and passing through the center of mass of the disc is equal to $I_{cm} = (1/2)mR^2$. Therefore

$$I_{edge} = (3/2)mR^2 . \qquad (16.2.20)$$

The kinetic energy is then

$$K = (1/2)I_{edge}\omega^2 = (3/4)mR^2\omega^2 . \qquad (16.2.21)$$

16.4 Conservation of Energy for Fixed Axis Rotation

Consider a closed system ($\Delta E_{system} = 0$) under action of only conservative internal forces. Then the change in the mechanical energy of the system is zero

$$\Delta E_m = \Delta U + \Delta K = (U_f + K_f) - (U_i + K_i) = 0 . \tag{16.3.1}$$

For fixed axis rotation with a component of angular velocity ω about the fixed axis, the change in kinetic energy is given by

$$\Delta K \equiv K_f - K_i = \frac{1}{2} I_S \omega_f^2 - \frac{1}{2} I_S \omega_i^2 , \tag{16.3.2}$$

where S is a point that lies on the fixed axis. Then conservation of energy implies that

$$U_f + \frac{1}{2} I_S \omega_f^2 = U_i + \frac{1}{2} I_S \omega_i^2 \tag{16.3.3}$$

Example 16.4 Energy and Pulley System

A wheel in the shape of a uniform disk of radius R and mass m_p is mounted on a frictionless horizontal axis. The wheel has moment of inertia about the center of mass $I_\mathrm{cm} = (1/2) m_\mathrm{p} R^2$. A massless cord is wrapped around the wheel and one end of the cord is attached to an object of mass m_2 that can slide up or down a frictionless inclined plane. The other end of the cord is attached to a second object of mass m_1 that hangs over the edge of the inclined plane. The plane is inclined from the horizontal by an angle θ (Figure 16.12). Once the objects are released from rest, the cord moves without slipping around the disk. Calculate the speed of block 2 as a function of distance that it moves down the inclined plane using energy techniques. Assume there are no energy losses due to friction and that the rope does not slip around the pulley

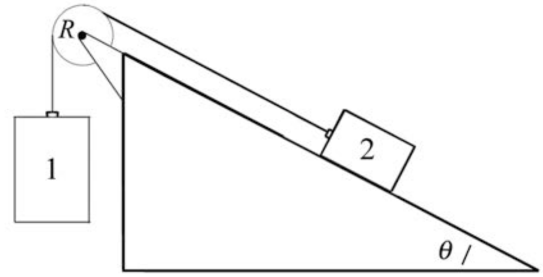


Figure 16.12 Pulley and blocks

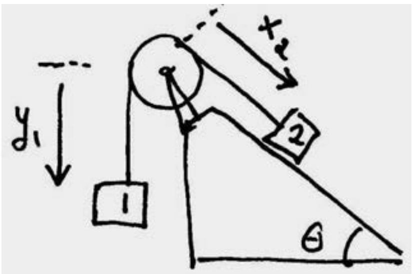
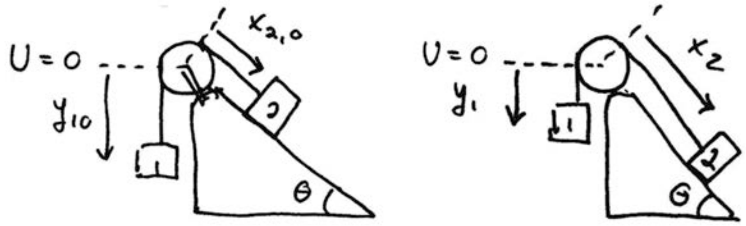


Figure 16.13 Coordinate system for pulley and blocks

Solution: Define a coordinate system as shown in Figure 16.13. Choose the zero for the gravitational potential energy at a height equal to the center of the pulley. In Figure 16.14 illustrates the energy diagrams for the initial state and a dynamic state at an arbitrary time when the blocks are sliding.



initial state dynamic state

Figure 16.14 Energy diagrams for initial state and dynamic state at arbitrary time

Then the initial mechanical energy is

$$E_i = U_i = -m_1 g y_{1,i} - m_2 g x_{2,i} \sin\theta \,. \tag{16.3.4}$$

The mechanical energy, when block 2 has moved a distance

$$d = x_2 - x_{2,i} \tag{16.3.5}$$

is given by

$$E = U + K = -m_1 g y_1 - m_2 g x_2 \sin\theta + \frac{1}{2} m_1 v_1{}^2 + \frac{1}{2} m_2 v_2{}^2 + \frac{1}{2} I_P \omega^2 \,. \tag{16.3.6}$$

The rope connects the two blocks, and so the blocks move at the same speed

$$v \equiv v_1 = v_2 \,. \tag{16.3.7}$$

The rope does not slip on the pulley; therefore as the rope moves around the pulley the tangential speed of the rope is equal to the speed of the blocks

$$v_{\tan} = R\omega = v \,. \tag{16.3.8}$$

Eq. (16.3.6) can now be simplified

$$E = U + K = -m_1 g y_1 - m_2 g x_2 \sin\theta + \frac{1}{2}\left(m_1 + m_2 + \frac{I_P}{R^2} \right) v^2 \,. \tag{16.3.9}$$

Because we have assumed that there is no loss of mechanical energy, we can set $E_i = E$ and find that

$$-m_1 g y_{1,i} - m_2 g x_{2,i} \sin\theta = -m_1 g y_1 - m_2 g x_2 \sin\theta + \frac{1}{2}\left(m_1 + m_2 + \frac{I_P}{R^2} \right) v^2, \quad (16.3.10)$$

which simplifies to

$$-m_1 g(y_{1,0} - y_1) + m_2 g(x_2 - x_{2,0}) \sin\theta = \frac{1}{2}\left(m_1 + m_2 + \frac{I_P}{R^2} \right) v^2. \quad (16.3.11)$$

We finally note that the movement of block 1 and block 2 are constrained by the relationship

$$d = x_2 - x_{2,i} = y_{1,i} - y_1. \quad (16.3.12)$$

Then Eq. (16.3.11) becomes

$$gd(-m_1 + m_2 \sin\theta) = \frac{1}{2}\left(m_1 + m_2 + \frac{I_P}{R^2} \right) v^2. \quad (16.3.13)$$

We can now solve for the speed as a function of distance $d = x_2 - x_{2,i}$ that block 2 has traveled down the incline plane

$$v = \sqrt{\frac{2gd(-m_1 + m_2 \sin\theta)}{\left(m_1 + m_2 + (I_P / R^2) \right)}}. \quad (16.3.14)$$

If we assume that the moment of inertial of the pulley is $I_{\text{cm}} = (1/2) m_{\text{p}} R^2$, then the speed becomes

$$v = \sqrt{\frac{2gd(-m_1 + m_2 \sin\theta)}{\left(m_1 + m_2 + (1/2) m_P \right)}}. \quad (16.3.15)$$

Example 16.5 Physical Pendulum

A physical pendulum consists of a uniform rod of mass m_1 pivoted at one end about the point S. The rod has length l_1 and moment of inertia I_1 about the pivot point. A disc of mass m_2 and radius r_2 with moment of inertia I_{cm} about its center of mass is rigidly attached a distance l_2 from the pivot point. The pendulum is initially displaced to an angle θ_i and then released from rest. (a) What is the moment of inertia of the physical pendulum about the pivot point S? (b) How far from the pivot point is the center of mass of the system? (c) What is the angular speed of the pendulum when the pendulum is at the bottom of its swing?

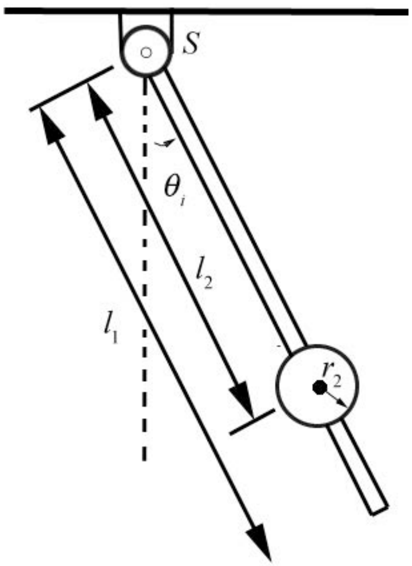


Figure 16.15 Rod and with fixed disc pivoted about the point S

Solution: a) The moment of inertia about the pivot point is the sum of the moment of inertia of the rod, given as I_1, and the moment of inertia of the disc about the pivot point. The moment of inertia of the disc about the pivot point is found from the parallel axis theorem,

$$I_{\text{disc}} = I_{\text{cm}} + m_2 l_2^2 . \tag{16.3.16}$$

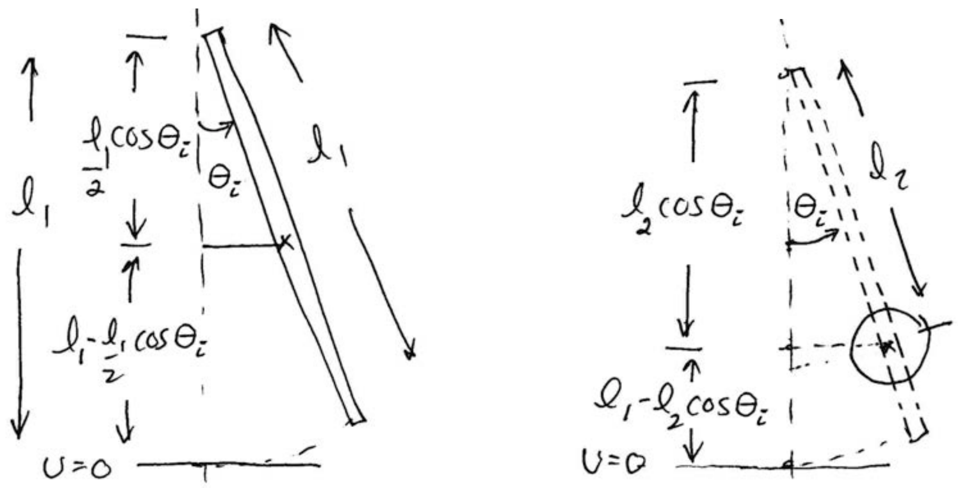
The moment of inertia of the system consisting of the rod and disc about the pivot point S is then

$$I_S = I_1 + I_{\text{disc}} = I_1 + I_{\text{cm}} + m_2 l_2^2 . \tag{16.3.17}$$

The center of mass of the system is located a distance from the pivot point

$$l_{\text{cm}} = \frac{m_1(l_1 / 2) + m_2 l_2}{m_1 + m_2} . \tag{16.3.18}$$

b) We can use conservation of mechanical energy, to find the angular speed of the pendulum at the bottom of its swing. Take the zero point of gravitational potential energy to be the point where the bottom of the rod is at its lowest point, that is, $\theta = 0$. The initial state energy diagram for the rod is shown in Figure 16.16a and the initial state energy diagram for the disc is shown in Figure 16.16b.



(a) (b)

Figure 16.16 (a) Initial state energy diagram for rod (b) Initial state energy diagram for disc

The initial mechanical energy is then

$$E_i = U_i = m_1 g(l_1 - \frac{l_1}{2}\cos\theta_i) + m_2 g(l_1 - l_2 \cos\theta_i)\,, \tag{16.3.19}$$

At the bottom of the swing, $\theta_f = 0$, and the system has angular velocity ω_f. The mechanical energy at the bottom of the swing is

$$E_f = U_f + K_f = m_1 g\frac{l_1}{2} + m_2 g(l_1 - l_2) + \frac{1}{2}I_S\omega_f^2\,, \tag{16.3.20}$$

with I_S as found in Equation (16.3.17). There are no non-conservative forces acting, so the mechanical energy is constant therefore equating the expressions in (16.3.19) and (16.3.20) we get that

$$m_1 g(l_1 - \frac{l_1}{2}\cos\theta_i) + m_2 g(l_1 - l_2\cos\theta_i) = m_1 g\frac{l_1}{2} + m_2 g(l_1 - l_2) + \frac{1}{2}I_S\omega_f^2\,, \tag{16.3.21}$$

This simplifies to

$$\left(\frac{m_1 l_1}{2} + m_2 l_2\right)g(1-\cos\theta_i) = \frac{1}{2}I_S\omega_f^2\,, \tag{16.3.22}$$

We now solve for ω_f (taking the positive square root to insure that we are calculating angular speed)

$$\omega_f = \sqrt{\frac{2\left(\dfrac{m_1 l_1}{2} + m_2 l_2\right) g(1-\cos\theta_i)}{I_S}}, \tag{16.3.23}$$

Finally we substitute in Eq.(16.3.17) in to Eq. (16.3.23) and find

$$\omega_f = \sqrt{\frac{2\left(\dfrac{m_1 l_1}{2} + m_2 l_2\right) g(1-\cos\theta_i)}{I_1 + I_{\text{cm}} + m_2 l_2^2}}. \tag{16.3.24}$$

Note that we can rewrite Eq. (16.3.22), using Eq. (16.3.18) for the distance between the center of mass and the pivot point, to get

$$(m_1 + m_2) l_{cm} g(1-\cos\theta_i) = \frac{1}{2} I_S \omega_f^2, \tag{16.3.25}$$

We can interpret this equation as follows. Treat the system as a point particle of mass $m_1 + m_2$ located at the center of mass l_{cm}. Take the zero point of gravitational potential energy to be the point where the center of mass is at its lowest point, that is, $\theta = 0$. Then

$$E_i = (m_1 + m_2) l_{cm} g(1-\cos\theta_i), \tag{16.3.26}$$

$$E_f = \frac{1}{2} I_S \omega_f^2. \tag{16.3.27}$$

Thus conservation of energy reproduces Eq. (16.3.25).

Appendix 16A: Proof of the Parallel Axis Theorem

Identify an infinitesimal volume element of mass dm. The vector from the point S to the mass element is $\vec{\mathbf{r}}_{S,dm}$, the vector from the center of mass to the mass element is $\vec{\mathbf{r}}_{dm}$, and the vector from the point S to the center of mass is $\vec{\mathbf{r}}_{S,\mathrm{cm}}$.

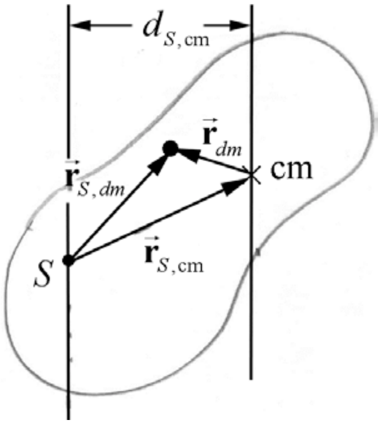


Figure 16A.1 Geometry of the parallel axis theorem.

From Figure 16A.1, we see that

$$\vec{\mathbf{r}}_{S,dm} = \vec{\mathbf{r}}_{S,\mathrm{cm}} + \vec{\mathbf{r}}_{dm} . \tag{16.A.1}$$

The notation gets complicated at this point. The vector $\vec{\mathbf{r}}_{dm}$ has a component vector $\vec{\mathbf{r}}_{\parallel,dm}$ parallel to the axis through the center of mass and a component vector $\vec{\mathbf{r}}_{\perp,dm}$ perpendicular to the axis through the center of mass. The magnitude of the perpendicular component vector is

$$\left|\vec{\mathbf{r}}_{\mathrm{cm},\perp,dm}\right| = r_{\perp,dm} . \tag{16.A.2}$$

The vector $\vec{\mathbf{r}}_{S,dm}$ has a component vector $\vec{\mathbf{r}}_{S,\parallel,dm}$ parallel to the axis through the point S and a component vector $\vec{\mathbf{r}}_{S,\perp,dm}$ perpendicular to the axis through the point S. The magnitude of the perpendicular component vector is

$$\left|\vec{\mathbf{r}}_{S,\perp,dm}\right| = r_{S,\perp,dm} . \tag{16.A.3}$$

The vector $\vec{\mathbf{r}}_{S,\mathrm{cm}}$ has a component vector $\vec{\mathbf{r}}_{S,\parallel,\mathrm{cm}}$ parallel to *both* axes and a perpendicular component vector $\vec{\mathbf{r}}_{S,\perp,\mathrm{cm}}$ that is perpendicular to *both* axes (the axes are parallel, of course). The magnitude of the perpendicular component vector is

$$\left|\vec{\mathbf{r}}_{S,\perp,\mathrm{cm}}\right| = d_{S,\mathrm{cm}} . \tag{16.A.4}$$

Equation (16.A.1) is now expressed as two equations,

$$\begin{aligned}\vec{\mathbf{r}}_{S,\perp,dm} &= \vec{\mathbf{r}}_{S,\perp,\text{cm}} + \vec{\mathbf{r}}_{\perp,dm}\\ \vec{\mathbf{r}}_{S,\|,dm} &= \vec{\mathbf{r}}_{S,\|,\text{cm}} + \vec{\mathbf{r}}_{\|,dm}.\end{aligned} \tag{16.A.5}$$

At this point, note that if we had simply decided that the two parallel axes are parallel to the z-direction, we could have saved some steps and perhaps spared some of the notation with the triple subscripts. However, we want a more general result, one valid for cases where the axes are not fixed, or when different objects in the same problem have different axes. For example, consider the turning bicycle, for which the two wheel axes will not be parallel, or a spinning top that *precesses* (wobbles). Such cases will be considered in later on, and we will show the general case of the parallel axis theorem in anticipation of use for more general situations.

The moment of inertia about the point S is

$$I_S = \int\limits_{\text{body}} dm (r_{S,\perp,dm})^2 . \tag{16.A.6}$$

From (16.A.5) we have

$$\begin{aligned}(r_{S,\perp,dm})^2 &= \vec{\mathbf{r}}_{S,\perp,dm} \cdot \vec{\mathbf{r}}_{S,\perp,dm}\\ &= (\vec{\mathbf{r}}_{S,\perp,\text{cm}} + \vec{\mathbf{r}}_{\perp,dm}) \cdot (\vec{\mathbf{r}}_{S,\perp,\text{cm}} + \vec{\mathbf{r}}_{\perp,dm})\\ &= d_{S,\text{cm}}^2 + (r_{\perp,dm})^2 + 2\,\vec{\mathbf{r}}_{S,\perp,\text{cm}} \cdot \vec{\mathbf{r}}_{\perp,dm}.\end{aligned} \tag{16.A.7}$$

Thus we have for the moment of inertia about S,

$$I_S = \int\limits_{\text{body}} dm\, d_{S,\text{cm}}^2 + \int\limits_{\text{body}} dm (r_{\perp,dm})^2 + 2 \int\limits_{\text{body}} dm (\vec{\mathbf{r}}_{S,\perp,\text{cm}} \cdot \vec{\mathbf{r}}_{\perp,dm}) . \tag{16.A.8}$$

In the first integral in Equation (16.A.8), $r_{S,\perp,\text{cm}} = d_{S,\text{cm}}$ is the distance between the parallel axes and is a constant. Therefore we can rewrite the integral as

$$d_{S,\text{cm}}^2 \int\limits_{\text{body}} dm \; = m\, d_{S,\text{cm}}^2 . \tag{16.A.9}$$

The second term in Equation (16.A.8) is the moment of inertia about the axis through the center of mass,

$$I_{\text{cm}} = \int\limits_{\text{body}} dm\, (r_{\perp,dm})^2 . \tag{16.A.10}$$

The third integral in Equation (16.A.8) is zero. To see this, note that the term $\vec{\mathbf{r}}_{S,\perp,\mathrm{cm}}$ is a constant and may be taken out of the integral,

$$2\int_{\mathrm{body}} dm\,(\vec{\mathbf{r}}_{S,\perp,\mathrm{cm}}\cdot\vec{\mathbf{r}}_{\perp,dm}) = \vec{\mathbf{r}}_{S,\perp,\mathrm{cm}}\cdot 2\int_{\mathrm{body}} dm\,\vec{\mathbf{r}}_{\perp,dm} \tag{16.A.11}$$

The integral $\int_{\mathrm{body}} dm\ \vec{\mathbf{r}}_{\perp,dm}$ is the perpendicular component of the position of the center of mass with respect to the center of mass, and hence $\vec{\mathbf{0}}$, with the result that

$$2\int_{\mathrm{body}} dm\,(\vec{\mathbf{r}}_{S,\perp,\mathrm{cm}}\cdot\vec{\mathbf{r}}_{\perp,dm}) = 0\,. \tag{16.A.12}$$

Thus, the moment of inertia about S is just the sum of the first two integrals in Equation (16.A.8)

$$I_S = I_{\mathrm{cm}} + m\,d_{S,\mathrm{cm}}^2\,, \tag{16.A.13}$$

proving the parallel axis theorem.

Chapter 17 Two Dimensional Rotational Dynamics

torque, n.

a. The twisting or rotary force in a piece of mechanism (as a measurable quantity); the moment of a system of forces producing rotation.

Oxford English Dictionary

17.1 Introduction

A body is called a *rigid body* if the distance between any two points in the body does not change in time. Rigid bodies, unlike point masses, can have forces applied at different points in the body. For most objects, treating as a rigid body is an idealization, but a very good one. In addition to forces applied at points, forces may be distributed over the entire body. Forces that are distributed over a body are difficult to analyze; however, for example, we regularly experience the effect of the gravitational force on bodies. Based on our experience observing the effect of the gravitational force on rigid bodies, we shall demonstrate that the gravitational force can be concentrated at a point in the rigid body called the *center of gravity*, which for small bodies (so that $\vec{\mathbf{g}}$ may be taken as constant within the body) is identical to the *center of mass* of the body.

Let's consider a rigid rod thrown in the air (Figure 17.1) so that the rod is spinning as its center of mass moves with velocity $\vec{\mathbf{v}}_{\text{cm}}$. We have explored the physics of translational motion; now, we wish to investigate the properties of rotational motion exhibited in the rod's motion, beginning with the notion that every particle is rotating about the center of mass with the same angular (rotational) velocity.

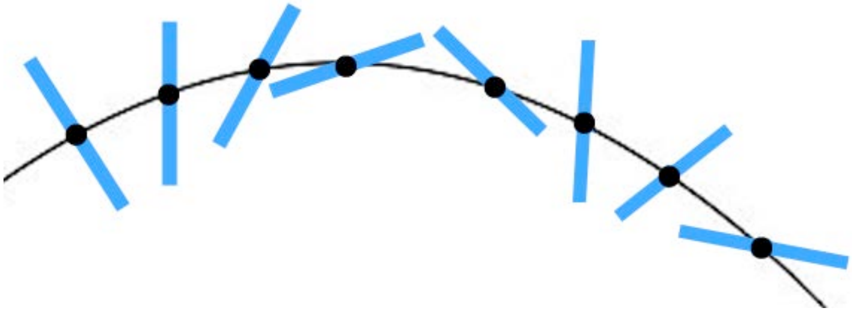


Figure 17.1 The center of mass of a thrown rigid rod follows a parabolic trajectory while the rod rotates about the center of mass.

We can use Newton's Second Law to predict how the center of mass will move. Because the only external force on the rod is the gravitational force (neglecting the action of air resistance), the center of mass of the body will move in a parabolic trajectory.

How was the rod induced to rotate? In order to spin the rod, we applied a torque with our fingers and wrist to one end of the rod as the rod was released. The applied torque is proportional to the angular acceleration. The constant of proportionality is the moment of inertia. When external forces and torques are present, the motion of a rigid body can be extremely complicated while it is translating and rotating in space.

In order to describe the relationship between torque, moment of inertia, and angular acceleration, we will introduce a new vector operation called the ***vector product*** also know as the "cross product" that takes any two vectors and generates a new vector. The vector product is a type of "multiplication" law that turns our vector space (law for addition of vectors) into a vector algebra (a vector algebra is a vector space with an additional rule for multiplication of vectors).

17.2 Vector Product (Cross Product)

Let $\vec{\mathbf{A}}$ *and* $\vec{\mathbf{B}}$ *be two vectors. Because any two non-parallel vectors form a plane, we denote the angle* θ *to be the angle between the vectors* $\vec{\mathbf{A}}$ *and* $\vec{\mathbf{B}}$ *as shown in Figure 17.2. The* ***magnitude of the vector product*** $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ *of the vectors* $\vec{\mathbf{A}}$ *and* $\vec{\mathbf{B}}$ *is defined to be product of the magnitude of the vectors* $\vec{\mathbf{A}}$ *and* $\vec{\mathbf{B}}$ *with the sine of the angle* θ *between the two vectors,*

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|\sin(\theta). \qquad (17.2.1)$$

The angle θ *between the vectors is limited to the values* $0 \le \theta \le \pi$ *ensuring that* $\sin(\theta) \ge 0$.

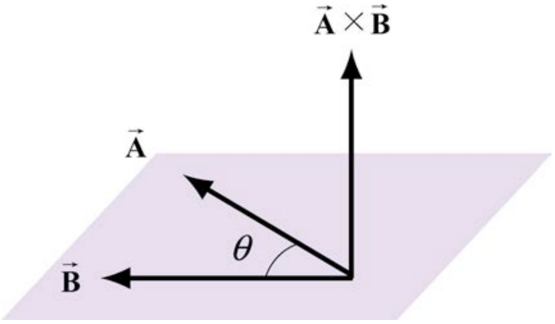


Figure 17.2 Vector product geometry.

The direction of the vector product is defined as follows. The vectors $\vec{\mathbf{A}}$ *and* $\vec{\mathbf{B}}$ *form a plane. Consider the direction perpendicular to this plane. There are two possibilities: we shall choose one of these two (the one shown in Figure 17.2) for the direction of the vector product* $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ *using a convention that is commonly called the "**right-hand rule**".*

17.2.1 Right-hand Rule for the Direction of Vector Product

The first step is to redraw the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ so that the tails are touching. Then draw an arc starting from the vector $\vec{\mathbf{A}}$ and finishing on the vector $\vec{\mathbf{B}}$. Curl your right fingers the same way as the arc. Your right thumb points in the direction of the vector product $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ (Figure 17.3).

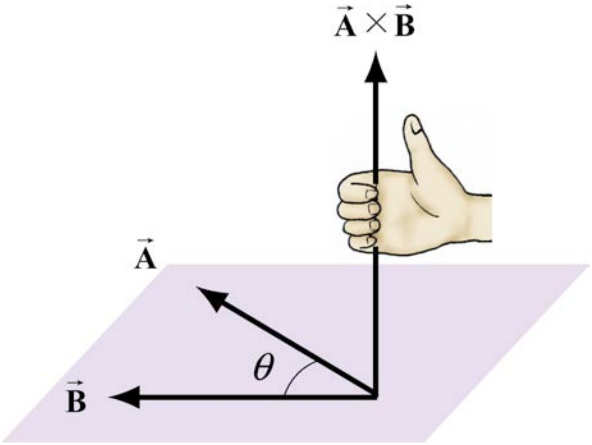


Figure 17.3 Right-Hand Rule.

You should remember that the direction of the vector product $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ is perpendicular to the plane formed by $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. We can give a geometric interpretation to the magnitude of the vector product by writing the magnitude as

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\right|\left(\left|\vec{\mathbf{B}}\right|\sin\theta\right). \tag{17.2.2}$$

The vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ form a parallelogram. The area of the parallelogram is equal to the height times the base, which is the magnitude of the vector product. In Figure 17.4, two different representations of the height and base of a parallelogram are illustrated. As depicted in Figure 17.4a, the term $\left|\vec{\mathbf{B}}\right|\sin\theta$ is the projection of the vector $\vec{\mathbf{B}}$ in the direction perpendicular to the vector $\vec{\mathbf{B}}$. We could also write the magnitude of the vector product as

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \left(\left|\vec{\mathbf{A}}\right|\sin\theta\right)\left|\vec{\mathbf{B}}\right|. \tag{17.2.3}$$

The term $\left|\vec{\mathbf{A}}\right|\sin\theta$ is the projection of the vector $\vec{\mathbf{A}}$ in the direction perpendicular to the vector $\vec{\mathbf{B}}$ as shown in Figure 17.4(b). The vector product of two vectors that are parallel (or anti-parallel) to each other is zero because the angle between the vectors is 0 (or π) and $\sin(0)=0$ (or $\sin(\pi)=0$). Geometrically, two parallel vectors do not have a unique component perpendicular to their common direction.

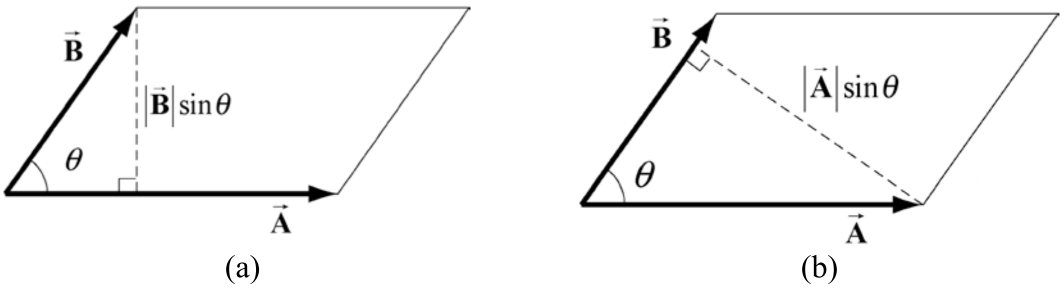


Figure 17.4 Projection of (a) $\vec{\mathbf{B}}$ perpendicular to $\vec{\mathbf{A}}$, (b) of $\vec{\mathbf{A}}$ perpendicular to $\vec{\mathbf{B}}$

17.2.2 Properties of the Vector Product

(1) The vector product is anti-commutative because changing the order of the vectors changes the direction of the vector product by the right hand rule:

$$\vec{\mathbf{A}}\times\vec{\mathbf{B}} = -\vec{\mathbf{B}}\times\vec{\mathbf{A}}\,. \qquad (17.2.4)$$

(2) The vector product between a vector $c\vec{\mathbf{A}}$ where c is a scalar and a vector $\vec{\mathbf{B}}$ is

$$c\vec{\mathbf{A}}\times\vec{\mathbf{B}} = c(\vec{\mathbf{A}}\times\vec{\mathbf{B}})\,. \qquad (17.2.5)$$

Similarly,

$$\vec{\mathbf{A}}\times c\vec{\mathbf{B}} = c(\vec{\mathbf{A}}\times\vec{\mathbf{B}})\,. \qquad (17.2.6)$$

(3) The vector product between the sum of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with a vector $\vec{\mathbf{C}}$ is

$$(\vec{\mathbf{A}}+\vec{\mathbf{B}})\times\vec{\mathbf{C}} = \vec{\mathbf{A}}\times\vec{\mathbf{C}}+\vec{\mathbf{B}}\times\vec{\mathbf{C}} \qquad (17.2.7)$$

Similarly,

$$\vec{\mathbf{A}}\times(\vec{\mathbf{B}}+\vec{\mathbf{C}}) = \vec{\mathbf{A}}\times\vec{\mathbf{B}}+\vec{\mathbf{A}}\times\vec{\mathbf{C}}\,. \qquad (17.2.8)$$

17.2.3 Vector Decomposition and the Vector Product: Cartesian Coordinates

We first calculate that the magnitude of vector product of the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$:

$$|\,\hat{\mathbf{i}}\times\hat{\mathbf{j}}\,|=|\,\hat{\mathbf{i}}\,||\,\hat{\mathbf{j}}\,|\sin(\pi/2)=1, \qquad (17.2.9)$$

because the unit vectors have magnitude $|\,\hat{\mathbf{i}}\,|=|\,\hat{\mathbf{j}}\,|=1$ and $\sin(\pi/2)=1$. By the right hand rule, the direction of $\hat{\mathbf{i}}\times\hat{\mathbf{j}}$ is in the $+\hat{\mathbf{k}}$ as shown in Figure 17.5. Thus $\hat{\mathbf{i}}\times\hat{\mathbf{j}}=\hat{\mathbf{k}}$.

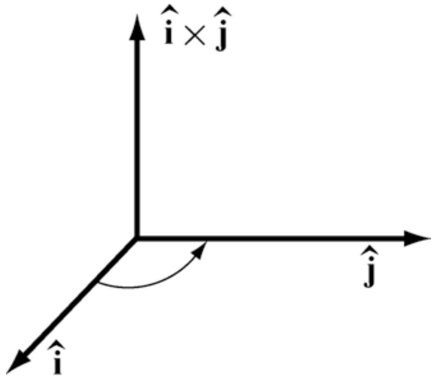


Figure 17.5 Vector product of $\hat{\mathbf{i}}\times\hat{\mathbf{j}}$

We note that the same rule applies for the unit vectors in the *y* and *z* directions,

$$\hat{\mathbf{j}}\times\hat{\mathbf{k}}=\hat{\mathbf{i}},\quad \hat{\mathbf{k}}\times\hat{\mathbf{i}}=\hat{\mathbf{j}}\,. \tag{17.2.10}$$

By the anti-commutatively property (1) of the vector product,

$$\hat{\mathbf{j}}\times\hat{\mathbf{i}}=-\hat{\mathbf{k}},\quad \hat{\mathbf{i}}\times\hat{\mathbf{k}}=-\hat{\mathbf{j}} \tag{17.2.11}$$

The vector product of the unit vector $\hat{\mathbf{i}}$ with itself is zero because the two unit vectors are parallel to each other, ($\sin(0)=0$),

$$|\,\hat{\mathbf{i}}\times\hat{\mathbf{i}}\,|=|\,\hat{\mathbf{i}}\,||\,\hat{\mathbf{i}}\,|\sin(0)=0\,. \tag{17.2.12}$$

The vector product of the unit vector $\hat{\mathbf{j}}$ with itself and the unit vector $\hat{\mathbf{k}}$ with itself are also zero for the same reason,

$$\left|\hat{\mathbf{j}}\times\hat{\mathbf{j}}\right|=0,\quad \left|\hat{\mathbf{k}}\times\hat{\mathbf{k}}\right|=0\,. \tag{17.2.13}$$

With these properties in mind we can now develop an algebraic expression for the vector product in terms of components. Let's choose a Cartesian coordinate system with the vector $\vec{\mathbf{B}}$ pointing along the positive *x*-axis with positive *x*-component B_x. Then the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ can be written as

$$\vec{\mathbf{A}}=A_x\,\hat{\mathbf{i}}+A_y\,\hat{\mathbf{j}}+A_z\,\hat{\mathbf{k}} \tag{17.2.14}$$

$$\vec{\mathbf{B}}=B_x\,\hat{\mathbf{i}}\,, \tag{17.2.15}$$

respectively. The vector product in vector components is

$$\vec{\mathbf{A}}\times\vec{\mathbf{B}}=(A_x\,\hat{\mathbf{i}}+A_y\,\hat{\mathbf{j}}+A_z\,\hat{\mathbf{k}})\times B_x\,\hat{\mathbf{i}}\,. \tag{17.2.16}$$

This becomes,

$$\begin{aligned}\vec{\mathbf{A}}\times\vec{\mathbf{B}} &= (A_x\ \hat{\mathbf{i}}\times B_x\ \hat{\mathbf{i}})+(A_y\ \hat{\mathbf{j}}\times B_x\ \hat{\mathbf{i}})+(A_z\ \hat{\mathbf{k}}\times B_x\ \hat{\mathbf{i}})\\ &= A_xB_x\ (\hat{\mathbf{i}}\times\hat{\mathbf{i}})+A_yB_x\ (\hat{\mathbf{j}}\times\hat{\mathbf{i}})+A_zB_x\ (\hat{\mathbf{k}}\times\hat{\mathbf{i}})\quad .\\ &= -A_yB_x\ \hat{\mathbf{k}}+A_zB_x\ \hat{\mathbf{j}}\end{aligned} \tag{17.2.17}$$

The vector component expression for the vector product easily generalizes for arbitrary vectors

$$\vec{\mathbf{A}} = A_x\ \hat{\mathbf{i}}+A_y\ \hat{\mathbf{j}}+A_z\ \hat{\mathbf{k}} \tag{17.2.18}$$

$$\vec{\mathbf{B}} = B_x\ \hat{\mathbf{i}}+B_y\ \hat{\mathbf{j}}+B_z\ \hat{\mathbf{k}}\,, \tag{17.2.19}$$

to yield

$$\vec{\mathbf{A}}\times\vec{\mathbf{B}} = (A_yB_z-A_zB_y)\,\hat{\mathbf{i}}+(A_zB_x-A_xB_z)\,\hat{\mathbf{j}}+(A_xB_y-A_yB_x)\,\hat{\mathbf{k}}\,. \tag{17.2.20}$$

17.2.4 Vector Decomposition and the Vector Product: Cylindrical Coordinates

Recall the cylindrical coordinate system, which we show in Figure 17.6. We have chosen two directions, radial and tangential in the plane, and a perpendicular direction to the plane.

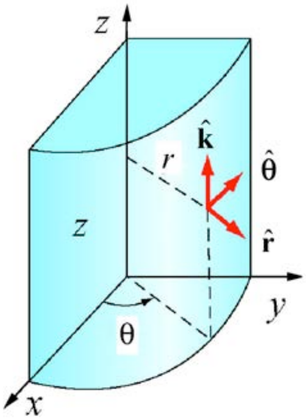


Figure 17.6 Cylindrical coordinates

The unit vectors are at right angles to each other and so using the right hand rule, the vector product of the unit vectors are given by the relations

$$\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} = \hat{\mathbf{k}} \tag{17.2.21}$$

$$\hat{\boldsymbol{\theta}}\times\hat{\mathbf{k}} = \hat{\mathbf{r}} \tag{17.2.22}$$

$$\hat{\mathbf{k}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}\,. \tag{17.2.23}$$

Because the vector product satisfies $\vec{\mathbf{A}}\times\vec{\mathbf{B}} = -\vec{\mathbf{B}}\times\vec{\mathbf{A}}$, we also have that

$$\hat{\boldsymbol{\theta}}\times\hat{\mathbf{r}} = -\hat{\mathbf{k}} \tag{17.2.24}$$

$$\hat{\mathbf{k}} \times \hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}} \tag{17.2.25}$$

$$\hat{\mathbf{r}} \times \hat{\mathbf{k}} = -\hat{\boldsymbol{\theta}}\,. \tag{17.2.26}$$

Finally

$$\hat{\mathbf{r}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \vec{\mathbf{0}}\,. \tag{17.2.27}$$

Example 17.1 Vector Products

Given two vectors, $\vec{\mathbf{A}} = 2\,\hat{\mathbf{i}} + -3\,\hat{\mathbf{j}} + 7\,\hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = 5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, find $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$.

Solution:

$$\begin{aligned}\vec{\mathbf{A}} \times \vec{\mathbf{B}} &= (A_y B_z - A_z B_y)\,\hat{\mathbf{i}} + (A_z B_x - A_x B_z)\,\hat{\mathbf{j}} + (A_x B_y - A_y B_x)\,\hat{\mathbf{k}} \\ &= ((-3)(2) - (7)(1))\,\hat{\mathbf{i}} + ((7)(5) - (2)(2))\,\hat{\mathbf{j}} + ((2)(1) - (-3)(5))\,\hat{\mathbf{k}} \\ &= -13\,\hat{\mathbf{i}} + 31\,\hat{\mathbf{j}} + 17\,\hat{\mathbf{k}}.\end{aligned}$$

Example 17.2 Law of Sines

For the triangle shown in Figure 17.7a, prove the law of sines, $\left|\vec{\mathbf{A}}\right| / \sin\alpha = \left|\vec{\mathbf{B}}\right| / \sin\beta = \left|\vec{\mathbf{C}}\right| / \sin\gamma$, using the vector product.

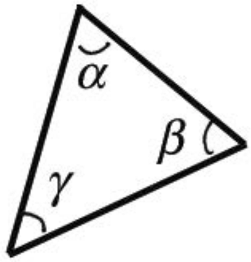


Figure 17.7 (a) Example 17.2

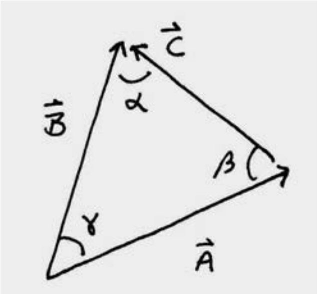


Figure 17.7 (b) Vector analysis

Solution: Consider the area of a triangle formed by three vectors $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, and $\vec{\mathbf{C}}$, where $\vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}} = 0$ (Figure 17.7b). Because $\vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}} = 0$, we have that $0 = \vec{\mathbf{A}} \times (\vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}}) = \vec{\mathbf{A}} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \vec{\mathbf{C}}$. Thus $\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{A}} \times \vec{\mathbf{C}}$ or $\left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}} \times \vec{\mathbf{C}}\right|$. From Figure 17.7b we see that $\left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right| \sin\gamma$ and $\left|\vec{\mathbf{A}} \times \vec{\mathbf{C}}\right| = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{C}}\right| \sin\beta$. Therefore $\left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right| \sin\gamma = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{C}}\right| \sin\beta$, and hence $\left|\vec{\mathbf{B}}\right| / \sin\beta = \left|\vec{\mathbf{C}}\right| / \sin\gamma$. A similar argument shows that $\left|\vec{\mathbf{B}}\right| / \sin\beta = \left|\vec{\mathbf{A}}\right| / \sin\alpha$ proving the law of sines.

Example 17.3 Unit Normal

Find a unit vector perpendicular to $\vec{\mathbf{A}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = -2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$.

Solution: The vector product $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ is perpendicular to both $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. Therefore the unit vectors $\hat{\mathbf{n}} = \pm\vec{\mathbf{A}}\times\vec{\mathbf{B}}/\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right|$ are perpendicular to both $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. We first calculate

$$\begin{aligned}\vec{\mathbf{A}}\times\vec{\mathbf{B}} &= (A_yB_z - A_zB_y)\,\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\,\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\,\hat{\mathbf{k}} \\ &= ((1)(3) - (-1)(-1))\,\hat{\mathbf{i}} + ((-1)(2) - (1)(3))\,\hat{\mathbf{j}} + ((1)(-1) - (1)(2))\,\hat{\mathbf{k}} \\ &= 2\,\hat{\mathbf{i}} - 5\,\hat{\mathbf{j}} - 3\,\hat{\mathbf{k}}.\end{aligned}$$

We now calculate the magnitude

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = (2^2 + 5^2 + 3^2)^{1/2} = (38)^{1/2} \quad .$$

Therefore the perpendicular unit vectors are

$$\hat{\mathbf{n}} = \pm\vec{\mathbf{A}}\times\vec{\mathbf{B}}/\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \pm(2\,\hat{\mathbf{i}} - 5\,\hat{\mathbf{j}} - 3\,\hat{\mathbf{k}})/(38)^{1/2}.$$

Example 17.4 Volume of Parallelepiped

Show that the volume of a parallelepiped with edges formed by the vectors $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, and $\vec{\mathbf{C}}$ is given by $\vec{\mathbf{A}}\cdot(\vec{\mathbf{B}}\times\vec{\mathbf{C}})$.

Solution: The volume of a parallelepiped is given by area of the base times height. If the base is formed by the vectors $\vec{\mathbf{B}}$ and $\vec{\mathbf{C}}$, then the area of the base is given by the magnitude of $\vec{\mathbf{B}}\times\vec{\mathbf{C}}$. The vector $\vec{\mathbf{B}}\times\vec{\mathbf{C}} = \left|\vec{\mathbf{B}}\times\vec{\mathbf{C}}\right|\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the base (Figure 17.8).

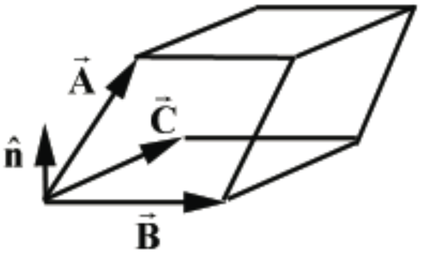


Figure 17.8 Example 17.4

The projection of the vector $\vec{\mathbf{A}}$ along the direction $\hat{\mathbf{n}}$ gives the height of the parallelepiped. This projection is given by taking the dot product of $\vec{\mathbf{A}}$ with a unit vector and is equal to $\vec{\mathbf{A}}\cdot\hat{\mathbf{n}} = \textit{height}$. Therefore

$$\vec{\mathbf{A}}\cdot(\vec{\mathbf{B}}\times\vec{\mathbf{C}}) = \vec{\mathbf{A}}\cdot(\left|\vec{\mathbf{B}}\times\vec{\mathbf{C}}\right|)\hat{\mathbf{n}} = (\left|\vec{\mathbf{B}}\times\vec{\mathbf{C}}\right|)\vec{\mathbf{A}}\cdot\hat{\mathbf{n}} = (\textit{area})(\textit{height}) = (\textit{volume}).$$

Example 17.5 Vector Decomposition

Let $\vec{\mathbf{A}}$ be an arbitrary vector and let $\hat{\mathbf{n}}$ be a unit vector in some fixed direction. Show that $\vec{\mathbf{A}} = (\vec{\mathbf{A}} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + (\hat{\mathbf{n}} \times \vec{\mathbf{A}}) \times \hat{\mathbf{n}}$.

Solution: Let $\vec{\mathbf{A}} = A_{\|}\hat{\mathbf{n}} + A_{\perp}\hat{\mathbf{e}}$ where $A_{\|}$ is the component $\vec{\mathbf{A}}$ in the direction of $\hat{\mathbf{n}}$, $\hat{\mathbf{e}}$ is the direction of the projection of $\vec{\mathbf{A}}$ in a plane perpendicular to $\hat{\mathbf{n}}$, and $A_{\perp}$ is the component of $\vec{\mathbf{A}}$ in the direction of $\hat{\mathbf{e}}$. Because $\hat{\mathbf{e}} \cdot \hat{\mathbf{n}} = 0$, we have that $\vec{\mathbf{A}} \cdot \hat{\mathbf{n}} = A_{\|}$. Note that

$$\hat{\mathbf{n}} \times \vec{\mathbf{A}} = \hat{\mathbf{n}} \times (A_{\|}\hat{\mathbf{n}} + A_{\perp}\hat{\mathbf{e}}) = \hat{\mathbf{n}} \times A_{\perp}\hat{\mathbf{e}} = A_{\perp}(\hat{\mathbf{n}} \times \hat{\mathbf{e}}).$$

The unit vector $\hat{\mathbf{n}} \times \hat{\mathbf{e}}$ lies in the plane perpendicular to $\hat{\mathbf{n}}$ and is also perpendicular to $\hat{\mathbf{e}}$. Therefore $(\hat{\mathbf{n}} \times \hat{\mathbf{e}}) \times \hat{\mathbf{n}}$ is also a unit vector that is parallel to $\hat{\mathbf{e}}$ (by the right hand rule. So $(\hat{\mathbf{n}} \times \vec{\mathbf{A}}) \times \hat{\mathbf{n}} = A_{\perp}\hat{\mathbf{e}}$. Thus

$$\vec{\mathbf{A}} = A_{\|}\hat{\mathbf{n}} + A_{\perp}\hat{\mathbf{e}} = (\vec{\mathbf{A}} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + (\hat{\mathbf{n}} \times \vec{\mathbf{A}}) \times \hat{\mathbf{n}}.$$

17.3 Torque

17.3.1 Definition of Torque about a Point

In order to understand the dynamics of a rotating rigid body we will introduce a new quantity, the torque. Let a force $\vec{\mathbf{F}}_P$ with magnitude $F = |\vec{\mathbf{F}}_P|$ act at a point P. Let $\vec{\mathbf{r}}_{S,P}$ be the vector from the point S to a point P, with magnitude $r = |\vec{\mathbf{r}}_{S,P}|$. The angle between the vectors $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$ is θ with $[0 \le \theta \le \pi]$ (Figure 17.9).

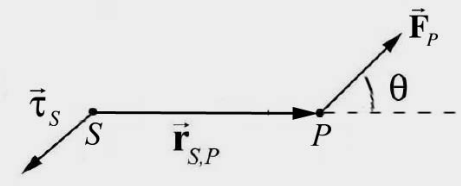


Figure 17.9 Torque about a point S due to a force acting at a point P

The torque about a point S due to force $\vec{\mathbf{F}}_P$ acting at P, is defined by

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,P} \times \vec{\mathbf{F}}_P. \qquad (17.2.28)$$

The magnitude of the torque about a point S due to force $\vec{\mathbf{F}}_P$ acting at P, is given by

$$\tau_S \equiv \left|\vec{\boldsymbol{\tau}}_S\right| = r\,F\sin\theta\,. \qquad (17.2.29)$$

The SI units for torque are $[\mathrm{N}\cdot\mathrm{m}]$. The direction of the torque is perpendicular to the plane formed by the vectors $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$ (for $[0<\theta<\pi]$), and by definition points in the direction of the unit normal vector to the plane $\hat{\mathbf{n}}_{RHR}$ as shown in Figure 17.10.

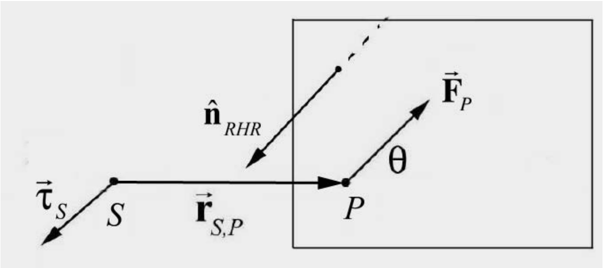


Figure 17.10 Vector direction for the torque

Figure 17.11 shows the two different ways of defining height and base for a parallelogram defined by the vectors $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$.

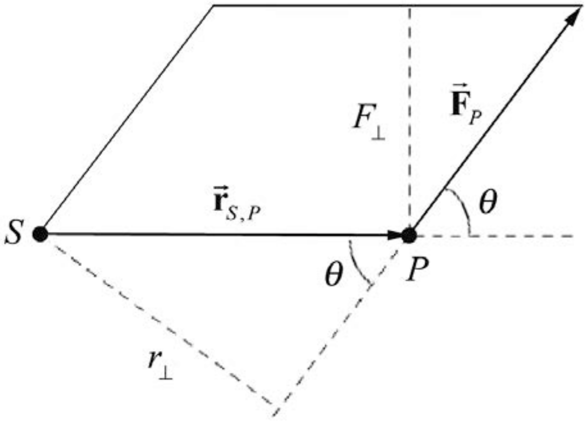


Figure 17.11 Area of the torque parallelogram.

Let $r_\perp = r\sin\theta$ and let $F_\perp = F\sin\theta$ be the component of the force $\vec{\mathbf{F}}_P$ that is perpendicular to the line passing from the point S to P. (Recall the angle θ has a range of values $0 \le \theta \le \pi$ so both $r_\perp \ge 0$ and $F_\perp \ge 0$.) Then the area of the parallelogram defined by $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$ is given by

$$\text{Area} = \tau_S = r_\perp F = r\,F_\perp = r\,F\sin\theta\,. \qquad (17.2.30)$$

We can interpret the quantity $r_\perp$ as follows.

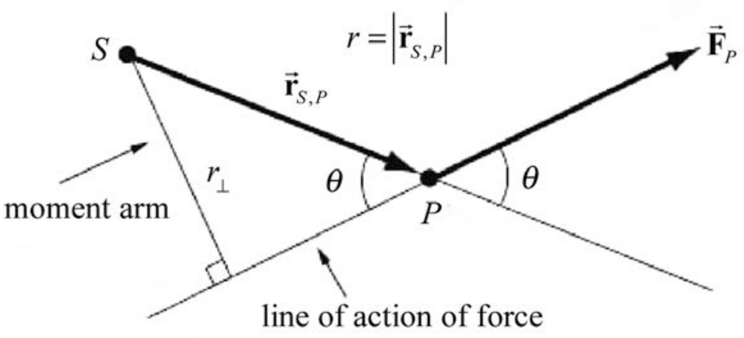


Figure 17.12 The moment arm about the point S and line of action of force passing through the point P

We begin by drawing the ***line of action of the force*** $\vec{\mathbf{F}}_P$. This is a straight line passing through P, parallel to the direction of the force $\vec{\mathbf{F}}_P$. Draw a perpendicular to this line of action that passes through the point S (Figure 17.12). The length of this perpendicular, $r_\perp = r\sin\theta$, is called ***the moment arm about the point S of the force*** $\vec{\mathbf{F}}_P$.

You should keep in mind three important properties of torque:

1. The torque is zero if the vectors $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$ are parallel $(\theta = 0)$ or anti-parallel $(\theta = \pi)$.

2. Torque is a vector whose direction and magnitude depend on the choice of a point S about which the torque is calculated.

3. The direction of torque is perpendicular to the plane formed by the two vectors, $\vec{\mathbf{F}}_P$ and $r = |\vec{\mathbf{r}}_{S,P}|$ (the vector from the point S to a point P).

17.3.2 Alternative Approach to Assigning a Sign Convention for Torque

In the case where all of the forces $\vec{\mathbf{F}}_i$ and position vectors $\vec{\mathbf{r}}_{i,P}$ are coplanar (or zero), we can, instead of referring to the direction of torque, assign a purely algebraic positive or negative sign to torque according to the following convention. We note that the arc in Figure 17.13a circles in counterclockwise direction. (Figures 17.13a and 17.13b use the simplifying assumption, for the purpose of the figure only, that the two vectors in question, $\vec{\mathbf{F}}_P$ and $\vec{\mathbf{r}}_{S,P}$ are perpendicular. The point S about which torques are calculated is not shown.)

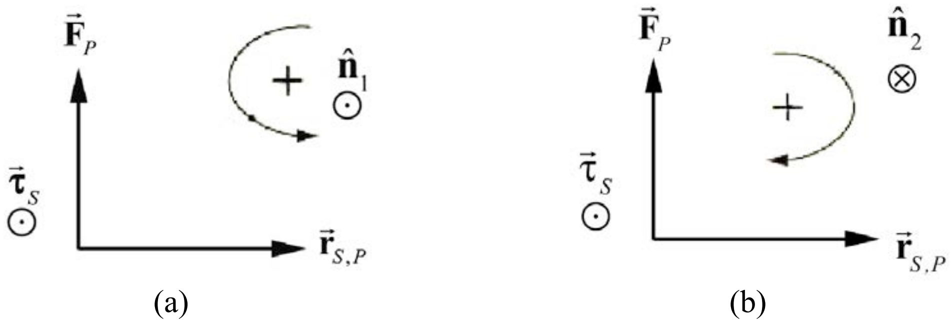


Figure 17.13 (a) Positive torque out of plane, **(b)** positive torque into plane

We can associate with this counterclockwise orientation a unit normal vector $\hat{\mathbf{n}}$ according to the right-hand rule: curl your right hand fingers in the counterclockwise direction and your right thumb will then point in the $\hat{\mathbf{n}}_1$ direction (Figure 17.13a). The arc in Figure 17.13b circles in the clockwise direction, and we associate this orientation with the unit normal $\hat{\mathbf{n}}_2$.

It's important to note that the terms "clockwise" and "counterclockwise" might be different for different observers. For instance, if the plane containing $\vec{\mathbf{F}}_P$ and $\vec{\mathbf{r}}_{S,P}$ is horizontal, an observer above the plane and an observer below the plane would disagree on the two terms. For a vertical plane, the directions that two observers on opposite sides of the plane would be mirror images of each other, and so again the observers would disagree.

1. Suppose we choose counterclockwise as positive. Then we assign a positive sign for the component of the torque when the torque is in the same direction as the unit normal $\hat{\mathbf{n}}_1$, i.e. $\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,P} \times \vec{\mathbf{F}}_P = +|\vec{\mathbf{r}}_{S,P}||\vec{\mathbf{F}}_P|\hat{\mathbf{n}}_1$, (Figure 17.13a).

2. Suppose we choose clockwise as positive. Then we assign a negative sign for the component of the torque in Figure 17.13b because the torque is directed opposite to the unit normal $\hat{\mathbf{n}}_2$, i.e. $\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,P} \times \vec{\mathbf{F}}_P = -|\vec{\mathbf{r}}_{S,P}||\vec{\mathbf{F}}_P|\hat{\mathbf{n}}_2$.

Example 17.6 Torque and Vector Product

Consider two vectors $\vec{\mathbf{r}}_{P,F} = x\hat{\mathbf{i}}$ with $x > 0$ and $\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_z\hat{\mathbf{k}}$ with $F_x > 0$ and $F_z > 0$. Calculate the torque $\vec{\mathbf{r}}_{P,F} \times \vec{\mathbf{F}}$.

Solution: We calculate the vector product noting that in a right handed choice of unit vectors, $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \vec{\mathbf{0}}$ and $\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$,

$$\vec{\mathbf{r}}_{P,F} \times \vec{\mathbf{F}} = x\hat{\mathbf{i}} \times (F_x\hat{\mathbf{i}} + F_z\hat{\mathbf{k}}) = (x\hat{\mathbf{i}} \times F_x\hat{\mathbf{i}}) + (x\hat{\mathbf{i}} \times F_z\hat{\mathbf{k}}) = -xF_z\,\hat{\mathbf{j}}.$$

Because $x > 0$ and $F_z > 0$, the direction of the vector product is in the negative y-direction.

Example 17.7 Calculating Torque

In Figure 17.14, a force of magnitude F is applied to one end of a lever of length L. What is the magnitude and direction of the torque about the point S?

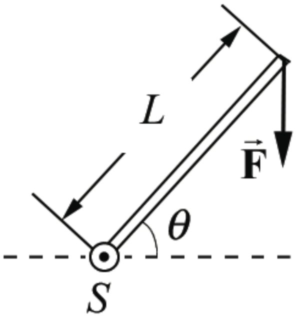


Figure 17.14 Example 17.7

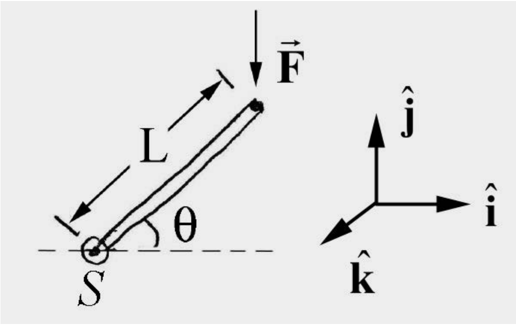


Figure 17.15 Coordinate system

Solution: Choose units vectors such that $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$, with $\hat{\mathbf{i}}$ pointing to the right and $\hat{\mathbf{j}}$ pointing up (Figure 17.15). The torque about the point S is given by $\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,F} \times \vec{\mathbf{F}}$, where $\vec{\mathbf{r}}_{SF} = L\cos\theta\hat{\mathbf{i}} + L\sin\theta\hat{\mathbf{j}}$ and $\vec{\mathbf{F}} = -F\hat{\mathbf{j}}$ then

$$\vec{\boldsymbol{\tau}}_S = (L\cos\theta\,\hat{\mathbf{i}} + L\sin\theta\,\hat{\mathbf{j}}) \times -F\,\hat{\mathbf{j}} = -FL\cos\theta\,\hat{\mathbf{k}}\,.$$

Example 17.8 Torque and the Ankle

A person of mass m is crouching with their weight evenly distributed on both tiptoes. The free-body force diagram on the skeletal part of the foot is shown in Figure 17.16. The normal force $\vec{\mathbf{N}}$ acts at the contact point between the foot and the ground. In this position, the tibia acts on the foot at the point S with a force $\vec{\mathbf{F}}$ of an unknown magnitude $F = |\vec{\mathbf{F}}|$ and makes an unknown angle β with the vertical. This force acts on the ankle a horizontal distance s from the point where the foot contacts the floor. The Achilles tendon also acts on the foot and is under considerable tension with magnitude $T \equiv |\vec{\mathbf{T}}|$ and acts at an angle α with the horizontal as shown in the figure. The tendon acts on the ankle a horizontal distance b from the point S where the tibia acts on the foot. You may ignore the weight of the foot. Let g be the gravitational constant. Compute the torque about the point S due to (a) the tendon force on the foot; (b) the force of the tibia on the foot; (c) the normal force of the floor on the foot.

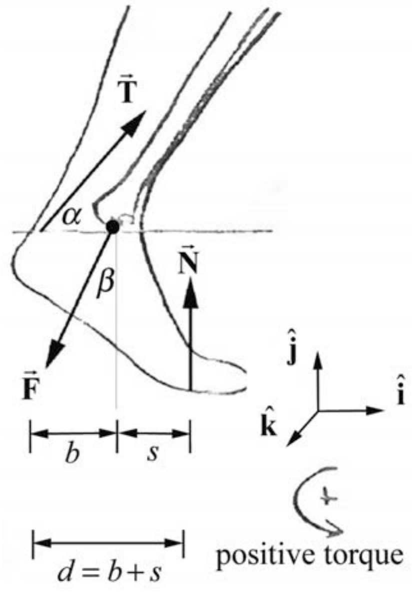


Figure 17.16 Force diagram and coordinate system for ankle

Solution: (a) We shall first calculate the torque due to the force of the Achilles tendon on the ankle. The tendon force has the vector decomposition $\vec{\mathbf{T}} = T\cos\alpha\hat{\mathbf{i}} + T\sin\alpha\hat{\mathbf{j}}$.

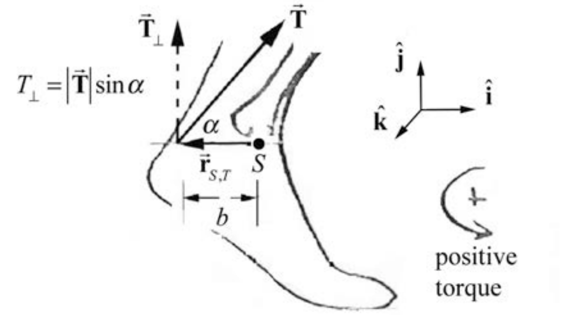


Figure 17.17 Torque diagram for tendon force on ankle

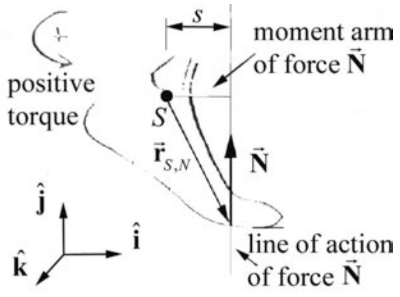


Figure 17.18 Torque diagram for normal force on ankle $\vec{\mathbf{r}}_{S,N}$

The vector from the point S to the point of action of the force is given by $\vec{\mathbf{r}}_{S,T} = -b\hat{\mathbf{i}}$ (Figure 17.17). Therefore the torque due to the force of the tendon $\vec{\mathbf{T}}$ on the ankle about the point S is then

$$\vec{\tau}_{S,T} = \vec{\mathbf{r}}_{S,T} \times \vec{\mathbf{T}} = -b\hat{\mathbf{i}} \times (T\cos\alpha\hat{\mathbf{i}} + T\sin\alpha\hat{\mathbf{j}}) = -bT\sin\alpha\,\hat{\mathbf{k}}\ .$$

(b) The torque diagram for the normal force is shown in Figure 17.18. The vector from the point S to the point where the normal force acts on the foot is given by

$\vec{\mathbf{r}}_{S,N} = (s\hat{\mathbf{i}} - h\hat{\mathbf{j}})$. Because the weight is evenly distributed on the two feet, the normal force on one foot is equal to half the weight, or $N = (1/2)mg$. The normal force is therefore given by $\vec{\mathbf{N}} = N\,\hat{\mathbf{j}} = (1/2)\,mg\,\hat{\mathbf{j}}$. Therefore the torque of the normal force about the point S is

$$\vec{\tau}_{S,N} = \vec{\mathbf{r}}_{S,N} \times N\,\hat{\mathbf{j}} = (s\hat{\mathbf{i}} - h\hat{\mathbf{j}}) \times N\,\hat{\mathbf{j}} = s\,N\,\hat{\mathbf{k}} = (1/2)\,s\,mg\,\hat{\mathbf{k}}\,.$$

(c) The force $\vec{\mathbf{F}}$ that the tibia exerts on the ankle will make no contribution to the torque about this point S since the tibia force acts at the point S and therefore the vector $\vec{\mathbf{r}}_{S,F} = \vec{\mathbf{0}}$.

17.4 Torque, Angular Acceleration, and Moment of Inertia

17.4.1 Torque Equation for Fixed Axis Rotation

For fixed-axis rotation, there is a direct relation between the component of the torque along the axis of rotation and angular acceleration. Consider the forces that act on the rotating body. Generally, the forces on different volume elements will be different, and so we will denote the force on the volume element of mass Δm_i by $\vec{\mathbf{F}}_i$. Choose the z-axis to lie along the axis of rotation. Divide the body into volume elements of mass Δm_i. Let the point S denote a specific point along the axis of rotation (Figure 17.19). Each volume element undergoes a tangential acceleration as the volume element moves in a circular orbit of radius $r_i = |\vec{\mathbf{r}}_i|$ about the fixed axis.

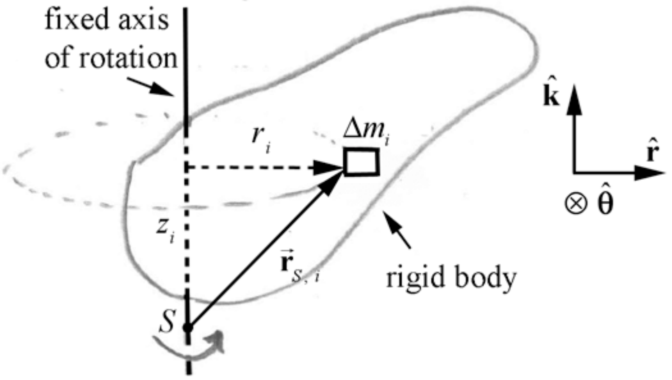


Figure 17.19: Volume element undergoing fixed-axis rotation about the z-axis.

The vector from the point S to the volume element is given by

$$\vec{\mathbf{r}}_{S,\,i} = z_i\,\hat{\mathbf{k}} + \vec{\mathbf{r}}_i = z_i\,\hat{\mathbf{k}} + r_i\,\hat{\mathbf{r}} \qquad (17.3.1)$$

where z_i is the distance along the axis of rotation between the point S and the volume element. The torque about S due to the force $\vec{\mathbf{F}}_i$ acting on the volume element is given by

$$\vec{\boldsymbol{\tau}}_{S,\,i} = \vec{\mathbf{r}}_{S,\,i} \times \vec{\mathbf{F}}_i . \tag{17.3.2}$$

Substituting Eq. (17.3.1) into Eq. (17.3.2) gives

$$\vec{\boldsymbol{\tau}}_{S,\,i} = (z_i\,\hat{\mathbf{k}} + r_i\,\hat{\mathbf{r}}) \times \vec{\mathbf{F}}_i . \tag{17.3.3}$$

For fixed-axis rotation, we are interested in the z-component of the torque, which must be the term

$$(\vec{\boldsymbol{\tau}}_{S,\,i})_z = (r_i\,\hat{\mathbf{r}} \times \vec{\mathbf{F}}_i)_z \tag{17.3.4}$$

because the vector product $z_i\,\hat{\mathbf{k}} \times \vec{\mathbf{F}}_i$ must be directed perpendicular to the plane formed by the vectors $\hat{\mathbf{k}}$ and $\vec{\mathbf{F}}_i$, hence perpendicular to the z-axis. The force acting on the volume element has components

$$\vec{\mathbf{F}}_i = F_{r,i}\,\hat{\mathbf{r}} + F_{\theta,i}\,\hat{\boldsymbol{\theta}} + F_{z,i}\,\hat{\mathbf{k}} . \tag{17.3.5}$$

The z-component $F_{z,i}$ of the force cannot contribute a torque in the z-direction, and so substituting Eq. (17.3.5) into Eq. (17.3.4) yields

$$(\vec{\boldsymbol{\tau}}_{S,\,i})_z = (r_i\,\hat{\mathbf{r}} \times (F_{r,i}\,\hat{\mathbf{r}} + F_{\theta,i}\,\hat{\boldsymbol{\theta}}))_z . \tag{17.3.6}$$

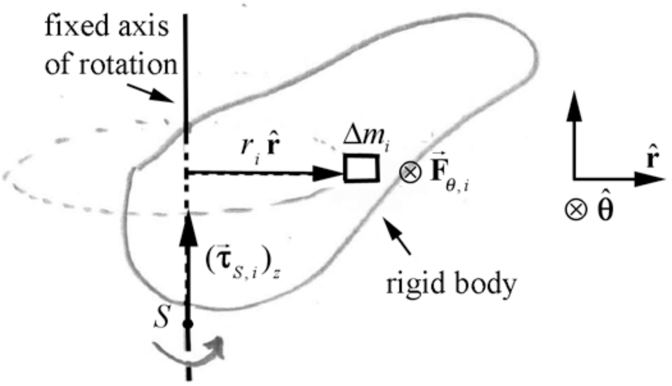


Figure 17.20 Tangential force acting on a volume element.

The radial force does not contribute to the torque about the z-axis, since

$$r_i\,\hat{\mathbf{r}} \times F_{r,i}\,\hat{\mathbf{r}} = \vec{\mathbf{0}} . \tag{17.3.7}$$

So, we are interested in the contribution due to torque about the z-axis due to the tangential component of the force on the volume element (Figure 17.20). The component of the torque about the z-axis is given by

$$(\vec{\boldsymbol{\tau}}_{S,i})_z = (r_i\,\hat{\mathbf{r}} \times F_{\theta,i}\,\hat{\boldsymbol{\theta}})_z = r_i\,F_{\theta,i}\,. \tag{17.3.8}$$

The z-component of the torque is directed upwards in Figure 17.20, where $F_{\theta,i}$ is positive (the tangential force is directed counterclockwise, as in the figure). Applying Newton's Second Law in the tangential direction,

$$F_{\theta,\,i} = \Delta m_i\, a_{\theta,\,i}\,. \tag{17.3.9}$$

Using our kinematics result that the tangential acceleration is $a_{\theta,\,i} = r_i\,\alpha_z$, where α_z is the z-component of angular acceleration, we have that

$$F_{\theta,\,i} = \Delta m_i\, r_i\, \alpha_z\,. \tag{17.3.10}$$

From Eq. (17.3.8), the component of the torque about the z-axis is then given by

$$(\vec{\boldsymbol{\tau}}_{S,i})_z = r_i\,F_{\theta,\,i} = \Delta m_i\; r_i^{\,2}\alpha_z\,. \tag{17.3.11}$$

The component of the torque about the z-axis is the summation of the torques on all the volume elements,

$$\begin{aligned}(\vec{\boldsymbol{\tau}}_S)_z &= \sum_{i=1}^{i=N} (\vec{\boldsymbol{\tau}}_{S,i})_z = \sum_{i=1}^{i=N} r_{\perp,\,i}\,F_{\theta,\,i} \\ &= \sum_{i=1}^{i=N} \Delta m_i r_i^{\,2}\alpha_z.\end{aligned} \tag{17.3.12}$$

Because each element has the same z-component of angular acceleration, α_z, the summation becomes

$$(\vec{\boldsymbol{\tau}}_S)_z = \left(\sum_{i=1}^{i=N} \Delta m_i\; r_i^{\,2}\right)\alpha_z\,. \tag{17.3.13}$$

Recalling our definition of the moment of inertia, (Chapter 16.3) the z-component of the torque is proportional to the z-component of angular acceleration,

$$\tau_{S,z} = I_S\,\alpha_z\,, \tag{17.3.14}$$

and the moment of inertia, I_S, is the constant of proportionality. The torque about the point S is the sum of the external torques and the internal torques

$$\vec{\boldsymbol{\tau}}_S = \vec{\boldsymbol{\tau}}_S^{\text{ ext}} + \vec{\boldsymbol{\tau}}_S^{\text{ int}} . \tag{17.3.15}$$

The external torque about the point S is the sum of the torques due to the net external force acting on each element

$$\vec{\boldsymbol{\tau}}_S^{\text{ext}} = \sum_{i=1}^{i=N} \vec{\boldsymbol{\tau}}_{S,i}^{\text{ext}} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_i^{\text{ext}} . \tag{17.3.16}$$

The internal torque arise from the torques due to the internal forces acting between pairs of elements

$$\vec{\boldsymbol{\tau}}_S^{\text{int}} = \sum_{i=1}^{N} \vec{\boldsymbol{\tau}}_{S,j}^{\text{int}} = \sum_{i=1}^{i=N} \sum_{\substack{j=1 \\ j \neq i}}^{j=N} \vec{\boldsymbol{\tau}}_{S,j,i}^{\text{int}} = \sum_{i=1}^{i=N} \sum_{\substack{j=1 \\ j \neq i}}^{j=N} \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_{j,i} . \tag{17.3.17}$$

We know by Newton’s Third Law that the internal forces cancel in pairs, $\vec{\mathbf{F}}_{j,i} = -\vec{\mathbf{F}}_{i,j}$, and hence the sum of the internal forces is zero

$$\vec{\mathbf{0}} = \sum_{i=1}^{i=N} \sum_{\substack{j=1 \\ j \neq i}}^{j=N} \vec{\mathbf{F}}_{j,i} . \tag{17.3.18}$$

Does the same statement hold about pairs of internal torques? Consider the sum of internal torques arising from the interaction between the i^{th} and j^{th} particles

$$\vec{\boldsymbol{\tau}}_{S,j,i}^{\text{int}} + \vec{\boldsymbol{\tau}}_{S,i,j}^{\text{int}} = \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_{j,i} + \vec{\mathbf{r}}_{S,j} \times \vec{\mathbf{F}}_{i,j} . \tag{17.3.19}$$

By the Newton’s Third Law this sum becomes

$$\vec{\boldsymbol{\tau}}_{S,j,i}^{\text{int}} + \vec{\boldsymbol{\tau}}_{S,i,j}^{\text{int}} = (\vec{\mathbf{r}}_{S,i} - \vec{\mathbf{r}}_{S,j}) \times \vec{\mathbf{F}}_{j,i} . \tag{17.3.20}$$

In the Figure 17.21, the vector $\vec{\mathbf{r}}_{S,i} - \vec{\mathbf{r}}_{S,j}$ points from the j^{th} element to the i^{th} element. If the internal forces between a pair of particles are directed along the line joining the two particles then the torque due to the internal forces cancel in pairs.

$$\vec{\boldsymbol{\tau}}_{S,j,i}^{\text{int}} + \vec{\boldsymbol{\tau}}_{S,i,j}^{\text{int}} = (\vec{\mathbf{r}}_{S,i} - \vec{\mathbf{r}}_{S,j}) \times \vec{\mathbf{F}}_{j,i} = \vec{\mathbf{0}} . \tag{17.3.21}$$

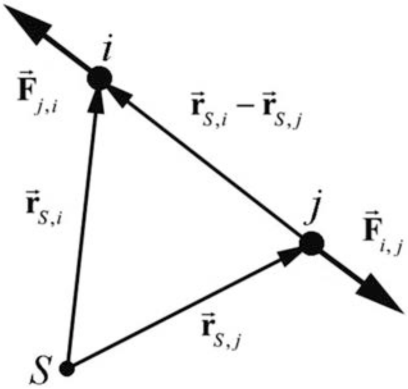


Figure 17.21 The internal force is directed along the line connecting the i^{th} and j^{th} particles

This is a stronger version of Newton's Third Law than we have so far since we have added the additional requirement regarding the direction of all the internal forces between pairs of particles. With this assumption, the torque is just due to the external forces

$$\vec{\boldsymbol{\tau}}_S = \vec{\boldsymbol{\tau}}_S^{\text{ext}} . \tag{17.3.22}$$

Thus Eq. (17.3.14) becomes

$$(\tau_S^{\text{ext}})_z = I_S \alpha_z , \tag{17.3.23}$$

This is very similar to Newton's Second Law: the total force is proportional to the acceleration,

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} . \tag{17.3.24}$$

where the mass, m, is the constant of proportionality.

17.4.2 Torque Acts at the Center of Gravity

Suppose a rigid body in static equilibrium consists of N particles labeled by the index $i = 1, 2, 3, ..., N$. Choose a coordinate system with a choice of origin O such that mass m_i has position $\vec{\mathbf{r}}_i$. Each point particle experiences a gravitational force $\vec{\mathbf{F}}_{\text{gravity},i} = m_i \vec{\mathbf{g}}$. The total torque about the origin is then zero (static equilibrium condition),

$$\vec{\boldsymbol{\tau}}_O = \sum_{i=1}^{i=N} \vec{\boldsymbol{\tau}}_{O,i} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_i \times \vec{\mathbf{F}}_{\text{gravity},i} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_i \times m_i \vec{\mathbf{g}} = \vec{\mathbf{0}} . \tag{17.3.25}$$

If the gravitational acceleration $\vec{\mathbf{g}}$ is assumed constant, we can rearrange the summation in Eq. (17.3.25) by pulling the constant vector $\vec{\mathbf{g}}$ out of the summation ($\vec{\mathbf{g}}$ appears in each term in the summation),

$$\vec{\boldsymbol{\tau}}_O = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_i \times m_i \vec{\mathbf{g}} = \left(\sum_{i=1}^{i=N} m_i \vec{\mathbf{r}}_i \right) \times \vec{\mathbf{g}} = \vec{\mathbf{0}}. \qquad (17.3.26)$$

We now use our definition of the center of the center of mass, Eq. (10.5.3), to rewrite Eq. (17.3.26) as

$$\vec{\boldsymbol{\tau}}_O = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_i \times m_i \vec{\mathbf{g}} = M_{\mathrm{T}} \vec{\mathbf{R}}_{\mathrm{cm}} \times \vec{\mathbf{g}} = \vec{\mathbf{R}}_{\mathrm{cm}} \times M_{\mathrm{T}} \vec{\mathbf{g}} = \vec{\mathbf{0}}. \qquad (17.3.27)$$

Thus the torque due to the gravitational force acting on each point-like particle is equivalent to the torque due to the gravitational force acting on a point-like particle of mass M_{T} located at a point in the body called the ***center of gravity***, which is equal to the center of mass of the body in the typical case in which the gravitational acceleration $\vec{\mathbf{g}}$ is constant throughout the body.

Example 17.9 Turntable

The turntable in Example 16.1, of mass $1.2\ \mathrm{kg}$ and radius $1.3 \times 10^{1}\ \mathrm{cm}$, has a moment of inertia $I_S = 1.01 \times 10^{-2}\ \mathrm{kg} \cdot \mathrm{m}^2$ about an axis through the center of the turntable and perpendicular to the turntable. The turntable is spinning at an initial constant frequency $f_i = 33\ \mathrm{cycles} \cdot \mathrm{min}^{-1}$. The motor is turned off and the turntable slows to a stop in $8.0\ \mathrm{s}$ due to frictional torque. Assume that the angular acceleration is constant. What is the magnitude of the frictional torque acting on the turntable?

Solution: We have already calculated the angular acceleration of the turntable in Example 16.1, where we found that

$$\alpha_z = \frac{\Delta \omega_z}{\Delta t} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{-3.5\ \mathrm{rad} \cdot \mathrm{s}^{-1}}{8.0\ \mathrm{s}} = -4.3 \times 10^{-1}\ \mathrm{rad} \cdot \mathrm{s}^{-2} \qquad (17.3.28)$$

and so the magnitude of the frictional torque is

$$\begin{aligned} \left| \tau_z^{\mathrm{fric}} \right| &= I_S \left| \alpha_z \right| = (1.01 \times 10^{-2}\ \mathrm{kg} \cdot \mathrm{m}^2)(4.3 \times 10^{-1}\ \mathrm{rad} \cdot \mathrm{s}^{-2}) \\ &= 4.3 \times 10^{-3}\ \mathrm{N} \cdot \mathrm{m}. \end{aligned} \qquad (17.3.29)$$

Example 17.10 Pulley and blocks

A pulley of mass m_{p}, radius R, and moment of inertia about its center of mass I_{cm}, is attached to the edge of a table. An inextensible string of negligible mass is wrapped around the pulley and attached on one end to block 1 that hangs over the edge of the table (Figure 17.22). The other end of the string is attached to block 2 that slides along a table.

The coefficient of sliding friction between the table and the block 2 is μ_k. Block 1 has mass m_1 and block 2 has mass m_2, with $m_1 > \mu_k m_2$. At time $t = 0$, the blocks are released from rest and the string does not slip around the pulley. At time $t = t_1$, block 1 hits the ground. Let g denote the gravitational constant. (a) Find the magnitude of the acceleration of each block. (b) How far did the block 1 fall before hitting the ground?

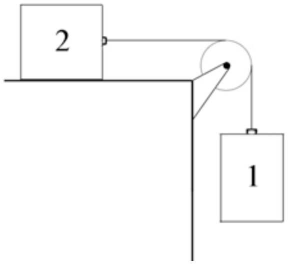


Figure 17.22 Example 17.10

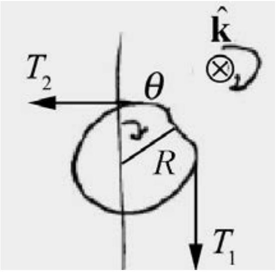


Figure 17.23 Torque diagram for pulley

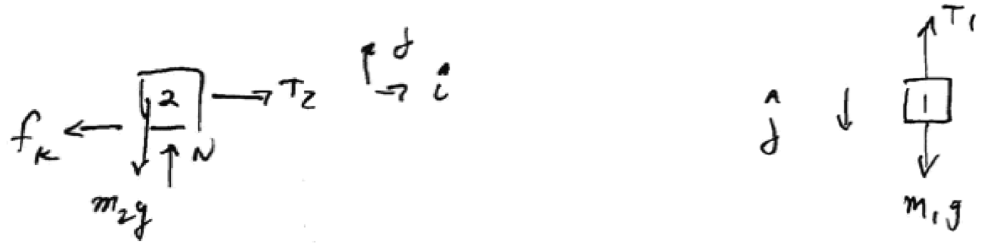
Solution: The torque diagram for the pulley is shown in the figure below where we choose $\hat{\mathbf{k}}$ pointing into the page. Note that the tensions in the string on either side of the pulley are not equal. The reason is that the pulley is massive. To understand why, remember that the difference in the magnitudes of the torques due to the tension on either side of the pulley is equal to the moment of inertia times the magnitude of the angular acceleration, which is non-zero for a massive pulley. So the tensions cannot be equal. From our torque diagram, the torque about the point O at the center of the pulley is given by

$$\vec{\boldsymbol{\tau}}_O = \vec{\mathbf{r}}_{O,1} \times \vec{\mathbf{T}}_1 + \vec{\mathbf{r}}_{O,2} \times \vec{\mathbf{T}}_2 = R(T_1 - T_2)\hat{\mathbf{k}}\,. \tag{17.3.30}$$

Therefore the torque equation (17.3.23) becomes

$$R(T_1 - T_2) = I_z \alpha_z\,. \tag{17.3.31}$$

The free body force diagrams on the two blocks are shown in Figure 17.23.



(a) (b)

Figure 17.23 Free-body force diagrams on (a) block 2, (b) block 1

Newton's Second Law on block 1 yields

$$m_1 g - T_1 = m_1 a_{y1} . \tag{17.3.32}$$

Newton's Second Law on block 2 in the $\hat{\mathbf{j}}$ direction yields

$$N - m_2 g = 0 . \tag{17.3.33}$$

Newton's Second Law on block 2 in the $\hat{\mathbf{i}}$ direction yields

$$T_2 - f_k = m_2 a_{x2} . \tag{17.3.34}$$

The kinetic friction force is given by

$$f_k = \mu_k N = \mu_k m_2 g \tag{17.3.35}$$

Therefore Eq. (17.3.34) becomes

$$T_2 - \mu_k m_2 g = m_2 a_{x2} . \tag{17.3.36}$$

Block 1 and block 2 are constrained to have the same acceleration so

$$a \equiv a_{x1} = a_{x2} . \tag{17.3.37}$$

We can solve Eqs. (17.3.32) and (17.3.36) for the two tensions yielding

$$T_1 = m_1 g - m_1 a , \tag{17.3.38}$$

$$T_2 = \mu_k m_2 g + m_2 a . \tag{17.3.39}$$

At point on the rim of the pulley has a tangential acceleration that is equal to the acceleration of the blocks so

$$a = a_\theta = R\alpha_z . \tag{17.3.40}$$

The torque equation (Eq. (17.3.31)) then becomes

$$T_1 - T_2 = \frac{I_z}{R^2} a . \tag{17.3.41}$$

Substituting Eqs. (17.3.38) and (17.3.39) into Eq. (17.3.41) yields

$$m_1 g - m_1 a - (\mu_k m_2 g + m_2 a) = \frac{I_z}{R^2} a , \tag{17.3.42}$$

which we can now solve for the accelerations of the blocks

$$a = \frac{m_1 g - \mu_k m_2 g}{m_1 + m_2 + I_z / R^2}. \tag{17.3.43}$$

Block 1 hits the ground at time t_1, therefore it traveled a distance

$$y_1 = \frac{1}{2}\left(\frac{m_1 g - \mu_k m_2 g}{m_1 + m_2 + I_z / R^2}\right) t_1^{\,2}. \tag{17.3.44}$$

Example 17.11 Experimental Method for Determining Moment of Inertia

A steel washer is mounted on a cylindrical rotor of radius $r = 12.7\ \text{mm}$. A massless string, with an object of mass $m = 0.055\ \text{kg}$ attached to the other end, is wrapped around the side of the rotor and passes over a massless pulley (Figure 17.24). Assume that there is a constant frictional torque about the axis of the rotor. The object is released and falls. As the object falls, the rotor undergoes an angular acceleration of magnitude α_1. After the string detaches from the rotor, the rotor coasts to a stop with an angular acceleration of magnitude α_2. Let $g = 9.8\ \text{m} \cdot \text{s}^{-2}$ denote the gravitational constant. Based on the data in the Figure 17.25, what is the moment of inertia I_R of the rotor assembly (including the washer) about the rotation axis?

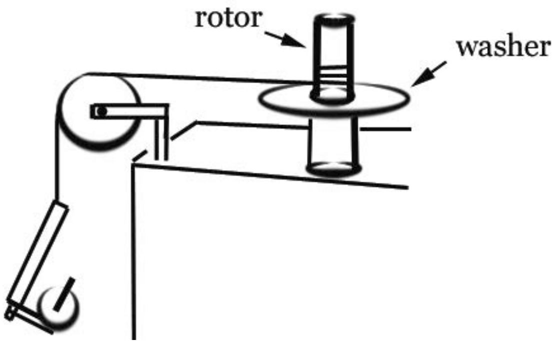


Figure 17.24 Steel washer, rotor, pulley, and hanging object

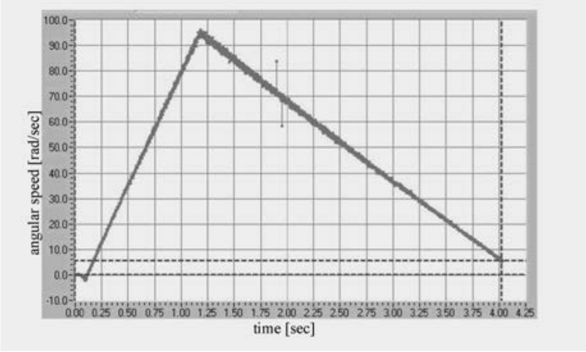
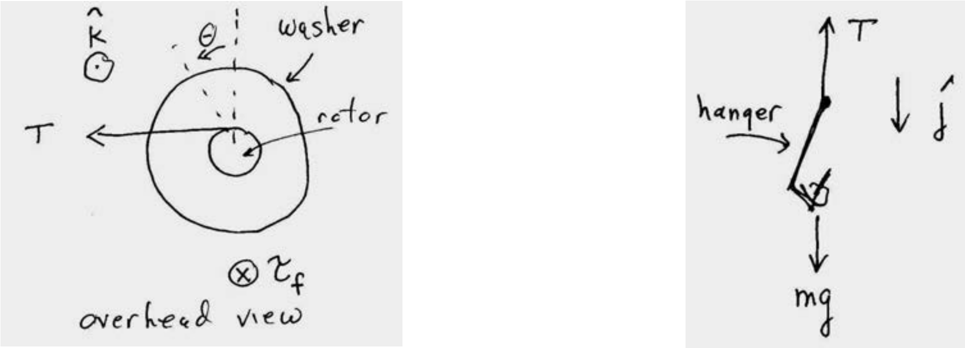


Figure 17.26 Graph of angular speed vs. time for falling object

Solution: We begin by drawing a force-torque diagram (Figure 17.26a) for the rotor and a free-body diagram for hanger (Figure 17.26b). (The choice of positive directions are indicated on the figures.) The frictional torque on the rotor is then given by $\vec{\boldsymbol{\tau}}_f = -\tau_f \hat{\mathbf{k}}$ where we use τ_f as the magnitude of the frictional torque. The torque about the center of the rotor due to the tension in the string is given by $\vec{\boldsymbol{\tau}}_T = rT\,\hat{\mathbf{k}}$ where r is the radius of the rotor. The angular acceleration of the rotor is given by $\vec{\boldsymbol{\alpha}}_1 = \alpha_1\,\hat{\mathbf{k}}$ and we expect that $\alpha_1 > 0$ because the rotor is speeding up.



(a) (b)

Figure 17.26 (a) Force-torque diagram on rotor and (b) free-body force diagram on hanging object

While the hanger is falling, the rotor-washer combination has a net torque due to the tension in the string and the frictional torque, and using the rotational equation of motion,

$$Tr - \tau_f = I_R \alpha_1 . \tag{17.4.1}$$

We apply Newton's Second Law to the hanger and find that

$$mg - T = ma_1 = m\alpha_1 r , \tag{17.4.2}$$

where $a_1 = r\alpha_1$ has been used to express the linear acceleration of the falling hanger to the angular acceleration of the rotor; that is, the string does not stretch. Before proceeding, it might be illustrative to multiply Eq. (17.4.2) by r and add to Eq. (17.4.1) to obtain

$$mgr - \tau_f = (I_R + mr^2)\alpha_1 . \tag{17.4.3}$$

Eq. (17.4.3) contains the unknown frictional torque, and this torque is determined by considering the slowing of the rotor/washer after the string has detached.

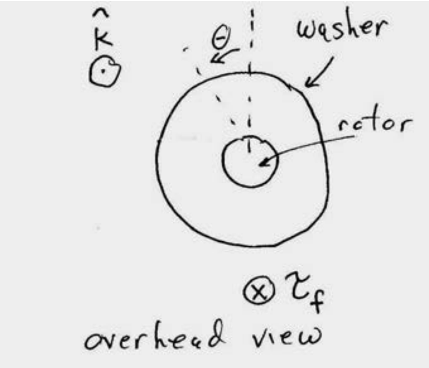


Figure 17.27 Torque diagram on rotor when string has detached

The torque on the system is just this frictional torque (Figure 17.27), and so

$$-\tau_f = I_R \alpha_2 \tag{17.4.4}$$

Note that in Eq. (17.4.4), $\tau_f > 0$ and $\alpha_2 < 0$. Subtracting Eq. (17.4.4) from Eq. (17.4.3) eliminates τ_f,

$$mgr = mr^2\alpha_1 + I_R(\alpha_1 - \alpha_2)\,. \tag{17.4.5}$$

We can now solve for I_R yielding

$$I_R = \frac{mr(g - r\alpha_1)}{\alpha_1 - \alpha_2}\,. \tag{17.4.6}$$

For a numerical result, we use the data collected during a trial run resulting in the graph of angular speed vs. time for the falling object shown in Figure 17.25. The values for α_1 and α_2 can be determined by calculating the slope of the two straight lines in Figure 17.28 yielding

$$\alpha_1 = (96\,\mathrm{rad}\cdot\mathrm{s}^{-1})/(1.15\,\mathrm{s}) = 83\ \mathrm{rad}\cdot\mathrm{s}^{-2},$$
$$\alpha_2 = -(89\,\mathrm{rad}\cdot\mathrm{s}^{-1})/(2.85\,\mathrm{s}) = -31\,\mathrm{rad}\cdot\mathrm{s}^{-2}.$$

Inserting these values into Eq. (17.4.6) yields

$$I_R = 5.3\times 10^{-5}\,\mathrm{kg}\cdot\mathrm{m}^2\,. \tag{17.4.7}$$

17.5 Torque and Rotational Work

When a constant torque $\tau_{s,z}$ is applied to an object, and the object rotates through an angle $\Delta\theta$ about a fixed z-axis through the center of mass, then the torque does an amount of work $\Delta W = \tau_{S,z}\,\Delta\theta$ on the object. By extension of the linear work-energy theorem, the amount of work done is equal to the change in the rotational kinetic energy of the object,

$$W_{\mathrm{rot}} = \frac{1}{2}I_{\mathrm{cm}}\omega_f^2 - \frac{1}{2}I_{\mathrm{cm}}\omega_i^2 = K_{\mathrm{rot},f} - K_{\mathrm{rot},i}\,. \tag{17.4.8}$$

The rate of doing this work is the rotational power exerted by the torque,

$$P_{\mathrm{rot}} \equiv \frac{dW_{\mathrm{rot}}}{dt} = \lim_{\Delta t\to 0}\frac{\Delta W_{\mathrm{rot}}}{\Delta t} = \tau_{S,z}\frac{d\theta}{dt} = \tau_{S,z}\,\omega_z\,. \tag{17.4.9}$$

17.5.1 Rotational Work

Consider a rigid body rotating about an axis. Each small element of mass Δm_i in the rigid body is moving in a circle of radius $(r_{S,i})_\perp$ about the axis of rotation passing through the point S. Each mass element undergoes a small angular displacement $\Delta\theta$ under the action of a tangential force, $\vec{\mathbf{F}}_{\theta,i} = F_{\theta,i}\,\hat{\boldsymbol{\theta}}$, where $\hat{\boldsymbol{\theta}}$ is the unit vector pointing in the tangential direction (Figure 17.20). The element will then have an associated displacement vector for this motion, $\Delta\vec{\mathbf{r}}_{S,i} = r_i\Delta\theta\,\hat{\boldsymbol{\theta}}$ and the work done by the tangential force is

$$\Delta W_{\mathrm{rot},i} = \vec{\mathbf{F}}_{\theta,i}\cdot\Delta\vec{\mathbf{r}}_{S,i} = (F_{\theta,i}\,\hat{\boldsymbol{\theta}})\cdot(r_i\Delta\theta\,\hat{\boldsymbol{\theta}}) = r_i F_{\theta,i}\Delta\theta\,. \tag{17.4.10}$$

Recall the result of Eq. (17.3.8) that the component of the torque (in the direction along the axis of rotation) about S due to the tangential force, $\vec{\mathbf{F}}_{\theta,i}$, acting on the mass element Δm_i is

$$(\tau_{S,i})_z = r_i F_{\theta,i}\,, \tag{17.4.11}$$

and so Eq. (17.4.10) becomes

$$\Delta W_{\mathrm{rot},\,i} = (\tau_{S,i})_z\Delta\theta\,. \tag{17.4.12}$$

Summing over all the elements yields

$$W_{\mathrm{rot}} = \sum_i \Delta W_{\mathrm{rot},\,i} = \sum_i \big((\tau_{S,i})_z\big)\Delta\theta = \tau_{S,z}\Delta\theta\,, \tag{17.4.13}$$

the rotational work is the product of the torque and the angular displacement. In the limit of small angles, $\Delta\theta \to d\theta$, $\Delta W_{\mathrm{rot}} \to dW_{\mathrm{rot}}$ and the differential rotational work is

$$dW_{\mathrm{rot}} = \tau_{S,z}d\theta\,. \tag{17.4.14}$$

We can integrate this amount of rotational work as the angle coordinate of the rigid body changes from some initial value $\theta = \theta_i$ to some final value $\theta = \theta_f$,

$$W_{\mathrm{rot}} = \int dW_{\mathrm{rot}} = \int_{\theta_i}^{\theta_f} \tau_{S,z}\,d\theta\,. \tag{17.4.15}$$

17.5.2 Rotational Work-Kinetic Energy Theorem

We will now show that the rotational work is equal to the change in rotational kinetic energy. We begin by substituting our result from Eq. (17.3.14) into Eq. (17.4.14) for the infinitesimal rotational work,

$$dW_{\mathrm{rot}} = I_S\alpha_z\,d\theta\,. \tag{17.4.16}$$

Recall that the rate of change of angular velocity is equal to the angular acceleration, $\alpha_z \equiv d\omega_z / dt$ and that the angular velocity is $\omega_z \equiv d\theta / dt$. Note that in the limit of small displacements,

$$\frac{d\omega_z}{dt} d\theta = d\omega_z \frac{d\theta}{dt} = d\omega_z \, \omega_z \,. \tag{17.4.17}$$

Therefore the infinitesimal rotational work is

$$dW_{\text{rot}} = I_S \alpha_z \; d\theta = I_S \frac{d\omega_z}{dt} d\theta = I_S \, d\omega_z \frac{d\theta}{dt} = I_S \, d\omega_z \, \omega_z \,. \tag{17.4.18}$$

We can integrate this amount of rotational work as the angular velocity of the rigid body changes from some initial value $\omega_z = \omega_{z,i}$ to some final value $\omega_z = \omega_{z,f}$,

$$W_{\text{rot}} = \int dW_{\text{rot}} = \int_{\omega_{z,i}}^{\omega_{z,f}} I_S \; d\omega_z \; \omega_z = \frac{1}{2} I_S \, \omega_{z,f}^2 - \frac{1}{2} I_S \, \omega_{z,i}^2 \,. \tag{17.4.19}$$

When a rigid body is rotating about a fixed axis passing through a point S in the body, there is both rotation and translation about the center of mass unless S is the center of mass. If we choose the point S in the above equation for the rotational work to be the center of mass, then

$$W_{\text{rot}} = \frac{1}{2} I_{\text{cm}} \omega_{\text{cm},f}^2 - \frac{1}{2} I_{\text{cm}} \omega_{\text{cm},i}^2 = K_{\text{rot},f} - K_{\text{rot},i} \equiv \Delta K_{\text{rot}} \,. \tag{17.4.20}$$

Note that because the z-component of the angular velocity of the center of mass appears as a square, we can just use its magnitude in Eq. (17.4.20).

17.5.3 Rotational Power

The rotational power is defined as the rate of doing rotational work,

$$P_{\text{rot}} \equiv \frac{dW_{\text{rot}}}{dt} \,. \tag{17.4.21}$$

We can use our result for the infinitesimal work to find that the rotational power is the product of the applied torque with the angular velocity of the rigid body,

$$P_{\text{rot}} \equiv \frac{dW_{\text{rot}}}{dt} = \tau_{S,z} \frac{d\theta}{dt} = \tau_{S,z} \omega_z \,. \tag{17.4.22}$$

Example 17.12 Work Done by Frictional Torque

A steel washer is mounted on the shaft of a small motor. The moment of inertia of the motor and washer is I_0. The washer is set into motion. When it reaches an initial angular velocity ω_0, at $t = 0$, the power to the motor is shut off, and the washer slows down during a time interval $\Delta t_1 = t_a$ until it reaches an angular velocity of ω_a at time t_a. At that instant, a second steel washer with a moment of inertia I_w is dropped on top of the first washer. Assume that the second washer is only in contact with the first washer. The collision takes place over a time $\Delta t_{\text{int}} = t_b - t_a$ after which the two washers and rotor rotate with angular speed ω_b. Assume the frictional torque on the axle (magnitude τ_f) is independent of speed, and remains the same when the second washer is dropped. (a) What angle does the rotor rotate through during the collision? (b) What is the work done by the friction torque from the bearings during the collision? (c) Write down an equation for conservation of energy. Can you solve this equation for ω_b? (d) What is the average rate that work is being done by the friction torque during the collision?

Solution: We begin by solving for the frictional torque during the first stage of motion when the rotor is slowing down. We choose a coordinate system shown in Figure 17.29.

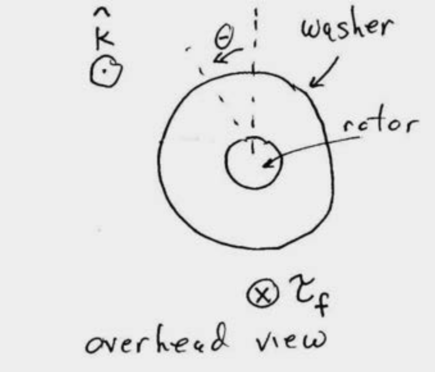


Figure 17.29 Coordinate system for Example 17.12

The component of average angular acceleration is given by

$$\alpha_1 = \frac{\omega_a - \omega_0}{t_a} < 0 .$$

We can use the rotational equation of motion, and find that the frictional torque satisfies

$$-\tau_f = I_0\left(\frac{\omega_a - \omega_0}{\Delta t_1}\right).$$

During the collision, the component of the average angular acceleration of the rotor is given by

$$\alpha_2 = \frac{\omega_b - \omega_a}{(\Delta t_{\text{int}})} < 0 .$$

The angle the rotor rotates through during the collision is (analogous to linear motion with constant acceleration)

$$\Delta\theta_2 = \omega_a \Delta t_{\text{int}} + \frac{1}{2}\alpha_2 \Delta t_{\text{int}}{}^2 = \omega_a \Delta t_{\text{int}} + \frac{1}{2}\left(\frac{\omega_b - \omega_a}{\Delta t_{\text{int}}}\right)\Delta t_{\text{int}}{}^2 = \frac{1}{2}(\omega_b + \omega_a)\Delta t_{\text{int}} > 0 .$$

The non-conservative work done by the bearing friction during the collision is

$$W_{f,b} = -\tau_f \Delta\theta_{rotor} = -\tau_f \frac{1}{2}(\omega_a + \omega_b)\Delta t_{\text{int}} .$$

Using our result for the frictional torque, the work done by the bearing friction during the collision is

$$W_{f,b} = \frac{1}{2} I_0 \left(\frac{\omega_a - \omega_0}{\Delta t_1}\right)(\omega_a + \omega_b)\Delta t_{\text{int}} < 0 .$$

The negative work is consistent with the fact that the kinetic energy of the rotor is decreasing as the rotor is slowing down. Using the work energy theorem during the collision the kinetic energy of the rotor has deceased by

$$W_{f,b} = \frac{1}{2}(I_0 + I_w)\omega_b^2 - \frac{1}{2} I_0 \omega_a^2 .$$

Using our result for the work, we have that

$$\frac{1}{2} I_0 \left(\frac{\omega_a - \omega_0}{\Delta t_1}\right)(\omega_a + \omega_b)\Delta t_{\text{int}} = \frac{1}{2}(I_0 + I_w)\omega_b^2 - \frac{1}{2} I_0 \omega_a^2 .$$

This is a quadratic equation for the angular speed ω_b of the rotor and washer immediately after the collision that we can in principle solve. However remember that we assumed that the frictional torque is independent of the speed of the rotor. Hence the best practice would be to measure ω_0, ω_a, ω_b, Δt_1, Δt_{int}, I_0, and I_w and then determine how closely our model agrees with conservation of energy. The rate of work done by the frictional torque is given by

$$P_f = \frac{W_{f,b}}{\Delta t_{\text{int}}} = \frac{1}{2} I_0 \left(\frac{\omega_a - \omega_0}{\Delta t_1}\right)(\omega_a + \omega_b) < 0 .$$

Chapter 18 Static Equilibrium

Chapter 18 Static Equilibrium

> *The proof of the correctness of a new rule can be attained by the repeated application of it, the frequent comparison with experience, the putting of it to the test under the most diverse circumstances. This process, would in the natural course of events, be carried out in time. The discoverer, however hastens to reach his goal more quickly. He compares the results that flow from his rule with all the experiences with which he is familiar, with all older rules, repeatedly tested in times gone by, and watches to see if he does not light on contradictions. In this procedure, the greatest credit is, as it should be, conceded to the oldest and most familiar experiences, the most thoroughly tested rules. Our instinctive experiences, those generalizations that are made involuntarily, by the irresistible force of the innumerable facts that press upon us, enjoy a peculiar authority; and this is perfectly warranted by the consideration that it is precisely the elimination of subjective caprice and of individual error that is the object aimed at.*[1]
>
> *Ernst Mach*

18.1 Introduction Static Equilibrium

When the vector sum of the forces acting on a point-like object is zero then the object will continue in its state of rest, or of uniform motion in a straight line. If the object is in uniform motion we can always change reference frames so that the object will be at rest. We showed that for a collection of point-like objects the sum of the external forces may be regarded as acting at the center of mass. So if that sum is zero the center of mass will continue in its state of rest, or of uniform motion in a straight line. We introduced the idea of a rigid body, and again showed that in addition to the fact that the sum of the external forces may be regarded as acting at the center of mass, forces like the gravitational force that acts at every point in the body may be treated as acting at the center of mass. However for an extended rigid body it matters where the force is applied because even though the sum of the forces on the body may be zero, a non-zero sum of torques on the body may still produce angular acceleration. In particular for fixed axis rotation, the torque along the axis of rotation on the object is proportional to the angular acceleration. It is possible that sum of the torques may be zero on a body that is not constrained to rotate about a fixed axis and the body may still undergo rotation. We will restrict ourselves to the special case in which in an inertial reference frame both the center of mass of the body is at rest and the body does not undergo any rotation, a condition that is called ***static equilibrium of an extended object.***

The two sufficient and necessary conditions for a rigid body to be in static equilibrium are:

---

[1] Ernst Mach, *The Science of Mechanics: A Critical and Historical Account of Its Development*, translated by Thomas J. McCormack, Sixth Edition with Revisions through the Ninth German Edition, Open Court Publishing, Illinois.

(1) The sum of the forces acting on the rigid body is zero,

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots = \vec{\mathbf{0}}. \tag{18.1.1}$$

(2) The vector sum of the torques about any point S in a rigid body is zero,

$$\vec{\boldsymbol{\tau}}_S = \vec{\boldsymbol{\tau}}_{S,1} + \vec{\boldsymbol{\tau}}_{S,2} + \cdots = \vec{\mathbf{0}}. \tag{18.1.2}$$

18.2 Lever Law

Let's consider a uniform rigid beam of mass m_b balanced on a pivot near the center of mass of the beam. We place two objects 1 and 2 of masses m_1 and m_2 on the beam, at distances d_1 and d_2 respectively from the pivot, so that the beam is static (that is, the beam is not rotating. See Figure 18.1.) We shall neglect the thickness of the beam and take the pivot point to be the center of mass.

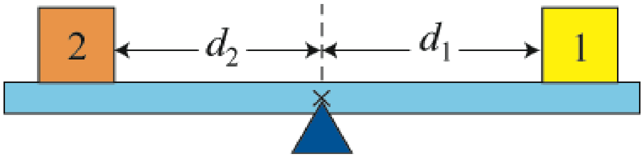


Figure 18.1 Pivoted Lever

Let's consider the forces acting on the beam. The earth attracts the beam downward. This gravitational force acts on every atom in the beam, but we can summarize its action by stating that the gravitational force $m_b\vec{\mathbf{g}}$ is concentrated at a point in the beam called the *center of gravity* of the beam, which is identical to the center of mass of the uniform beam. There is also a contact force $\vec{\mathbf{F}}_{\text{pivot}}$ between the pivot and the beam, acting upwards on the beam at the pivot point. The objects 1 and 2 exert normal forces downwards on the beam, $\vec{\mathbf{N}}_{1,b} \equiv \vec{\mathbf{N}}_1$, and $\vec{\mathbf{N}}_{2,b} \equiv \vec{\mathbf{N}}_2$, with magnitudes N_1, and N_2, respectively. Note that the normal forces are not the gravitational forces acting on the objects, but contact forces between the beam and the objects. (In this case, they are mathematically the same, due to the horizontal configuration of the beam and the fact that all objects are in static equilibrium.) The distances d_1 and d_2 are called the *moment arms* with respect to the pivot point for the forces $\vec{\mathbf{N}}_1$ and $\vec{\mathbf{N}}_2$, respectively. The force diagram on the beam is shown in Figure 18.2. Note that the pivot force $\vec{\mathbf{F}}_{\text{pivot}}$ and the force of gravity $m_b\vec{\mathbf{g}}$ each has a zero moment arm about the pivot point.

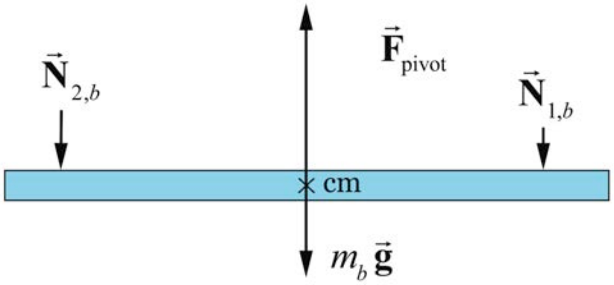
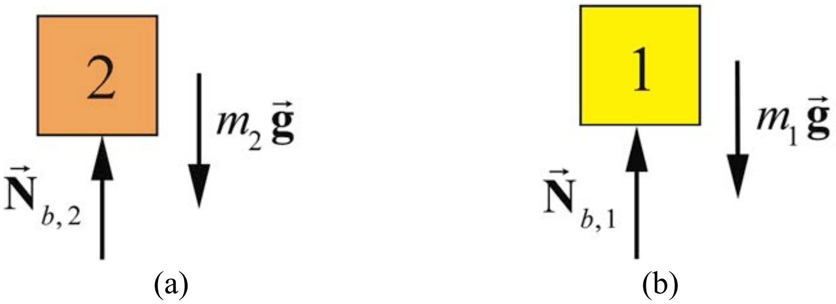


Figure 18.2 Free-body diagram on beam

Because we assume the beam is not moving, the sum of the forces in the vertical direction acting on the beam is therefore zero,

$$F_{\text{pivot}} - m_b g - N_1 - N_2 = 0\,. \tag{18.2.1}$$

The force diagrams on the objects are shown in Figure 18.3. Note the magnitude of the normal forces on the objects are also N_1 and N_2 since these are each part of an action-reaction pair, $\vec{\mathbf{N}}_{1,b} = -\vec{\mathbf{N}}_{b,1}$, and $\vec{\mathbf{N}}_{2,b} = -\vec{\mathbf{N}}_{b,2}$.



(a) (b)

Figure 18.3 Free-body force diagrams for each body.

The condition that the forces sum to zero is not sufficient to completely predict the motion of the beam. All we can deduce is that the center of mass of the system is at rest (or moving with a uniform velocity). In order for the beam not to rotate the sum of the torques about any point must be zero. In particular the sum of the torques about the pivot point must be zero. Because the moment arm of the gravitational force and the pivot force is zero, only the two normal forces produce a torque on the beam. If we choose out of the page as positive direction for the torque (or equivalently counterclockwise rotations are positive) then the condition that the sum of the torques about the pivot point is zero becomes

$$d_2 N_2 - d_1 N_1 = 0\,. \tag{18.2.2}$$

The magnitudes of the two torques about the pivot point are equal, a condition known as the lever law.

Lever Law: *A beam of length l is balanced on a pivot point that is placed directly beneath the center of mass of the beam. The beam will not undergo rotation if the product of the normal force with the moment arm to the pivot is the same for each body,*

$$d_1 N_1 = d_2 N_2 \,. \tag{18.2.3}$$

Example 18.1 Lever Law

Suppose a uniform beam of length $l = 1.0\text{ m}$ and mass $m_\text{B} = 2.0\text{ kg}$ is balanced on a pivot point, placed directly beneath the center of the beam. We place body 1 with mass $m_1 = 0.3\text{ kg}$ a distance $d_1 = 0.4\text{ m}$ to the right of the pivot point, and a second body 2 with $m_2 = 0.6\text{ kg}$ a distance d_2 to the left of the pivot point, such that the beam neither translates nor rotates. (a) What is the force $\vec{\mathbf{F}}_\text{pivot}$ that the pivot exerts on the beam? (b) What is the distance d_2 that maintains static equilibrium?

Solution: a) By Newton's Third Law, the beam exerts equal and opposite normal forces of magnitude N_1 on body 1, and N_2 on body 2. The condition for force equilibrium applied separately to the two bodies yields

$$N_1 - m_1 \, g = 0 \,, \tag{18.2.4}$$

$$N_2 - m_2 \, g = 0 \,. \tag{18.2.5}$$

Thus the total force acting on the beam is zero,

$$F_\text{pivot} - (m_b + m_1 + m_2) g = 0 \,, \tag{18.2.6}$$

and the pivot force is

$$\begin{aligned} F_\text{pivot} &= (m_b + m_1 + m_2) g \\ &= (2.0\text{ kg} + 0.3\text{ kg} + 0.6\text{ kg})(9.8\text{ m}\cdot\text{s}^{-2}) = 2.8 \times 10^1\text{ N}. \end{aligned} \tag{18.2.7}$$

b) We can compute the distance d_2 from the Lever Law,

$$d_2 = \frac{d_1 N_1}{N_2} = \frac{d_1 m_1 g}{m_2 g} = \frac{d_1 m_1}{m_2} = \frac{(0.4\text{ m})(0.3\text{ kg})}{0.6\text{ kg}} = 0.2\text{ m} \,. \tag{18.2.8}$$

18.3 Generalized Lever Law

We can extend the Lever Law to the case in which two external forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ are acting on the pivoted beam at angles θ_1 and θ_2 with respect to the horizontal as shown in the Figure 18.4. Throughout this discussion the angles will be limited to the range $[0 \le \theta_1, \theta_2 \le \pi]$. We shall again neglect the thickness of the beam and take the pivot point to be the center of mass.

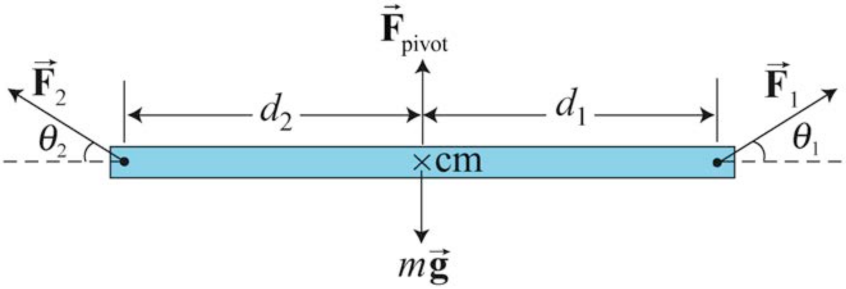


Figure 18.4 Forces acting at angles to a pivoted beam.

The forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ can be decomposed into separate vectors components respectively $(\vec{\mathbf{F}}_{1,\|}, \vec{\mathbf{F}}_{1,\perp})$ and $(\vec{\mathbf{F}}_{2,\|}, \vec{\mathbf{F}}_{2,\perp})$, where $\vec{\mathbf{F}}_{1,\|}$ and $\vec{\mathbf{F}}_{2,\|}$ are the horizontal vector projections of the two forces with respect to the direction formed by the length of the beam, and $\vec{\mathbf{F}}_{1,\perp}$ and $\vec{\mathbf{F}}_{2,\perp}$ are the perpendicular vector projections respectively to the beam (Figure 18.5), with

$$\vec{\mathbf{F}}_1 = \vec{\mathbf{F}}_{1,\|} + \vec{\mathbf{F}}_{1,\perp}, \tag{18.3.1}$$

$$\vec{\mathbf{F}}_2 = \vec{\mathbf{F}}_{2,\|} + \vec{\mathbf{F}}_{2,\perp}. \tag{18.3.2}$$

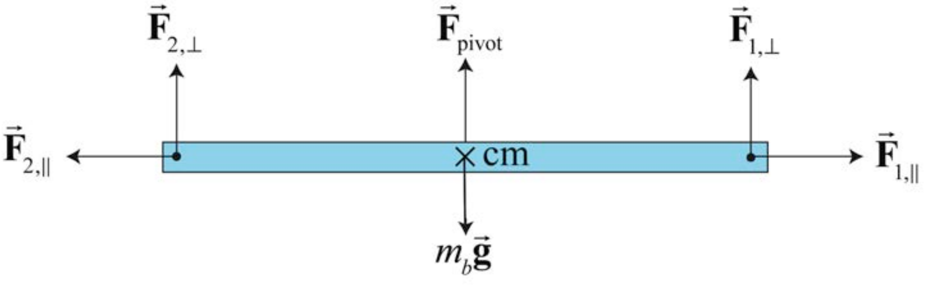


Figure 18.5 Vector decomposition of forces.

The horizontal components of the forces are

$$F_{1,\|} = F_1 \cos\theta_1, \tag{18.3.3}$$

$$F_{2,\|} = -F_2 \cos\theta_2, \tag{18.3.4}$$

where our choice of positive horizontal direction is to the right. Neither horizontal force component contributes to possible rotational motion of the beam. The sum of these horizontal forces must be zero,

$$F_1 \cos\theta_1 - F_2 \cos\theta_2 = 0\,. \tag{18.3.5}$$

The perpendicular component forces are

$$F_{1,\perp} = F_1 \sin\theta_1\,, \tag{18.3.6}$$

$$F_{2,\perp} = F_2 \sin\theta_2\,, \tag{18.3.7}$$

where the positive vertical direction is upwards. The perpendicular components of the forces must also sum to zero,

$$F_{\text{pivot}} - m_b g + F_1 \sin\theta_1 + F_2 \sin\theta_2 = 0\,. \tag{18.3.8}$$

Only the vertical components $F_{1,\perp}$ and $F_{2,\perp}$ of the external forces are involved in the lever law (but the horizontal components must balance, as in Equation (18.3.5), for equilibrium). Then the Lever Law can be extended as follows.

> **Generalized Lever Law** *A beam of length l is balanced on a pivot point that is placed directly beneath the center of mass of the beam. Suppose a force $\vec{\mathbf{F}}_1$ acts on the beam a distance d_1 to the right of the pivot point. A second force $\vec{\mathbf{F}}_2$ acts on the beam a distance d_2 to the left of the pivot point. The beam will remain in static equilibrium if the following two conditions are satisfied:*
>
> *1) The total force on the beam is zero,*
>
> *2) The product of the magnitude of the perpendicular component of the force with the distance to the pivot is the same for each force,*
>
> $$d_1 \left|F_{1,\perp}\right| = d_2 \left|F_{2,\perp}\right|. \tag{18.3.9}$$

The Generalized Lever Law can be stated in an equivalent form,

$$d_1 F_1 \sin\theta_1 = d_2 F_2 \sin\theta_2\,. \tag{18.3.10}$$

We shall now show that the generalized lever law can be reinterpreted as the statement that the vector sum of the torques about the pivot point S is zero when there are just two forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ acting on our beam as shown in Figure 18.6.

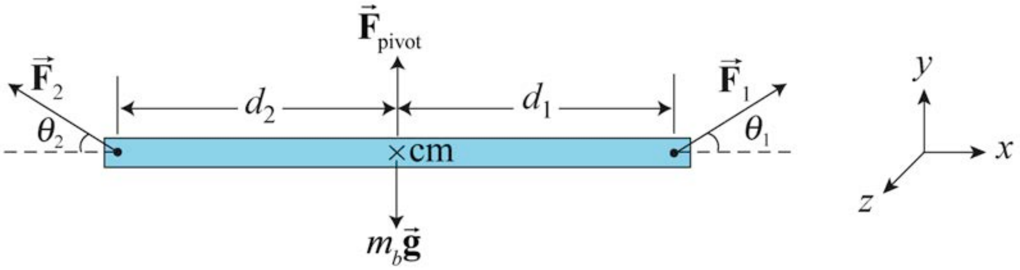


Figure 18.6 Force and torque diagram.

Let's choose the positive z-direction to point out of the plane of the page then torque pointing out of the page will have a positive z-component of torque (counterclockwise rotations are positive). From our definition of torque about the pivot point, the magnitude of torque due to force $\vec{\mathbf{F}}_1$ is given by

$$\tau_{S,1} = d_1 F_1 \sin\theta_1 . \tag{18.3.11}$$

From the right hand rule this is out of the page (in the counterclockwise direction) so the component of the torque is positive, hence,

$$(\tau_{S,1})_z = d_1 F_1 \sin\theta_1 . \tag{18.3.12}$$

The torque due to $\vec{\mathbf{F}}_2$ about the pivot point is into the page (the clockwise direction) and the component of the torque is negative and given by

$$(\tau_{S,2})_z = -d_2 F_2 \sin\theta_2 . \tag{18.3.13}$$

The z-component of the torque is the sum of the z-components of the individual torques and is zero,

$$(\tau_{S,\text{total}})_z = (\tau_{S,1})_z + (\tau_{S,2})_z = d_1 F_1 \sin\theta_1 - d_2 F_2 \sin\theta_2 = 0 , \tag{18.3.14}$$

which is equivalent to the Generalized Lever Law, Equation (18.3.10),

$$d_1 F_1 \sin\theta_1 = d_2 F_2 \sin\theta_2 .$$

18.4 Worked Examples

Example 18.2 Suspended Rod

A uniform rod of length $l = 2.0\text{ m}$ and mass $m = 4.0\text{ kg}$ is hinged to a wall at one end and suspended from the wall by a cable that is attached to the other end of the rod at an angle of $\beta = 30^o$ to the rod (see Figure 18.7). Assume the cable has zero mass. There is a contact force at the pivot on the rod. The magnitude and direction of this force is unknown. One of the most difficult parts of these types of problems is to introduce an angle for the pivot force and then solve for that angle if possible. In this problem you will solve for the magnitude of the tension in the cable and the direction and magnitude of the pivot force. (a) What is the tension in the cable? (b) What angle does the pivot force make with the beam? (c) What is the magnitude of the pivot force?

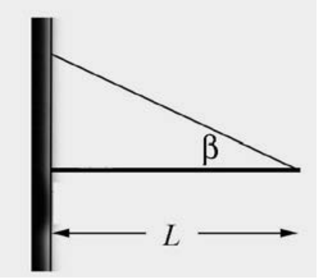


Figure 18.7 Example 18.2

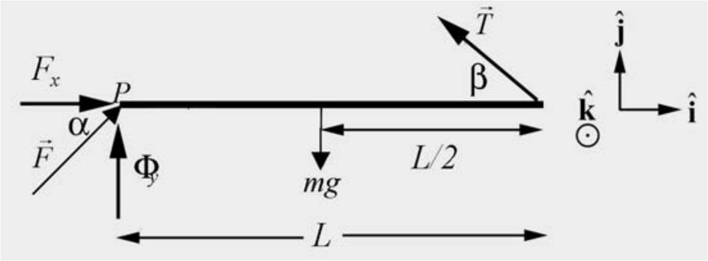


Figure 18.8 Force and torque diagram.

Solution: a) The force diagram is shown in Figure 18.8. Take the positive $\hat{\mathbf{i}}$-direction to be to the right in the figure above, and take the positive $\hat{\mathbf{j}}$-direction to be vertically upward. The forces on the rod are: the gravitational force $m\vec{\mathbf{g}} = -m\,g\,\hat{\mathbf{j}}$, acting at the center of the rod; the force that the cable exerts on the rod, $\vec{\mathbf{T}} = T(-\cos\beta\,\hat{\mathbf{i}} + \sin\beta\,\hat{\mathbf{j}})$, acting at the right end of the rod; and the pivot force $\vec{\mathbf{F}}_{\text{pivot}} = F(\cos\alpha\,\hat{\mathbf{i}} + \sin\alpha\,\hat{\mathbf{j}})$, acting at the left end of the rod. If $0 < \alpha < \pi/2$, the pivot force is directed up and to the right in the figure. If $0 > \alpha > -\pi/2$, the pivot force is directed down and to the right. We have no reason, at this point, to expect that α will be in either of the quadrants, but it must be in one or the other.

For static equilibrium, the sum of the forces must be zero, and hence the sums of the components of the forces must be zero,

$$\begin{aligned} 0 &= -T\cos\beta + F\cos\alpha \\ 0 &= -m\,g + T\sin\beta + F\sin\alpha. \end{aligned} \tag{18.4.1}$$

With respect to the pivot point, and taking positive torques to be counterclockwise, the gravitational force exerts a negative torque of magnitude $m\,g(l/2)$ and the cable exerts a positive torque of magnitude $T\,l\sin\beta$. The pivot force exerts no torque about the pivot. Setting the sum of the torques equal to zero then gives

$$\begin{aligned} 0 &= T\,l\sin\beta - m\,g(l/2) \\ T &= \frac{m\,g}{2\sin\beta}. \end{aligned} \tag{18.4.2}$$

This result has many features we would expect; proportional to the weight of the rod and inversely proportional to the sine of the angle made by the cable with respect to the horizontal. Inserting numerical values gives

$$T = \frac{mg}{2\sin\beta} = \frac{(4.0\,\mathrm{kg})(9.8\,\mathrm{m}\cdot\mathrm{s}^{-2})}{2\sin 30^{\circ}} = 39.2\,\mathrm{N}. \qquad (18.4.3)$$

There are many ways to find the angle α. Substituting Eq. (18.4.2) for the tension into both force equations in Eq. (18.4.1) yields

$$\begin{aligned} F\cos\alpha &= T\cos\beta = (mg/2)\cot\beta \\ F\sin\alpha &= mg - T\sin\beta = mg/2. \end{aligned} \qquad (18.4.4)$$

In Eq. (18.4.4), dividing one equation by the other, we see that $\tan\alpha = \tan\beta$, $\alpha = \beta$.

The horizontal forces on the rod must cancel. The tension force and the pivot force act with the same angle (but in opposite horizontal directions) and hence must have the same magnitude,

$$F = T = 39.2\,\mathrm{N}. \qquad (18.4.5)$$

As an alternative, if we had not done the previous parts, we could find torques about the point where the cable is attached to the wall. The cable exerts no torque about this point and the y-component of the pivot force exerts no torque as well. The moment arm of the x-component of the pivot force is $l\tan\beta$ and the moment arm of the weight is $l/2$. Equating the magnitudes of these two torques gives

$$F\cos\alpha\, l\tan\beta = mg\frac{l}{2},$$

equivalent to the first equation in Eq. (18.4.4). Similarly, evaluating torques about the right end of the rod, the cable exerts no torques and the x-component of the pivot force exerts no torque. The moment arm of the y-component of the pivot force is l and the moment arm of the weight is $l/2$. Equating the magnitudes of these two torques gives

$$F\sin\alpha\, l = mg\frac{l}{2},$$

reproducing the second equation in Eq. (18.4.4). The point of this alternative solution is to show that choosing a different origin (or even more than one origin) in order to remove an unknown force from the torques equations might give a desired result more directly.

Example 18.3 Person Standing on a Hill

A person is standing on a hill that is sloped at an angle of α with respect to the horizontal (Figure 18.9). The person's legs are separated by a distance d, with one foot uphill and one downhill. The center of mass of the person is at a distance h above the ground, perpendicular to the hillside, midway between the person's feet. Assume that the coefficient of static friction between the person's feet and the hill is sufficiently large that the person will not slip. (a) What is the magnitude of the normal force on each foot? (b) How far must the feet be apart so that the normal force on the upper foot is just zero? This is the moment when the person starts to rotate and fall over.

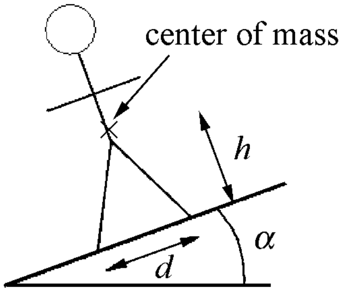


Figure 18.9 Person standing on hill

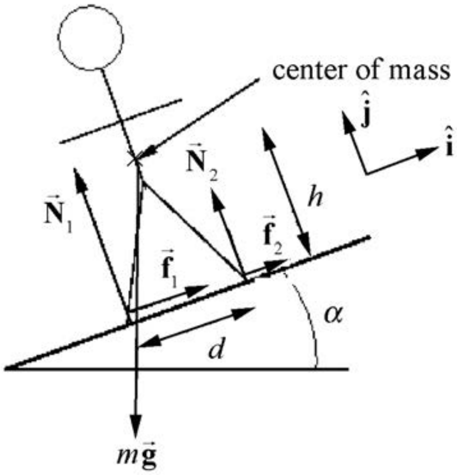


Figure 18.10 Free-body force diagram for person standing on hill

Solution: The force diagram on the person is shown in Figure 18.10. Note that the contact forces have been decomposed into components perpendicular and parallel to the hillside. A choice of unit vectors and positive direction for torque is also shown. Applying Newton's Second Law to the two components of the net force,

$$\hat{\mathbf{j}}: \quad N_1 + N_2 - mg\cos\alpha = 0 \tag{18.4.6}$$

$$\hat{\mathbf{i}}: \quad f_1 + f_2 - mg\sin\alpha = 0\,. \tag{18.4.7}$$

These two equations imply that

$$N_1 + N_2 = mg\cos\alpha \tag{18.4.8}$$

$$f_1 + f_2 = mg\sin\alpha\,. \tag{18.4.9}$$

Evaluating torques about the center of mass,

$$h(f_1 + f_2) + (N_2 - N_1)\frac{d}{2} = 0\,. \tag{18.4.10}$$

Equation (18.4.10) can be rewritten as

$$N_1 - N_2 = \frac{2h(f_1 + f_2)}{d}. \tag{18.4.11}$$

Substitution of Equation (18.4.9) into Equation (18.4.11) yields

$$N_1 - N_2 = \frac{2h(mg\sin\alpha)}{d}. \tag{18.4.12}$$

We can solve for N_1 by adding Equations (18.4.8) and (18.4.12), and then dividing by 2, yielding

$$N_1 = \frac{1}{2}mg\cos\alpha + \frac{h(mg\sin\alpha)}{d} = mg\left(\frac{1}{2}\cos\alpha + \frac{h}{d}\sin\alpha\right). \tag{18.4.13}$$

Similarly, we can solve for N_2 by subtracting Equation (18.4.12) from Equation (18.4.8) and dividing by 2, yielding

$$N_2 = mg\left(\frac{1}{2}\cos\alpha - \frac{h}{d}\sin\alpha\right). \tag{18.4.14}$$

The normal force N_2 as given in Equation (18.4.14) vanishes when

$$\frac{1}{2}\cos\alpha = \frac{h}{d}\sin\alpha, \tag{18.4.15}$$

which can be solved for the minimum distance between the legs,

$$d = 2h(\tan\alpha). \tag{18.4.16}$$

It should be noted that no specific model for the frictional force was used, that is, no coefficient of static friction entered the problem. The two frictional forces f_1 and f_2 were not determined separately; only their sum entered the above calculations.

Example 18.4 The Knee

A man of mass $m = 70\,\text{kg}$ is about to start a race. Assume the runner's weight is equally distributed on both legs. The patellar ligament in the knee is attached to the upper tibia and runs over the kneecap. When the knee is bent, a tensile force, $\vec{\mathbf{T}}$, that the ligament exerts on the upper tibia, is directed at an angle of $\theta = 40°$ with respect to the horizontal. The femur exerts a force $\vec{\mathbf{F}}$ on the upper tibia. The angle, α, that this force makes with the vertical will vary and is one of the unknowns to solve for. Assume that the ligament is connected a distance, $d = 3.8\,\text{cm}$, directly below the contact point of the femur on the tibia. The contact point between the foot and the ground is a distance $s = 3.6\times10^{1}\text{cm}$ from the vertical line passing through contact point of the femur on the tibia. The center of mass of the lower leg lies a distance $x = 1.8\times10^{1}\text{cm}$ from this same vertical line. Suppose the mass m_L of the lower leg is a 1/10 of the mass of the body (Figure 18.11). (a) Find the magnitude T of the force $\vec{\mathbf{T}}$ of the patellar ligament on the tibia. (b) Find the direction (the angle α) of the force $\vec{\mathbf{F}}$ of the femur on the tibia. (c) Find the magnitude F of the force $\vec{\mathbf{F}}$ of the femur on the tibia.

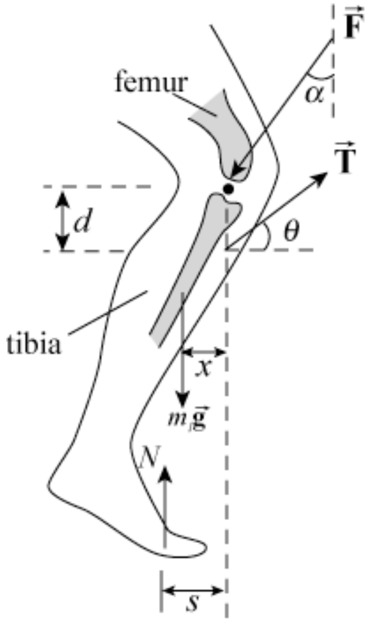


Figure 18.11 Example 18.4

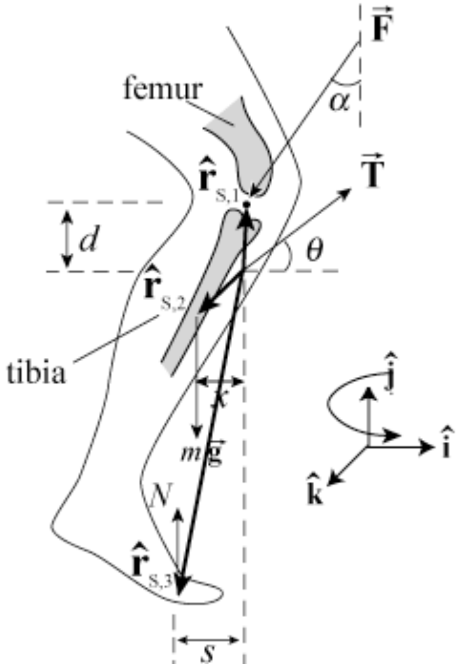


Figure 18.12 Torque-force diagram for knee

Solutions: a) Choose the unit vector $\hat{\mathbf{i}}$ to be directed horizontally to the right and $\hat{\mathbf{j}}$ directed vertically upwards. The first condition for static equilibrium, Eq. (18.1.1), that the sum of the forces is zero becomes

$$\hat{\mathbf{i}}: -F\sin\alpha + T\cos\theta = 0. \qquad (18.4.17)$$

$$\hat{\mathbf{j}}: N - F\cos\alpha + T\sin\theta - (1/10)mg = 0. \qquad (18.4.18)$$

Because the weight is evenly distributed on the two feet, the normal force on one foot is equal to half the weight, or

$$N = (1/2)mg; \qquad (18.4.19)$$

Equation (18.4.18) becomes

$$\begin{aligned}\hat{\mathbf{j}}:\ (1/2)mg - F\cos\alpha + T\sin\theta - (1/10)mg &= 0\\ (2/5)mg - F\cos\alpha + T\sin\theta &= 0.\end{aligned} \qquad (18.4.20)$$

The torque-force diagram on the knee is shown in Figure 18.12. Choose the point of action of the ligament on the tibia as the point S about which to compute torques. Note that the tensile force, $\vec{\mathbf{T}}$, that the ligament exerts on the upper tibia will make no contribution to the torque about this point S. This may help slightly in doing the calculations. Choose counterclockwise as the positive direction for the torque; this is the positive $\hat{\mathbf{k}}$ - direction. Then the torque due to the force $\vec{\mathbf{F}}$ of the femur on the tibia is

$$\vec{\boldsymbol{\tau}}_{S,1} = \vec{\mathbf{r}}_{S,1} \times \vec{\mathbf{F}} = d\,\hat{\mathbf{j}} \times (-F\sin\alpha\,\hat{\mathbf{i}} - F\cos\alpha\,\hat{\mathbf{j}}) = d\,F\sin\alpha\,\hat{\mathbf{k}}\,. \tag{18.4.21}$$

The torque due to the mass of the leg is

$$\vec{\boldsymbol{\tau}}_{S,2} = \vec{\mathbf{r}}_{S,2} \times (-mg/10)\,\hat{\mathbf{j}} = (-x\,\hat{\mathbf{i}} - y_L\hat{\mathbf{j}}) \times (-mg/10)\,\hat{\mathbf{j}} = (1/10)x\,mg\,\hat{\mathbf{k}}\,. \tag{18.4.22}$$

The torque due to the normal force of the ground is

$$\vec{\boldsymbol{\tau}}_{S,3} = \vec{\mathbf{r}}_{S,3} \times N\,\hat{\mathbf{j}} = (-s\,\hat{\mathbf{i}} - y_N\,\hat{\mathbf{j}}) \times N\,\hat{\mathbf{j}} = -s\,N\,\hat{\mathbf{k}} = -(1/2)s\,mg\,\hat{\mathbf{k}}\,. \tag{18.4.23}$$

(In Equations (18.4.22) and (18.4.23), y_L and y_N are the vertical displacements of the point where the weight of the leg and the normal force with respect to the point S; as can be seen, these quantities do not enter directly into the calculations.) The condition that the sum of the torques about the point S vanishes, Eq. (18.1.2),

$$\vec{\boldsymbol{\tau}}_{S,\text{ total}} = \vec{\boldsymbol{\tau}}_{S,1} + \vec{\boldsymbol{\tau}}_{S,2} + \vec{\boldsymbol{\tau}}_{S,3} = \vec{\mathbf{0}}\,, \tag{18.4.24}$$

becomes

$$d\,F\sin\alpha\,\hat{\mathbf{k}} + (1/10)x\,mg\,\hat{\mathbf{k}} - (1/2)s\,mg\,\hat{\mathbf{k}} = \vec{\mathbf{0}}\,. \tag{18.4.25}$$

The three equations in the three unknowns are summarized below:

$$\begin{aligned} -F\sin\alpha + T\cos\theta &= 0 \\ (2/5)mg - F\cos\alpha + T\sin\theta &= 0 \\ d\,F\sin\alpha + (1/10)x\,mg - (1/2)s\,mg &= 0. \end{aligned} \tag{18.4.26}$$

The horizontal force equation, the first in (18.4.26), implies that

$$F\sin\alpha = T\cos\theta\,. \tag{18.4.27}$$

Substituting this into the torque equation, the third equation of (18.4.26), yields

$$d\,T\cos\theta + (1/10)x\,mg - s(1/2)mg = 0\,. \tag{18.4.28}$$

Note that Equation (18.4.28) is the equation that would have been obtained if we had chosen the contact point between the tibia and the femur as the point about which to determine torques. Had we chosen this point, we would have saved one minor algebraic step. We can solve this Equation (18.4.28) for the magnitude T of the force $\vec{\mathbf{T}}$ of the patellar ligament on the tibia,

$$T = \frac{s(1/2)mg - (1/10)x\,mg}{d\cos\theta}. \tag{18.4.29}$$

Inserting numerical values into Equation (18.4.29),

$$\begin{aligned} T &= (70\,\text{kg})(9.8\,\text{m}\cdot\text{s}^{-2})\frac{(3.6\times10^{-1}\text{m})(1/2) - (1/10)(1.8\times10^{-1}\text{m})}{(3.8\times10^{-2}\text{m})\cos(40°)} \\ &= 3.8\times10^{3}\,\text{N}. \end{aligned} \tag{18.4.30}$$

b) We can now solve for the direction α of the force $\vec{\mathbf{F}}$ of the femur on the tibia as follows. Rewrite the two force equations in (18.4.26) as

$$\begin{aligned} F\cos\alpha &= (2/5)mg + T\sin\theta \\ F\sin\alpha &= T\cos\theta. \end{aligned} \tag{18.4.31}$$

Dividing these equations yields

$$\frac{F\cos\alpha}{F\sin\alpha} = \text{cotan}\,\alpha = \frac{(2/5)mg + T\sin\theta}{T\cos\theta}, \tag{18.4.32}$$

And so

$$\begin{aligned} \alpha &= \text{cotan}^{-1}\left(\frac{(2/5)mg + T\sin\theta}{T\cos\theta}\right) \\ \alpha &= \text{cotan}^{-1}\left(\frac{(2/5)(70\,\text{kg})(9.8\,\text{m}\cdot\text{s}^{-2}) + (3.4\times10^{3}\,\text{N})\sin(40°)}{(3.4\times10^{3}\text{N})\cos(40°)}\right) = 47°. \end{aligned} \tag{18.4.33}$$

c) We can now use the horizontal force equation to calculate the magnitude F of the force of the femur $\vec{\mathbf{F}}$ on the tibia from Equation (18.4.27),

$$F = \frac{(3.8\times10^{3}\,\text{N})\cos(40°)}{\sin(47°)} = 4.0\times10^{3}\,\text{N}. \tag{18.4.34}$$

Note you can find a symbolic expression for α that did not involve the intermediate numerical calculation of the tension. This is rather complicated algebraically; basically, the last two equations in (18.4.26) are solved for F and T in terms of α, θ and the other variables (Cramer's Rule is suggested) and the results substituted into the first of (18.4.26). The resulting expression is

$$\begin{aligned}\cot\alpha &= \frac{(s/2 - x/10)\sin(40^\circ) + ((2d/5)\cos(40^\circ))}{(s/2 - x/10)\cos(40^\circ)} \\ &= \tan(40^\circ) + \frac{2d/5}{s/2 - x/10}\end{aligned} \qquad (18.4.35)$$

which leads to the same numerical result, $\alpha = 47^\circ$.

Appendix 18A The Torques About Any Two Points are Equal for a Body in Static Equilibrium

When the net force on a body is zero, the torques about any two points are equal. To show this, consider any two points A and B. Choose a coordinate system with origin O and denote the constant vector from A to B by $\vec{\mathbf{r}}_{A,B}$. Suppose a force $\vec{\mathbf{F}}_i$ is acting at the point $\vec{\mathbf{r}}_{\mathrm{O},i}$. The vector from the point A to the point where the force acts is denoted by $\vec{\mathbf{r}}_{A,\,i}$, and the vectors from the point B to the point where the force acts is denoted by $\vec{\mathbf{r}}_{B,\,i}$.

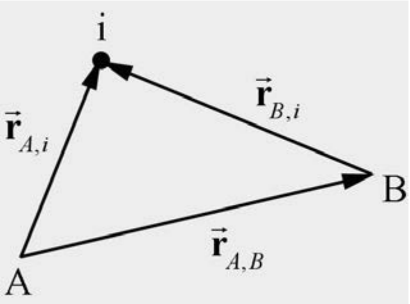


Figure 18A.1 Location of body i with respect to the points A and B.

In Figure 18A.1, the position vectors satisfy

$$\vec{\mathbf{r}}_{A,i} = \vec{\mathbf{r}}_{A,B} + \vec{\mathbf{r}}_{B,i}. \tag{18.A.1}$$

The sum of the torques about the point A is given by

$$\vec{\boldsymbol{\tau}}_A = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{A,i} \times \vec{\mathbf{F}}_i . \tag{18.A.2}$$

The sum of the torques about the point B is given by

$$\vec{\boldsymbol{\tau}}_B = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{B,i} \times \vec{\mathbf{F}}_i . \tag{18.A.3}$$

We can now substitute Equation (18.A.1) into Equation (18.A.2) and find that

$$\vec{\boldsymbol{\tau}}_A = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{A,i} \times \vec{\mathbf{F}}_i = \sum_{i=1}^{i=N} (\vec{\mathbf{r}}_{A,B} + \vec{\mathbf{r}}_{B,i}) \times \vec{\mathbf{F}}_i = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{A,B} \times \vec{\mathbf{F}}_i + \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{B,i} \times \vec{\mathbf{F}}_i . \tag{18.A.4}$$

In the next-to-last term in Equation (18.A.4), the vector $\vec{\mathbf{r}}_{A,B}$ is constant and so may be taken outside the summation,

$$\sum_{i=1}^{i=N} \vec{\mathbf{r}}_{A,B} \times \vec{\mathbf{F}}_i = \vec{\mathbf{r}}_{A,B} \times \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i \,. \tag{18.A.5}$$

We are assuming that there is no net force on the body, and so the sum of the forces on the body is zero,

$$\sum_{i=1}^{i=N} \vec{\mathbf{F}}_i = \vec{\mathbf{0}} \,. \tag{18.A.6}$$

Therefore the torque about point A, Equation (18.A.2), becomes

$$\vec{\boldsymbol{\tau}}_A = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{B,i} \times \vec{\mathbf{F}}_i \;= \vec{\boldsymbol{\tau}}_B \,. \tag{18.A.7}$$

For static equilibrium problems, the result of Equation (18.A.7) tells us that it does not matter which point we use to determine torques. In fact, note that the position of the chosen origin did not affect the result at all. Choosing the point about which to calculate torques (variously called "A", "B", "S" or sometimes "O") so that unknown forces do not exert torques about that point may often greatly simplify calculations.

Chapter 19 Angular Momentum

Chapter 19 Angular Momentum

> *The situation, in brief, is that newtonian physics is incapable of predicting conservation of angular momentum, but no isolated system has yet been encountered experimentally for which angular momentum is not conserved. We conclude that conservation of angular momentum is an independent physical law, and until a contradiction is observed, our physical understanding must be guided by it.* [1]
>
> *Dan Kleppner*

19.1 Introduction

When we consider a system of objects, we have shown that the external force, acting at the center of mass of the system, is equal to the time derivative of the total momentum of the system,

$$\vec{\mathbf{F}}^{\text{ext}} = \frac{d\vec{\mathbf{p}}_{\text{sys}}}{dt} . \tag{19.1.1}$$

We now introduce the rotational analog of Equation (19.1.1). We will first introduce the concept of angular momentum for a point-like particle of mass m with linear momentum $\vec{\mathbf{p}}$ about a point S, defined by the equation

$$\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}} , \tag{19.1.2}$$

where $\vec{\mathbf{r}}_S$ is the vector from the point S to the particle. We will show in this chapter that the torque about the point S acting on the particle is equal to the rate of change of the angular momentum about the point S of the particle,

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt} . \tag{19.1.3}$$

Equation (19.1.3) generalizes to any body undergoing rotation.

We shall concern ourselves first with the special case of rigid body undergoing fixed axis rotation about the z-axis with angular velocity $\vec{\boldsymbol{\omega}} = \omega_z \hat{\mathbf{k}}$. We divide up the rigid body into N elements labeled by the index i, $i = 1,2,\ldots N$, the i^{th} element having mass m_i and position vector $\vec{\mathbf{r}}_{S,i}$. The rigid body has a moment of inertia I_S about some point S on the fixed axis, (often taken to be the *z*-axis, but not always) which rotates with angular velocity $\vec{\boldsymbol{\omega}}$ about this axis. The angular momentum is then the vector sum of the individual angular momenta,

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[1] Kleppner, Daniel, An Introduction to Mechanics (1973), p. 307.

$$\vec{\mathbf{L}}_S = \sum_{i=1}^{i=N} \vec{\mathbf{L}}_{S,i} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{p}}_i \tag{19.1.4}$$

When the rotation axis is the z-axis the z-component of the angular momentum, $L_{S,z}$, about the point S is then given by

$$L_{S,z} = I_S \omega_z . \tag{19.1.5}$$

We shall show that the z-component of the torque about the point S, $\tau_{S,z}$, is then the time derivative of the z-component of angular momentum about the point S,

$$\tau_{S,z} = \frac{dL_{S,z}}{dt} = I_S \frac{d\omega_z}{dt} = I_S \alpha_z . \tag{19.1.6}$$

19.2 Angular Momentum about a Point for a Particle

19.2.1 Angular Momentum for a Point Particle

Consider a point-like particle of mass m moving with a velocity $\vec{\mathbf{v}}$ (Figure 19.1) with momentum $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$.

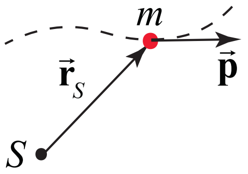

Figure 19.1 A point-like particle and its angular momentum about S.

Consider a point S located anywhere in space. Let $\vec{\mathbf{r}}_S$ denote the vector from the point S to the location of the object.

> *Define the angular momentum $\vec{\mathbf{L}}_S$ about the point S of a point-like particle as the vector product of the vector from the point S to the location of the object with the momentum of the particle,*

$$\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}} . \tag{19.2.1}$$

The derived SI units for angular momentum are $[\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-1}] = [\text{N} \cdot \text{m} \cdot \text{s}] = [\text{J} \cdot \text{s}]$. There is no special name for this set of units.

Because angular momentum is defined as a vector, we begin by studying its magnitude and direction. The magnitude of the angular momentum about S is given by

$$\left|\vec{\mathbf{L}}_S\right| = \left|\vec{\mathbf{r}}_S\right|\left|\vec{\mathbf{p}}\right|\sin\theta\,, \qquad (19.2.2)$$

where θ is the angle between the vectors and $\vec{\mathbf{p}}$, and lies within the range $[0 \le \theta \le \pi]$ Analogous to the magnitude of torque, there are two ways to determine the magnitude of the angular momentum about S.

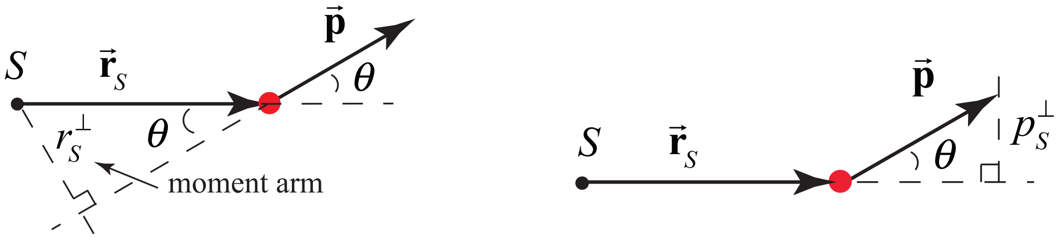

Figure 19.2 (a) Moment arm. (b) Perpendicular component of momentum.

Define the *moment arm*, $r_S^{\perp}$, (Figure 19.2 (a)), as the perpendicular distance from the point S to the line defined by the direction of the momentum. Then

$$r_S^{\perp} = \left|\vec{\mathbf{r}}_S\right|\sin\theta\,. \qquad (19.2.3)$$

Hence the magnitude of the angular momentum is the product of the moment arm with the magnitude of the momentum,

$$\left|\vec{\mathbf{L}}_S\right| = r_S^{\perp}\left|\vec{\mathbf{p}}\right|. \qquad (19.2.4)$$

Alternatively, let **Error! Objects cannot be created from editing field codes.** denote the magnitude of the component of the momentum perpendicular to the line defined by the direction of the vector $\vec{\mathbf{r}}_S$. From the geometry shown in Figure 19.2 (b),

$$p_S^{\perp} = \left|\vec{\mathbf{p}}\right|\sin\theta\,. \qquad (19.2.5)$$

Thus the magnitude of the angular momentum is the product of the distance from S to the particle with $p_S^{\perp}$,

$$\left|\vec{\mathbf{L}}_S\right| = \left|\vec{\mathbf{r}}_S\right| p_S^{\perp}\,. \qquad (19.2.6)$$

19.2.2 Right-Hand-Rule for the Direction of the Angular Momentum

We shall define the direction of the angular momentum about the point S by a right hand rule. Draw the vectors $\vec{\mathbf{r}}_S$ and $\vec{\mathbf{p}}$ so their tails are touching. Then draw an arc starting from the vector $\vec{\mathbf{r}}_S$ and finishing on the vector $\vec{\mathbf{p}}$. (There are two such arcs; choose the shorter one.) This arc is either in the clockwise or counterclockwise direction. Curl the fingers of your right hand in the same direction as the arc. Your right thumb points in the direction of the angular momentum.

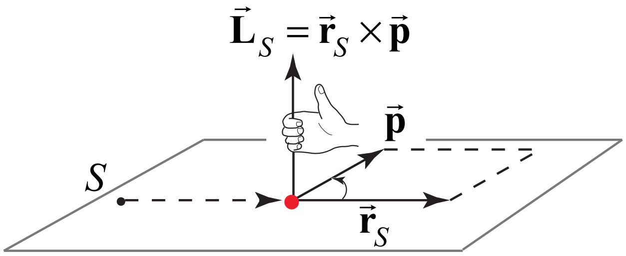

Figure 19.3 The right hand rule for determining the direction of angular momentum about S.

Remember that, as in all vector products, the direction of the angular momentum about S is perpendicular to the plane formed by $\vec{\mathbf{r}}_S$ and $\vec{\mathbf{p}}$.

Example 19.1 Angular Momentum: Constant Velocity

A particle of mass $m = 2.0\,\text{kg}$ moves as shown in Figure 19.4 with a uniform velocity $\vec{\mathbf{v}} = 3.0\ \text{m}\cdot\text{s}^{-1}\ \hat{\mathbf{i}} + 3.0\ \text{m}\cdot\text{s}^{-1}\ \hat{\mathbf{j}}$. At time t, the particle passes through the point $(2.0\ \text{m}, 3.0\ \text{m})$. Find the direction and the magnitude of the angular momentum about the point S (the origin) at time t.

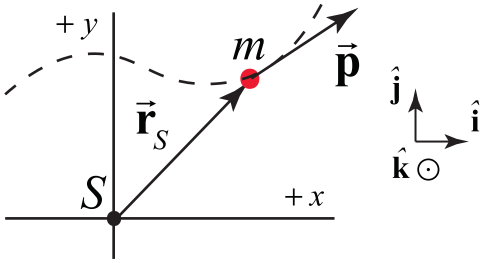

Figure 19.4 Example 19.4

Solution: Choose Cartesian coordinates with unit vectors shown in the figure above. The vector from the point S to the location of the particle is $\vec{\mathbf{r}}_S = 2.0\ \text{m}\ \hat{\mathbf{i}} + 3.0\ \text{m}\ \hat{\mathbf{j}}$. The angular momentum vector $\vec{\mathbf{L}}_O$ of the particle about the origin S is given by:

$$\begin{aligned}\vec{\mathbf{L}}_S &= \vec{\mathbf{r}}_S \times \vec{\mathbf{p}} = \vec{\mathbf{r}}_S \times m\vec{\mathbf{v}}\\ &= (2.0\text{m}\ \hat{\mathbf{i}} + 3.0\text{m}\ \hat{\mathbf{j}}) \times (2\text{kg})(3.0\text{m}\cdot\text{s}^{-1}\hat{\mathbf{i}} + 3.0\text{m}\cdot\text{s}^{-1}\hat{\mathbf{j}})\\ &= 0 + 12\,\text{kg}\cdot\text{m}^2\cdot\text{s}^{-1}\,\hat{\mathbf{k}} - 18\,\text{kg}\cdot\text{m}^2\cdot\text{s}^{-1}(-\hat{\mathbf{k}}) + \vec{\mathbf{0}}\\ &= -6\,\text{kg}\cdot\text{m}^2\cdot\text{s}^{-1}\,\hat{\mathbf{k}}.\end{aligned}$$

In the above, the relations $\vec{\mathbf{i}} \times \vec{\mathbf{j}} = \vec{\mathbf{k}},\ \vec{\mathbf{j}} \times \vec{\mathbf{i}} = -\vec{\mathbf{k}},\ \vec{\mathbf{i}} \times \vec{\mathbf{i}} = \vec{\mathbf{j}} \times \vec{\mathbf{j}} = \vec{\mathbf{0}}$ were used.

Example 19.2 Angular Momentum and Circular Motion

A particle of mass m moves in a circle of radius R about the z-axis in the x-y plane defined by $z = 0$ with angular velocity $\vec{\omega} = \omega_z \hat{\mathbf{k}}$, $\omega_z > 0$, (Figure 19.5). Find the magnitude and the direction of the angular momentum $\vec{\mathbf{L}}_S$ relative to the point S lying at the center of the circular orbit, (the origin).

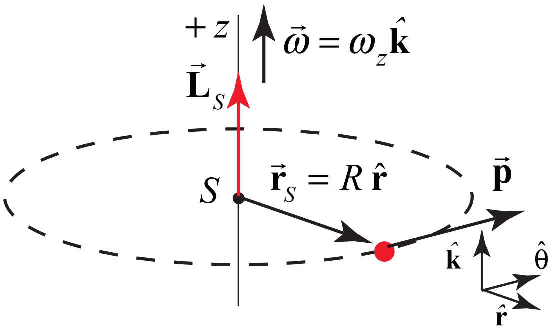

Figure 19.5 Example 19.2

Solution: The velocity of the particle is given by $\vec{\mathbf{v}} = R\omega_z\,\hat{\boldsymbol{\theta}}$. The vector from the center of the circle (the point S) to the object is given by $\vec{\mathbf{r}}_S = R\,\hat{\mathbf{r}}$. The angular momentum about the center of the circle is the vector product

$$\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}} = \vec{\mathbf{r}}_S \times m\vec{\mathbf{v}} = Rmv\,\hat{\mathbf{k}} = RmR\omega_z\,\hat{\mathbf{k}} = mR^2\omega_z\,\hat{\mathbf{k}} = I_S\vec{\omega}\,.$$

The magnitude is $\left|\vec{\mathbf{L}}_S\right| = mR^2\omega_z$, and the direction is in the $+\hat{\mathbf{k}}$-direction. For the particle, the moment of inertia about the z-axis is $I_S = mR^2$, therefore the angular momentum about S is

$$\vec{\mathbf{L}}_S = I_S\vec{\omega}\,.$$

The fact that $\vec{\mathbf{L}}_S$ is in the same direction as the angular velocity is due to the fact that the point S lies on the plane of motion.

Example 19.3 Angular Momentum About a Point along Central Axis for Circular Motion

A particle of mass m moves in a circle of radius R with angular velocity $\vec{\omega} = \omega_z \hat{\mathbf{k}}$, $\omega_z > 0$, about the z- axis in a plane parallel to but a distance h above the x-y plane (Figure 19.6). Find the magnitude and the direction of the angular momentum $\vec{\mathbf{L}}_S$ relative to the point S (the origin).

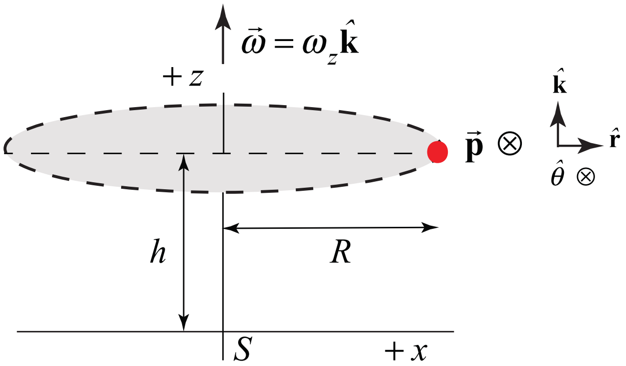

Figure 19.6 Example 19.3

Solution: The easiest way to calculate $\vec{\mathbf{L}}_S$ is to use cylindrical coordinates. We begin by writing the two vectors $\vec{\mathbf{r}}_S$ and $\vec{\mathbf{p}}$ in polar coordinates. We start with the vector from point S (the origin) to the location of the moving object, $\vec{\mathbf{r}}_S = R\hat{\mathbf{r}} + h\hat{\mathbf{k}}$. The momentum vector is tangent to the circular orbit so $\vec{\mathbf{p}} = m\vec{\mathbf{v}} = mR\omega_z\hat{\boldsymbol{\theta}}$. Using the fact that $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{k}}$ and $\hat{\mathbf{k}} \times \hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}}$, the angular momentum about point S is

$$\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}} = (R\hat{\mathbf{r}} + h\hat{\mathbf{k}}) \times mR\omega_z\hat{\boldsymbol{\theta}} = mR^2\omega_z\hat{\mathbf{k}} - hmR\omega_z\hat{\mathbf{r}}$$

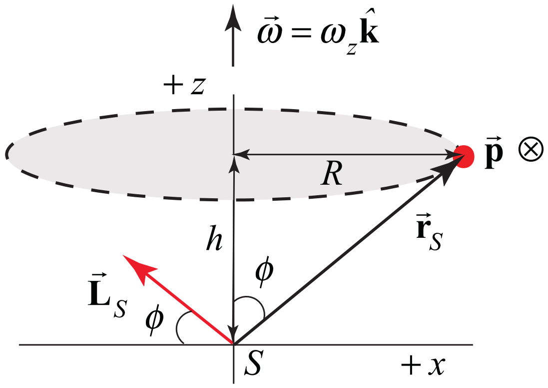

Figure 19.7 Angular momentum about the point S

The magnitude of $\vec{\mathbf{L}}_S$ is given by

$$\left|\vec{\mathbf{L}}_S\right| = ((mR^2\omega_z)^2 + (hmR\omega_z)^2)^{1/2} = mR\omega_z(h^2 + R^2)^{1/2}$$

The direction of $\vec{\mathbf{L}}_S$ is given by (Figure 19.7)

$$-\frac{L_{S,z}}{L_{S,r}} = \frac{R}{h} = \tan\phi$$

We also present a geometric argument. Suppose the particle has coordinates (x, y, h). The angular momentum about the origin is given by $\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}}$. The vectors $\vec{\mathbf{r}}_S$ and $\vec{\mathbf{p}}$ are perpendicular to each other so the angular momentum is perpendicular to the plane formed by those two vectors. Recall that the speed $v = R\omega_z$. Suppose the vector $\vec{\mathbf{r}}_S$ forms an angle ϕ with the z-axis. Then $\vec{\mathbf{L}}_S$ forms an angle ϕ with respect to the $x-y$ plane as shown in the figure above. The magnitude of $\vec{\mathbf{L}}_S$ is

$$\left|\vec{\mathbf{L}}_S\right| = \left|\vec{\mathbf{r}}_S\right| m \left|\vec{\mathbf{v}}\right| = (h^2 + R^2)^{1/2} mR\omega_z$$

The magnitude of $\vec{\mathbf{L}}_S$ is constant, but its direction is changing as the particle moves in a circular orbit about the z-axis, sweeping out a cone as shown in Figure 19.8. We draw the vector $\vec{\mathbf{L}}_S$ at the origin because it is defined at that point.

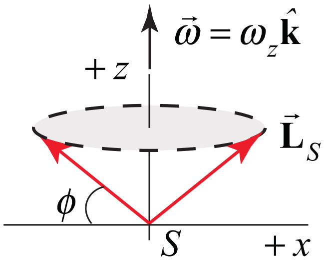

Figure 19.8 Direction of angular momentum about the point S sweeps out a cone

The important point to keep in mind regarding this calculation is that for any point along the z-axis not at the center of the circular orbit of a single particle, the angular momentum about that point does not point along the z-axis but it is has a non-zero component in the $x-y$ plane (or in the $-\hat{\mathbf{r}}$ direction if you use polar coordinates). The z-component of the angular momentum about any point along the z-axis is independent of the location of that point along the axis.

19.3 Torque and the Time Derivative of Angular Momentum about a Point for a Particle

We will now show that the torque about a point S is equal to the time derivative of the angular momentum about S,

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt}. \qquad (19.3.1)$$

Take the time derivative of the angular momentum about S,

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \frac{d}{dt}\left(\vec{\mathbf{r}}_S \times \vec{\mathbf{p}}\right). \qquad (19.3.2)$$

In this equation we are taking the time derivative of a vector product of two vectors. There are two important facts that will help us simplify this expression. First, the time derivative of the vector product of two vectors satisfies the product rule,

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \frac{d}{dt}(\vec{\mathbf{r}}_S \times \vec{\mathbf{p}}) = \left(\left(\frac{d\vec{\mathbf{r}}_S}{dt}_{s}\right) \times \vec{\mathbf{p}}\right) + \left(\vec{\mathbf{r}}_S \times \left(\frac{d\vec{\mathbf{p}}}{dt}\right)\right). \qquad (19.3.3)$$

Second, the first term on the right hand side vanishes,

$$\frac{d\vec{\mathbf{r}}_S}{dt} \times \vec{\mathbf{p}} = \vec{\mathbf{v}} \times m\,\vec{\mathbf{v}} = \vec{\mathbf{0}}. \qquad (19.3.4)$$

The rate of angular momentum change about the point S is then

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \vec{\mathbf{r}}_S \times \frac{d\vec{\mathbf{p}}}{dt}. \qquad (19.3.5)$$

From Newton's Second Law, the force on the particle is equal to the derivative of the linear momentum,

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}. \qquad (19.3.6)$$

Therefore the rate of change in time of angular momentum about the point S is

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \vec{\mathbf{r}}_S \times \vec{\mathbf{F}}. \qquad (19.3.7)$$

Recall that the torque about the point S due to the force $\vec{\mathbf{F}}$ acting on the particle is

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{F}}. \qquad (19.3.8)$$

Combining the expressions in (19.3.7) and (19.3.8), it is readily seen that the torque about the point S is equal to the rate of change of angular momentum about the point S,

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt}. \qquad (19.3.9)$$

19.4 Conservation of Angular Momentum about a Point

So far we have introduced two conservation principles, showing that energy is constant for closed systems (no change in energy in the surroundings) and linear momentum is constant isolated system. The change in mechanical energy of a closed system is

$$W_{nc} = \Delta E_m = \Delta K + \Delta U, \qquad \text{(closed system)}. \tag{19.3.10}$$

If the non-conservative work done in the system is zero, then the mechanical energy is constant,

$$0 = W_{\text{nc}} = \Delta E_{\text{mechanical}} = \Delta K + \Delta U, \quad \text{(closed system)}. \tag{19.3.11}$$

The conservation of linear momentum arises from Newton's Second Law applied to systems,

$$\vec{\mathbf{F}}^{\text{ext}} = \sum_{i=1}^{N} \frac{d}{dt}\vec{\mathbf{p}}_i = \frac{d}{dt}\vec{\mathbf{p}}_{\text{sys}} \tag{19.3.12}$$

Thus if the external force in any direction is zero, then the component of the momentum of the system in that direction is a constant. For example, if there are no external forces in the x-and y-directions then

$$\begin{aligned} \vec{\mathbf{0}} &= (\vec{\mathbf{F}}^{\text{ext}})_x = \frac{d}{dt}(\vec{\mathbf{p}}_{\text{sys}})_x \\ \vec{\mathbf{0}} &= (\vec{\mathbf{F}}^{\text{ext}})_y = \frac{d}{dt}(\vec{\mathbf{p}}_{\text{sys}})_y. \end{aligned} \tag{19.3.13}$$

We can now use our relation between torque about a point S and the change of the angular momentum about S, Eq. (19.3.9), to introduce a new conservation law. Suppose we can find a point S such that torque about the point S is zero,

$$\vec{\mathbf{0}} = \vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt}, \tag{19.3.14}$$

then the angular momentum about the point S is a constant vector, and so the change in angular momentum is zero,

$$\Delta\vec{\mathbf{L}}_S \equiv \vec{\mathbf{L}}_{S,f} - \vec{\mathbf{L}}_{S,i} = \vec{\mathbf{0}}. \tag{19.3.15}$$

Thus when the torque about a point S is zero, the final angular momentum about S is equal to the initial angular momentum,

$$\vec{\mathbf{L}}_{S,f} = \vec{\mathbf{L}}_{S,i}. \tag{19.3.16}$$

Example 19.4 Meteor Flyby of Earth

A meteor of mass m is approaching Earth as shown in the figure. The radius of Earth is R_E. The mass of Earth is M_E. Assume that the meteor started very far away from Earth with speed v_i and at a perpendicular distance h from the axis of symmetry of the orbit. At some later time the meteor just grazes Earth (Figure 19.9). You may ignore all other gravitational forces except due to Earth. Find the distance h. Hint: What quantities are constant for this orbit?

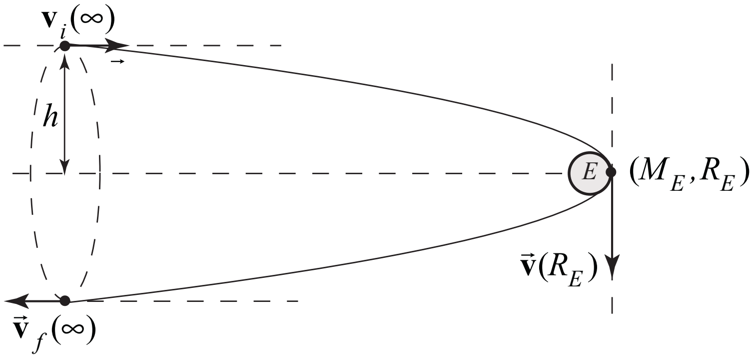

Figure 19.9 Meteor flyby of earth

Solution: In this problem both energy and angular momentum about the center of Earth are constant (see below for justification).

The meteor's mass is so much small than the mass of Earth that we will assume that the earth's motion is not affected by the meteor. We'll also need to neglect any air resistance when the meteor approaches Earth. Choose the center of Earth, (point S) to calculate the torque and angular momentum. The force on the meteor is

$$\vec{\mathbf{F}}^G_{E,m} = -\frac{GmM_E}{r^2}\hat{\mathbf{r}}$$

The vector from the center of Earth to the meteor is $\vec{\mathbf{r}}_S = r\hat{\mathbf{r}}$. The torque about S is zero because they two vectors are anti-parallel

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{F}}^G_{E,m} = r\hat{\mathbf{r}} \times -\frac{GmM_E}{r^2}\hat{\mathbf{r}} = \vec{\mathbf{0}}$$

Therefore the angular momentum about the center of Earth is a constant.

The initial angular momentum is

$$\vec{\mathbf{L}}_{S,i} = \vec{\mathbf{r}}_{S,i} \times m\vec{\mathbf{v}}_i = (x_i\hat{\mathbf{i}} + h\hat{\mathbf{j}}) \times mv_i\hat{\mathbf{i}} = -hmv_i\hat{\mathbf{k}}$$

When the meteor just grazes Earth, the angular momentum is

$$\vec{\mathbf{L}}_{S,E} = \vec{\mathbf{r}}_{S,E} \times m\vec{\mathbf{v}}_p = R_E\hat{\mathbf{i}} \times mv_p(-\hat{\mathbf{j}}) = -R_Emv_p\hat{\mathbf{k}}$$

where we have used v_p for the speed of the meteor at its nearest approach to Earth. The constancy of angular momentum requires that

$$mv_i h = mv_p R_E$$

In order to solve for h, we need to find v_p. Because we are neglecting all forces on the meteor other than Earth's gravity, mechanical energy is constant, and

$$\frac{1}{2}mv_i^2 = \frac{1}{2}mv_p^2 - \frac{GmM_E}{R_E},$$

where we have taken the meteor to have speed v_i at a distance "very far away from Earth" to mean that we neglect any gravitational potential energy in the meteor-Earth system, when $r \to \infty$, $U(r) = -GmM_E / r \to 0$. From the angular momentum condition, $v_p = v_i h / R_E$ and therefore the energy condition can be rewritten as

$$v_i^2 = v_i^2 \left(\frac{h}{R_E} \right)^2 - \frac{2GM_E}{R_E}$$

which we solve for the impact parameter h

$$h = \sqrt{R_E^2 + \frac{2GM_E R_E}{v_i^2}}\ .$$

19.5 Angular Impulse and Change in Angular Momentum

If there is a total applied torque $\vec{\boldsymbol{\tau}}_S$ about a point S over an interval of time $\Delta t = t_f - t_i$, then the torque applies an ***angular impulse*** about a point S, given by

$$\vec{\mathbf{J}}_S = \int_{t_i}^{t_f} \vec{\boldsymbol{\tau}}_S \, dt . \qquad (19.4.1)$$

Because $\vec{\boldsymbol{\tau}}_S = d\vec{\mathbf{L}}_S^{\text{total}} / dt$, the angular impulse about S is equal to the change in angular momentum about S,

$$\vec{\mathbf{J}}_S = \int_{t_i}^{t_f} \vec{\boldsymbol{\tau}}_S \, dt = \int_{t_i}^{t_f} \frac{d\vec{\mathbf{L}}_S}{dt} dt = \Delta \vec{\mathbf{L}}_S = \vec{\mathbf{L}}_{S,f} - \vec{\mathbf{L}}_{S,i} . \qquad (19.4.2)$$

This result is the rotational analog to linear impulse, which is equal to the change in momentum,

$$\vec{\mathbf{I}} = \int_{t_i}^{t_f} \vec{\mathbf{F}}\, dt = \int_{t_i}^{t_f} \frac{d\vec{\mathbf{p}}}{dt} dt = \Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i\,. \tag{19.4.3}$$

19.6 Angular Momentum of a System of Particles

We now calculate the angular momentum about the point S associated with a system of N point particles. Label each individual particle by the index j, $j = 1, 2, \cdots, N$. Let the j^{th} particle have mass m_j and velocity $\vec{\mathbf{v}}_j$. The momentum of an individual particle is then $\vec{\mathbf{p}}_j = m_j \vec{\mathbf{v}}_j$. Let $\vec{\mathbf{r}}_{S,j}$ be the vector from the point S to the j^{th} particle, and let θ_j be the angle between the vectors $\vec{\mathbf{r}}_{S,j}$ and $\vec{\mathbf{p}}_j$ (Figure 19.10).

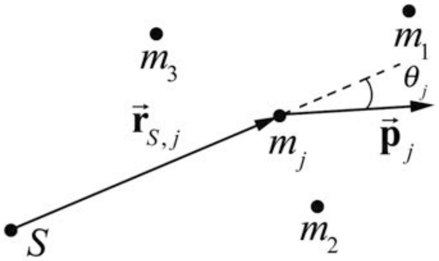

Figure 19.10 System of particles

The angular momentum $\vec{\mathbf{L}}_{S,j}$ of the j^{th} particle is

$$\vec{\mathbf{L}}_{S,j} = \vec{\mathbf{r}}_{S,j} \times \vec{\mathbf{p}}_j\,. \tag{19.5.1}$$

The angular momentum for the system of particles is the vector sum of the individual angular momenta,

$$\vec{\mathbf{L}}_S^{\text{sys}} = \sum_{j=1}^{j=N} \vec{\mathbf{L}}_{S,j} = \sum_{j=1}^{j=N} \vec{\mathbf{r}}_{S,j} \times \vec{\mathbf{p}}_j\,. \tag{19.5.2}$$

The change in the angular momentum of the system of particles about a point S is given by

$$\frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} = \frac{d}{dt}\sum_{j=1}^{j=N} \vec{\mathbf{L}}_{S,j} = \sum_{j=1}^{j=N}\left(\frac{d\vec{\mathbf{r}}_{S,j}}{dt} \times \vec{\mathbf{p}}_j + \vec{\mathbf{r}}_{S,j} \times \frac{d\vec{\mathbf{p}}_j}{dt}\right). \tag{19.5.3}$$

Because the velocity of the j^{th} particle is $\vec{\mathbf{v}}_{S,j} = d\vec{\mathbf{r}}_{S,j} / dt$, the first term in the parentheses vanishes (the cross product of a vector with itself is zero because they are parallel to each other)

$$\frac{d\vec{\mathbf{r}}_{S,j}}{dt} \times \vec{\mathbf{p}}_j = \vec{\mathbf{v}}_{S,j} \times m_j \vec{\mathbf{v}}_{S,j} = 0 . \tag{19.5.4}$$

Substitute Eq. (19.5.4) and $\vec{\mathbf{F}}_j = d\vec{\mathbf{p}}_j / dt$ into Eq. (19.5.3) yielding

$$\frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} = \sum_{j=1}^{j=N} \left(\vec{\mathbf{r}}_{S,j} \times \frac{d\vec{\mathbf{p}}_j}{dt} \right) = \sum_{j=1}^{j=N} \left(\vec{\mathbf{r}}_{S,j} \times \vec{\mathbf{F}}_j \right). \tag{19.5.5}$$

Because

$$\sum_{j=1}^{j=N} \left(\vec{\mathbf{r}}_{S,j} \times \vec{\mathbf{F}}_j \right) = \sum_{j=1}^{j=N} \vec{\boldsymbol{\tau}}_{S,j} = \vec{\boldsymbol{\tau}}_S^{\ \text{ext}} + \vec{\boldsymbol{\tau}}_S^{\ \text{int}} \tag{19.5.6}$$

We have already shown in Chapter 17.4 that when we assume all internal forces are directed along the line connecting the two interacting objects then the internal torque about the point S is zero,

$$\vec{\boldsymbol{\tau}}_S^{\text{int}} = \vec{\mathbf{0}} . \tag{19.5.7}$$

Eq. (19.5.6) simplifies to

$$\sum_{j=1}^{j=N} \left(\vec{\mathbf{r}}_{S,j} \times \vec{\mathbf{F}}_j \right) = \sum_{j=1}^{j=N} \vec{\boldsymbol{\tau}}_{S,j} = \vec{\boldsymbol{\tau}}_S^{\ \text{ext}} . \tag{19.5.8}$$

Therefore Eq. (19.5.5) becomes

$$\vec{\boldsymbol{\tau}}_S^{\text{ext}} = \frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} . \tag{19.5.9}$$

The external torque about the point S is equal to the time derivative of the angular momentum of the system about that point.

Example 19.5 Angular Momentum of Two Particles undergoing Circular Motion

Two identical particles of mass m move in a circle of radius R, with angular velocity $\vec{\omega} = \omega_z \hat{\mathbf{k}}$, $\omega_z > 0$, ω about the z-axis in a plane parallel to but a distance h above the x-y plane. The particles are located on opposite sides of the circle (Figure 19.11). Find the magnitude and the direction of the angular momentum about the point S (the origin).

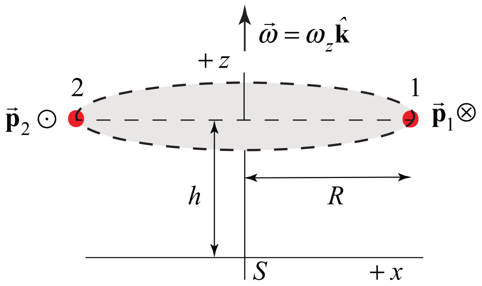

Figure 19.11 Example 19.5

Solution: The angular momentum about the origin is the sum of the contributions from each object. The calculation of each contribution will be identical to the calculation in Example 19.3

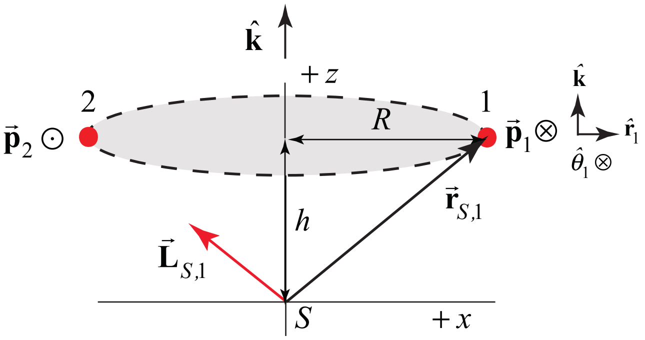

Figure 19.12 Angular momentum of particle 1 about origin

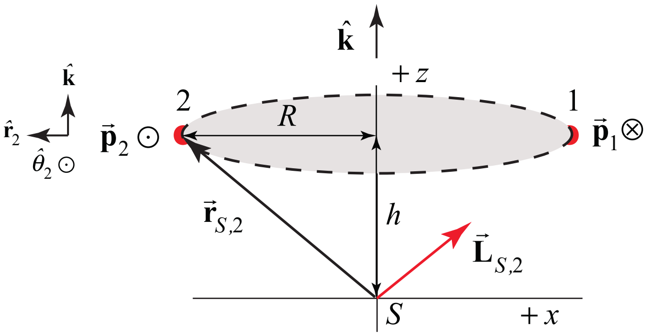

Figure 19.13 Angular momentum of particle 2 about origin

For particle 1 (Figure 19.12), the angular momentum about the point S is

$$\vec{\mathbf{L}}_{S,1} = \vec{\mathbf{r}}_{S,1} \times \vec{\mathbf{p}}_1 = (R\hat{\mathbf{r}}_1 + h\hat{\mathbf{k}}) \times mR\omega_z\hat{\boldsymbol{\theta}}_1 = mR^2\omega_z\hat{\mathbf{k}} - hmR\omega_z\hat{\mathbf{r}}_1 .$$

For particle 2, (Figure 19.13), the angular momentum about the point S is

$$\vec{\mathbf{L}}_{S,2} = \vec{\mathbf{r}}_{S,2} \times \vec{\mathbf{p}}_2 = (R\hat{\mathbf{r}}_2 + h\hat{\mathbf{k}}) \times mR\omega_z\hat{\boldsymbol{\theta}}_2 = mR^2\omega_z\hat{\mathbf{k}} - hmR\omega_z\hat{\mathbf{r}}_2 .$$

Because the particles are located on opposite sides of the circle, $\hat{\mathbf{r}}_1 = -\hat{\mathbf{r}}_2$. The vector sum only points along the z-axis and is equal to

$$\vec{\mathbf{L}}_S = \vec{\mathbf{L}}_{S,1} + \vec{\mathbf{L}}_{S,2} = 2mR^2\omega_z\hat{\mathbf{k}} . \qquad (19.5.10)$$

The two angular momentum vectors are shown in Figure 19.14.

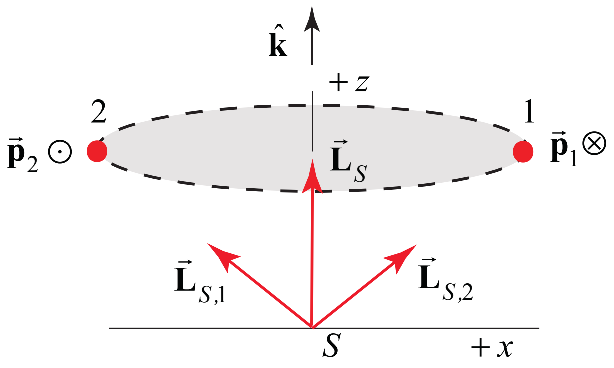

Figure 19.14 Angular momentum about the point S of both particles and their sum

The moment of inertia of the two particles about the z-axis is given by $I_S = 2mR^2$. Therefore $\vec{\mathbf{L}}_S = I_S\vec{\omega}$. The important point about this example is that the two objects are symmetrically distributed with respect to the z-axis (opposite sides of the circular orbit). Therefore the angular momentum about any point S along the z-axis has the same value $\vec{\mathbf{L}}_S = 2mr^2\omega\hat{\mathbf{k}}$, which is constant in magnitude and points in the $+z$-direction. This result generalizes to any rigid body in which the mass is distributed symmetrically about the axis of rotation.

Example 19.6 Angular Momentum of a System of Particles about Different Points

Consider a system of N particles, and two points A and B (Figure 19.15). The angular momentum of the j^{th} particle about the point A is given by

$$\vec{\mathbf{L}}_{A,j} = \vec{\mathbf{r}}_{A,j} \times m_j \vec{\mathbf{v}}_j . \tag{19.5.11}$$

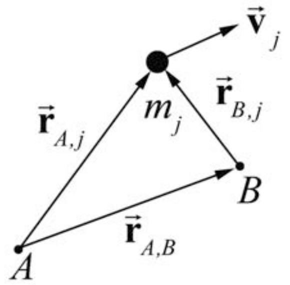

Figure 19.15 Vector triangle relating position of object and points A and B

The angular momentum of the system of particles about the point A is given by the sum

$$\vec{\mathbf{L}}_A = \sum_{j=1}^{N} \vec{\mathbf{L}}_{A,j} = \sum_{j=1}^{N} \vec{\mathbf{r}}_{A,j} \times m_j \vec{\mathbf{v}}_j \tag{19.5.12}$$

The angular momentum about the point B can be calculated in a similar way and is given by

$$\vec{\mathbf{L}}_B = \sum_{j=1}^{N} \vec{\mathbf{L}}_{B,j} = \sum_{j=1}^{N} \vec{\mathbf{r}}_{B,j} \times m_j \vec{\mathbf{v}}_j \,. \tag{19.5.13}$$

From Figure 19.15, the vectors

$$\vec{\mathbf{r}}_{A,j} = \vec{\mathbf{r}}_{B,j} + \vec{\mathbf{r}}_{A,B} \,. \tag{19.5.14}$$

We can substitute Eq. (19.5.14) into Eq. (19.5.12) yielding

$$\vec{\mathbf{L}}_A = \sum_{j=1}^{N} (\vec{\mathbf{r}}_{B,j} + \vec{\mathbf{r}}_{A,B}) \times m_j \vec{\mathbf{v}}_j = \sum_{j=1}^{N} \vec{\mathbf{r}}_{B,j} \times m_j \vec{\mathbf{v}}_j + \sum_{j=1}^{N} \vec{\mathbf{r}}_{A,B} \times m_j \vec{\mathbf{v}}_j \,. \tag{19.5.15}$$

The first term in Eq. (19.5.15) is the angular momentum about the point B. The vector $\vec{\mathbf{r}}_{A,B}$ is a constant and so can be pulled out of the sum in the second term, and Eq. (19.5.15) becomes

$$\vec{\mathbf{L}}_A = \vec{\mathbf{L}}_B + \vec{\mathbf{r}}_{A,B} \times \sum_{j=1}^{N} m_j \vec{\mathbf{v}}_j \tag{19.5.16}$$

The sum in the second term is the momentum of the system

$$\vec{\mathbf{p}}_{\text{sys}} = \sum_{j=1}^{N} m_j \vec{\mathbf{v}}_j \,. \tag{19.5.17}$$

Therefore the angular momentum about the points A and B are related by

$$\vec{\mathbf{L}}_A = \vec{\mathbf{L}}_B + \vec{\mathbf{r}}_{A,B} \times \vec{\mathbf{p}}_{\text{sys}} \tag{19.5.18}$$

Thus if the momentum of the system is zero, the angular momentum is the same about any point.

$$\vec{\mathbf{L}}_A = \vec{\mathbf{L}}_B, \qquad (\vec{\mathbf{p}}_{\text{sys}} = \vec{\mathbf{0}}) \,. \tag{19.5.19}$$

In particular, the momentum of a system of particles is zero by definition in the center of mass reference frame because in that reference frame $\vec{\mathbf{p}}_{\text{sys}} = \vec{\mathbf{0}}$. Hence the angular momentum is the same about any point in the center of mass reference frame.

19.7 Angular Momentum and Torque for Fixed Axis Rotation

We have shown that, for fixed axis rotation, the component of torque that causes the angular velocity to change is the rotational analog of Newton's Second Law,

$$\vec{\boldsymbol{\tau}}_S^{\text{ext}} = I_S \vec{\boldsymbol{\alpha}}\,. \tag{19.5.20}$$

We shall now see that this is a special case of the more general result

$$\vec{\boldsymbol{\tau}}_S^{\text{ext}} = \frac{d}{dt}\vec{\mathbf{L}}_S^{\text{sys}}\,. \tag{19.5.21}$$

Consider a rigid body rotating about a fixed axis passing through the point S and take the fixed axis of rotation to be the z-axis. Recall that all the points in the rigid body rotate about the z-axis with the same angular velocity $\vec{\boldsymbol{\omega}} = (d\theta / dt)\hat{\mathbf{k}} = \omega_z\,\hat{\mathbf{k}}$. In a similar fashion, all points in the rigid body have the same angular acceleration, $\vec{\boldsymbol{\alpha}} = (d^2\theta / dt^2)\,\hat{\mathbf{k}} = \alpha_z\,\hat{\mathbf{k}}$. Let the point S lie somewhere along the z-axis.

As before, the body is divided into individual elements. We calculate the contribution of each element to the angular momentum about the point S, and then sum over all the elements. The summation will become an integral for a continuous body.

Each individual element has a mass Δm_j and is moving in a circle of radius $r_{S,j}^{\perp}$ about the axis of rotation. Let $\vec{\mathbf{r}}_{S,j}$ be the vector from the point S to the element. The momentum of the element, $\vec{\mathbf{p}}_j$, is tangent to this circle (Figure 19.16).

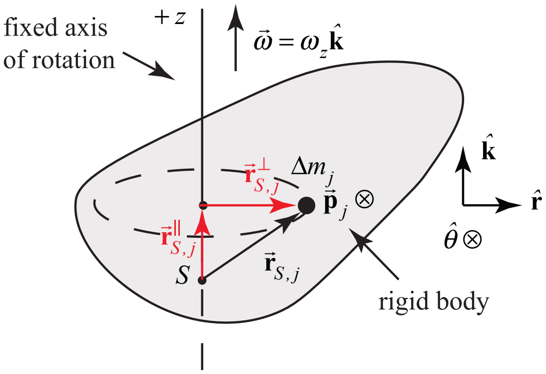

Figure 19.16 Geometry of instantaneous rotation.

The angular momentum of the j^{th} element about the point S is given by $\vec{\mathbf{L}}_{S,j} = \vec{\mathbf{r}}_{S,j} \times \vec{\mathbf{p}}_j$. The vector $\vec{\mathbf{r}}_{S,j}$ can be decomposed into parallel and perpendicular components with respect to the axis of rotation $\vec{\mathbf{r}}_{S,j} = \vec{\mathbf{r}}_{S,j}^{\parallel} + \vec{\mathbf{r}}_{S,j}^{\perp}$ (Figure 19.16), where $r_{S,j}^{\perp} = \left|\vec{\mathbf{r}}_{S,j}^{\perp}\right|$ and $r_{S,j}^{\parallel} = \left|\vec{\mathbf{r}}_{S,j}^{\parallel}\right|$. The momentum is given by $\vec{\mathbf{p}}_j = \Delta m_j\, r_{S,j}^{\perp}\omega_z\hat{\theta}$. Then the angular momentum about the point S is

$$\begin{aligned}\vec{\mathbf{L}}_{S,j} &= \vec{\mathbf{r}}_{S,j} \times \vec{\mathbf{p}}_j = (r^{\perp}_{S,j}\hat{\mathbf{r}} + r^{\parallel}_{S,j}\hat{\mathbf{k}}) \times (\Delta m_j r^{\perp}_{S,j}\omega_z\hat{\theta}) \\ &= \Delta m_j (r^{\perp}_{S,j})^2 \omega_z \hat{\mathbf{k}} - \Delta m_j r^{\parallel}_{S,j} r^{\perp}_{S,j} \omega_z \hat{\mathbf{r}}\end{aligned} . \qquad (19.5.22)$$

In the last expression in Equation (19.5.22), the second term has a direction that is perpendicular to the z-axis. Therefore the z-component of the angular momentum about the point S, $(L_{S,j})_z$, arises entirely from the second term, $\vec{\mathbf{r}}^{\perp}_{S,j} \times \vec{\mathbf{p}}_j$. Therefore the z-component of the angular momentum about S is

$$(L_{S,j})_z = \Delta m_j (r^{\perp}_{S,j})^2 \omega_z . \qquad (19.5.23)$$

The z-component of the angular momentum of the system about S is the summation over all the elements,

$$L^{\text{sys}}_{S,z} = \sum_j (L_{S,j})_z = \sum_j \Delta m_j (r^{\perp}_{S,j})^2 \omega_z . \qquad (19.5.24)$$

For a continuous mass distribution the summation becomes an integral over the body,

$$L^{\text{sys}}_{S,z} = \int_{\text{body}} dm\,(r_{dm})^2 \omega_z , \qquad (19.5.25)$$

where r_{dm} is the distance form the fixed z-axis to the infinitesimal element of mass dm. The moment of inertia of a rigid body about a fixed z-axis passing through a point S is given by an integral over the body

$$I_S = \int_{\text{body}} dm\,(r_{dm})^2 . \qquad (19.5.26)$$

Thus the z-component of the angular momentum about S for a fixed axis that passes through S in the z-direction is proportional to the z-component of the angular velocity, ω_z,

$$L^{\text{sys}}_{S,z} = I_S\,\omega_z . \qquad (19.5.27)$$

For fixed axis rotation, our result that torque about a point is equal to the time derivative of the angular momentum about that point,

$$\vec{\boldsymbol{\tau}}^{\text{ext}}_S = \frac{d}{dt}\vec{\mathbf{L}}^{\text{sys}}_S , \qquad (19.5.28)$$

can now be resolved in the z-direction,

$$\tau^{\text{ext}}_{S,z} = \frac{dL^{\text{sys}}_{S,z}}{dt} = \frac{d}{dt}(I_S\,\omega_z) = I_S\frac{d\omega_z}{dt} = I_S\frac{d^2\theta}{dt^2} = I_S\,\alpha_z , \qquad (19.5.29)$$

in agreement with our earlier result that the z -component of torque about the point S is equal to the product of moment of inertia about I_S, and the z-component of the angular acceleration, α_z.

Example 19.6 Circular Ring

A circular ring of radius R, and mass M is rotating about the z-axis in a plane parallel to but a distance h above the x-y plane. The z-component of the angular velocity is ω_z (Figure 19.17). Find the magnitude and the direction of the angular momentum $\vec{\mathbf{L}}_S$ along at any point S on the central z-axis.

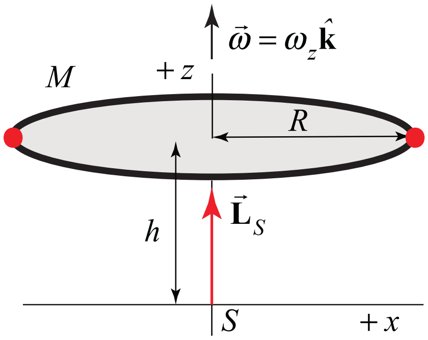

Figure 19.17 Example 19.6

Solution: Use the same symmetry argument as we did in Example 19.5. The ring can be thought of as made up of pairs of point like objects on opposite sides of the ring each of mass m (Figure 19.18).

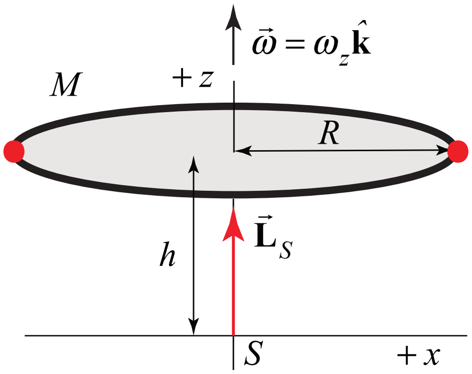

Figure 19.18 Ring as a sum of pairs of symmetrically distributed particles

Each pair has a non-zero z-component of the angular momentum taken about any point S along the z-axis, $\vec{\mathbf{L}}_S^{pair} = \vec{\mathbf{L}}_{S,1} + \vec{\mathbf{L}}_{S,2} = 2mR^2\omega_z\hat{\mathbf{k}} = m^{pair}R^2\omega_z\hat{\mathbf{k}}$. The angular momentum of the ring about the point S is then the sum over all the pairs

$$\vec{\mathbf{L}}_S = \sum_{pairs} m^{pair} R^2 \omega_z \hat{\mathbf{k}} = MR^2\omega_z\hat{\mathbf{k}}. \qquad (19.5.30)$$

Recall that the moment of inertia of a ring is given by

$$I_S = \int_{\text{body}} dm\,(r_{dm})^2 = MR^2 . \qquad (19.5.31)$$

For the symmetric ring, the angular momentum about S points in the direction of the angular velocity and is equal to

$$\vec{\mathbf{L}}_S = I_S \omega_z \hat{\mathbf{k}} \ . \qquad (19.5.32)$$

19.8 Principle of Conservation of Angular Momentum

Consider a system of particles. We begin with the result that we derived in Section 19.7 that the torque about a point S is equal to the time derivative of the angular momentum about that point S,

$$\vec{\boldsymbol{\tau}}_S^{\text{ext}} = \frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} . \qquad (19.5.33)$$

With this assumption, the torque due to the external forces is equal to the rate of change of the angular momentum

$$\vec{\boldsymbol{\tau}}_S^{\text{ext}} = \frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} . \qquad (19.5.34)$$

> ***Principle of Conservation of Angular Momentum***
>
> *If the external torque acting on a system is zero, then the angular momentum of the system is constant. So for any change of state of the system the change in angular momentum is zero*
>
> $$\Delta\vec{\mathbf{L}}_S^{\text{sys}} \equiv (\vec{\mathbf{L}}_S^{\text{sys}})_f - (\vec{\mathbf{L}}_S^{\text{sys}})_i = \vec{\mathbf{0}} . \qquad (19.5.35)$$
>
> *Equivalently the angular momentum is constant*
>
> $$(\vec{\mathbf{L}}_S^{\text{sys}})_f = (\vec{\mathbf{L}}_S^{\text{sys}})_i . \qquad (19.5.36)$$

So far no isolated system has been encountered such that the angular momentum is not constant so our assumption that internal torques cancel is pairs can be taken as an experimental observation.

Example 19.7 Collision Between Pivoted Rod and Object

A point-like object of mass m_1 moving with constant speed v_i strikes a rigid uniform rod of length l and mass m_2 that is hanging by a frictionless pivot from the ceiling. Immediately after striking the rod, the object continues forward but its speed decreases to $v_i / 2$ (Figure 19.19). The moment of inertia of the rod about its center of mass is $I_{cm} = (1/12)m_2 l^2$. Gravity acts with acceleration g downward. (a) For what value of v_i will the rod just touch the ceiling on its first swing? (b) For what ratio m_2 / m_1 will the collision be elastic?

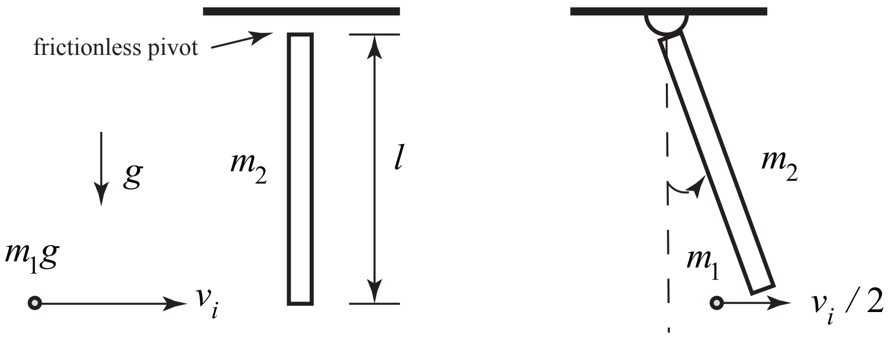

Figure 19.19 Example 19.7

Solution: We begin by identifying our system, which consists of the object and the uniform rod. We identify three states; an initial state i: immediately before the collision, state a: immediately after the collision, and state f: the instant the rod touches the ceiling when the final angular speed is zero. We would like to know if any of our fundamental quantities: momentum, energy, and angular momentum, are constant during these state changes, state i to state a, state a to state f.

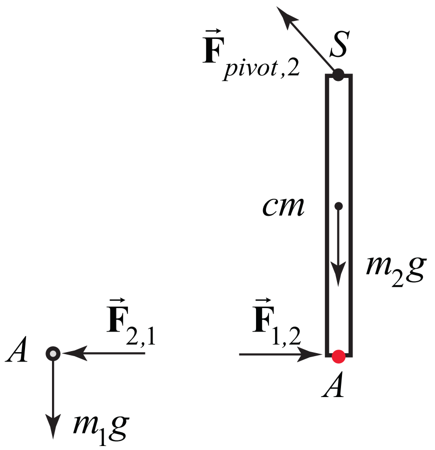

Figure 19.20 Free-body force diagrams on particle and rod

We start with the transition from state i to state a. The pivot force holding the rod to the ceiling is an external force acting at the pivot point S. There is also the gravitational force acting at the center of mass of the rod and on the object. There are also internal forces due to the collision of the rod and the object at point A (Figure 19.20).

The external force means that momentum is not constant. The point of action of the external pivot force is fixed and so does no work. However, we do not know whether or not the collision is elastic and so we cannot assume that mechanical energy is constant. Choose the pivot point S as the point about which to calculate torque, then the torque diagrams are shown in Figure 19.21.

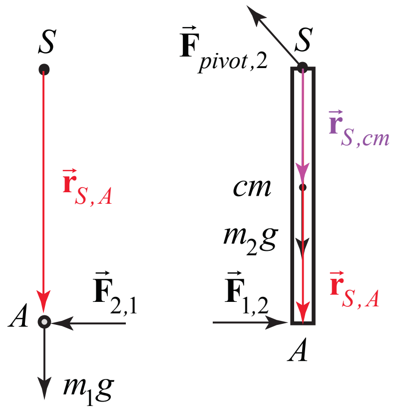

Figure 19.21 Torque diagrams on particle and rod with torque calculated about pivot point S

The torque on the system about the pivot S is then the sum of terms

$$\vec{\boldsymbol{\tau}}_S^{\text{sys}} = \vec{\mathbf{r}}_{S,S} \times \vec{\mathbf{F}}_{pivot,2} + \vec{\mathbf{r}}_{S,A} \times \vec{\mathbf{F}}_{1,2} + \vec{\mathbf{r}}_{S,A} \times \vec{\mathbf{F}}_{2,1} + \vec{\mathbf{r}}_{S,cm} \times m_2 \vec{\mathbf{g}} + \vec{\mathbf{r}}_{S,A} \times m_1 \vec{\mathbf{g}} \ . \quad (19.5.37)$$

The external pivot force does not contribute any torque because $\vec{\mathbf{r}}_{S,S} = \vec{\mathbf{0}}$. The internal forces between the rod and the object are equal in magnitude and opposite in direction, $\vec{\mathbf{F}}_{1,2} = -\vec{\mathbf{F}}_{2,1}$ (Newton's Third Law), and so their contributions to the torque add to zero. If the collision is instantaneous then the gravitational force is parallel to $\vec{\mathbf{r}}_{S,cm}$ and $\vec{\mathbf{r}}_{S,A}$ so the two gravitational torques are zero. Therefore the torque on the system about the pivot point is zero, $\vec{\boldsymbol{\tau}}_S^{\text{sys}} = \vec{\mathbf{0}}$. Thus the angular momentum about the pivot point is constant,

$$\vec{\mathbf{L}}_{S,i}^{\text{sys}} = \vec{\mathbf{L}}_{S,a}^{\text{sys}} \, . \quad (19.5.38)$$

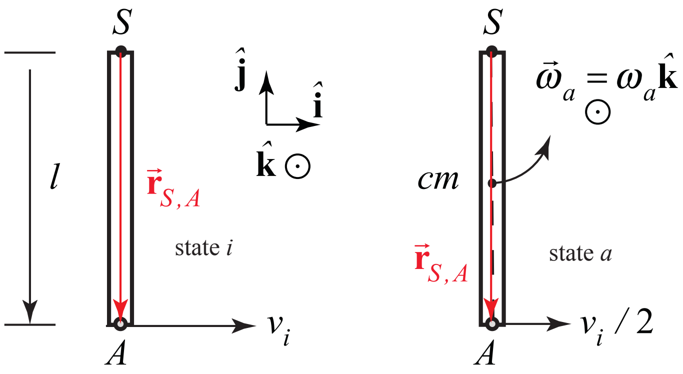

Figure 19.22 Angular momentum diagram

In order to calculate the angular momentum we draw a diagram showing the momentum of the object and the angular speed of the rod in (Figure 19.22). The angular momentum about S immediately before the collision is

$$\vec{\mathbf{L}}^{\text{sys}}_{S,i} = \vec{\mathbf{r}}_{S,1} \times m_1 \vec{\mathbf{v}}_i = l(-\hat{\mathbf{j}}) \times m_1 v_i \hat{\mathbf{i}} = l m_1 v_i \hat{\mathbf{k}}\,.$$

The angular momentum about S immediately after the collision is

$$\vec{\mathbf{L}}^{\text{sys}}_{S,a} = \vec{\mathbf{r}}_{S,1} \times m_1 \vec{\mathbf{v}}_i / 2 + I_S \vec{\boldsymbol{\omega}}_a = l(-\hat{\mathbf{j}}) \times m_1 (v_i / 2)\hat{\mathbf{i}} + I_S \omega_a \hat{\mathbf{k}} = (l m_1 v_i / 2)\hat{\mathbf{k}} + I_S \omega_a \hat{\mathbf{k}}\,.$$

Therefore the condition that the angular momentum about S is constant during the collision becomes

$$l m_1 v_i \hat{\mathbf{k}} = (l m_1 v_i / 2 + I_S \omega_a)\hat{\mathbf{k}}\,.$$

We can solve for the angular speed immediately after the collision

$$\omega_a = \frac{l m_1 v_i}{2 I_S}\,.$$

By the parallel axis theorem the moment of inertial of a uniform rod about the pivot point is

$$I_S = m_2 (l/2)^2 + I_{cm} = (1/4) m_2 l^2 + (1/12) m_2 l^2 = (1/3) m_2 l^2\,. \quad (19.5.39)$$

Therefore the angular speed immediately after the collision is

$$\omega_2 = \frac{3 m_1 v_i}{2 m_2 l}\,. \quad (19.5.40)$$

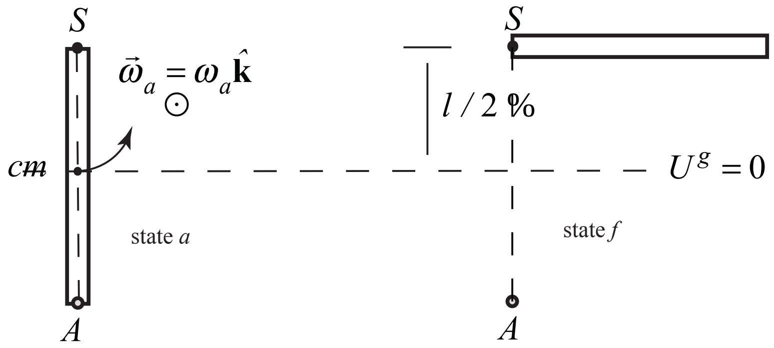

Figure 19.23 Energy diagram for transition from state a to state f.

For the transition from state a to state f, we know that the gravitational force is conservative and the pivot force does no work so mechanical energy is constant.

$$E_a^{mech} = E_f^{mech}$$

We draw an energy diagram only for the rod because the kinetic energy for the particle is not changing between states a and f, (Figure 19.23), with a choice of zero for the potential energy at the center of mass. The mechanical energy of the rod and particle immediately after the collision is

$$E_a^{mech} = \frac{1}{2} I_S \omega_a^2 + \frac{1}{2} m_1 (v_i / 2)^2 .$$

Using our results for the moment of inertia I_S (Eq. (19.5.39)) and ω_2 (Eq. (19.5.40)), we have that

$$E_a^{mech} = \frac{1}{2}(1/3) m_2 l^2 \left(\frac{3 m_1 v_i}{2 m_2 l} \right)^2 + \frac{1}{2} m_1 (v_i / 2)^2 = \frac{3 {m_1}^2 {v_i}^2}{8 m_2} + \frac{1}{2} m_1 (v_i / 2)^2 . \quad (19.5.41)$$

The mechanical energy when the rod just reaches the ceiling when the final angular speed is zero is then

$$E_f^{mech} = m_2 g (l / 2) + \frac{1}{2} m_1 (v_i / 2)^2 .$$

Then the condition that the mechanical energy is constant becomes

$$\frac{3 {m_1}^2 {v_i}^2}{8 m_2} + \frac{1}{2} m_1 (v_i / 2)^2 = m_2 g (l / 2) + \frac{1}{2} m_1 (v_i / 2)^2 . \quad (19.5.42)$$

We can now solve Eq. (19.5.42) for the initial speed of the object

$$v_i = \frac{m_2}{m_1} \sqrt{\frac{4 g l}{3}} . \quad (19.5.43)$$

We now return to the transition from state i to state a. and determine the constraint on the mass ratio in order for the collision to be elastic. The mechanical energy before the collision is

$$E_i^{mech} = \frac{1}{2} m_1 {v_i}^2 . \quad (19.5.44)$$

If we impose the condition that the collision is elastic then

$$E_i^{mech} = E_a^{mech} . \quad (19.5.45)$$

Substituting Eqs. (19.5.41) and (19.5.44) into Eq. (19.5.45) yields

$$\frac{1}{2}m_1 v_i^{\,2} = \frac{3m_1^{\,2}v_i^{\,2}}{8m_2} + \frac{1}{2}m_1(v_i/2)^2 .$$

This simplifies to

$$\frac{3}{8}m_1 v_i^{\,2} = \frac{3m_1^{\,2}v_i^{\,2}}{8m_2}$$

Hence we can solve for the mass ratio necessary to ensure that the collision is elastic if the final speed of the object is half it's initial speed

$$\frac{m_2}{m_1} = 1 . \tag{19.5.46}$$

Notice that this mass ratio is independent of the initial speed of the object.

19.9 External Angular Impulse and Change in Angular Momentum

> *Define the* ***external angular impulse*** *about a point S applied as the integral of the external torque about S*

$$\vec{\mathbf{J}}_S^{\text{ext}} \equiv \int_{t_i}^{t_f} \vec{\boldsymbol{\tau}}_S^{\text{ext}}\, dt . \tag{19.5.47}$$

Then the external angular impulse about S is equal to the change in angular momentum

$$\vec{\mathbf{J}}_S^{\text{ext}} \equiv \int_{t_i}^{t_f} \vec{\boldsymbol{\tau}}_S^{\text{ext}}\, dt = \int_{t_i}^{t_f} \frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} dt = \vec{\mathbf{L}}_{S,f}^{\text{sys}} - \vec{\mathbf{L}}_{S,i}^{\text{sys}} . \tag{19.5.48}$$

Notice that this is the rotational analog to our statement about impulse and momentum,

$$\vec{\mathbf{I}}_S^{\text{ext}} \equiv \int_{t_i}^{t_f} \vec{\mathbf{F}}^{\text{ext}}\, dt = \int_{t_i}^{t_f} \frac{d\vec{\mathbf{p}}_{\text{sys}}}{dt} dt = \vec{\mathbf{p}}_{\text{sys},f} - \vec{\mathbf{p}}_{\text{sys},i} . \tag{19.5.49}$$

Example 19.8 Angular Impulse on Steel Washer

A steel washer is mounted on the shaft of a small motor. The moment of inertia of the motor and washer is I_0. The washer is set into motion. When it reaches an initial angular speed ω_0, at $t = 0$, the power to the motor is shut off, and the washer slows down until it reaches an angular speed of ω_a at time t_a. At that instant, a second steel washer with a moment of inertia I_w is dropped on top of the first washer. Assume that the second washer is only in contact with the first washer. The collision takes place over a time $\Delta t_{\text{int}} = t_b - t_a$. Assume the frictional torque on the axle is independent of speed, and remains the same when the second washer is dropped. The two washers continue to slow down during the time interval $\Delta t_2 = t_f - t_b$ until they stop at time $t = t_f$. (a) What is the angular acceleration while the washer and motor are slowing down during the interval $\Delta t_1 = t_a$? (b) Suppose the collision is nearly instantaneous, $\Delta t_{\text{int}} = (t_b - t_a) \simeq 0$. What is the angular speed ω_b of the two washers immediately after the collision is finished (when the washers rotate together)?

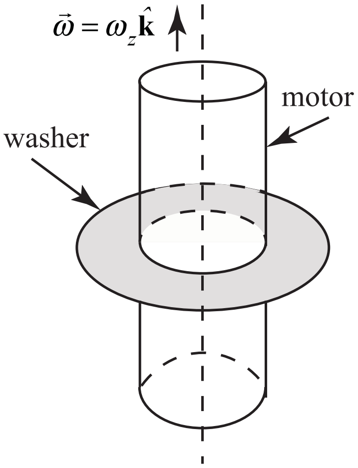

Figure 19.24 Example 19.8

Now suppose the collision is not instantaneous but that the frictional torque is independent of the speed of the rotor. (c) What is the angular impulse during the collision? (d) What is the angular velocity ω_b of the two washers immediately after the collision is finished (when the washers rotate together)? (e) What is the angular deceleration α_2 after the collision?

Solution: a) The angular acceleration of the motor and washer from the instant when the power is shut off until the second washer was dropped is given by

$$\alpha_1 = \frac{\omega_a - \omega_0}{\Delta t_1} < 0. \qquad (19.5.50)$$

(b) If the collision is nearly instantaneous, then there is no angular impulse and therefore the z-component of the angular momentum about the rotation axis of the motor remains constant

$$0 = \Delta L_z = L_{f,z} - L_{0,z} = (I_0 + I_{\text{w}})\omega_b - I_0 \omega_a. \qquad (19.5.51)$$

We can solve Eq. (19.5.51) for the angular speed ω_b of the two washers immediately after the collision is finished

$$\omega_b = \frac{I_0}{I_0 + I_{\text{w}}} \omega_a \,. \qquad (19.5.52)$$

(c) The angular acceleration found in part a) is due to the frictional torque in the motor.

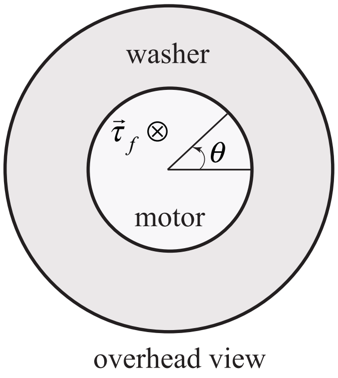

Figure 19.25 Frictional torque in the motor

Let $\vec{\boldsymbol{\tau}}_f = -\tau_f \hat{\mathbf{k}}$ where τ_f is the magnitude of the frictional torque (Figure 19.25) then

$$-\tau_f = I_0 \alpha_1 = \frac{I_0(\omega_a - \omega_0)}{\Delta t_1} \,. \qquad (19.5.53)$$

During the collision with the second washer, the frictional torque exerts an angular impulse (pointing along the z-axis in the figure),

$$J_z = -\int_{t_a}^{t_b} \tau_f \, dt = -\tau_f \, \Delta t_{\text{int}} = I_0(\omega_a - \omega_0) \frac{\Delta t_{\text{int}}}{\Delta t_1} \,. \qquad (19.5.54)$$

(d) The z-component of the angular momentum about the rotation axis of the motor changes during the collision,

$$\Delta L_z = L_{f,z} - L_{0,z} = (I_0 + I_{\text{w}})\omega_b - I_0 \omega_a . \qquad (19.5.55)$$

The change in the z-component of the angular momentum is equal to the z-component of the angular impulse

$$J_z = \Delta L_z \,. \qquad (19.5.56)$$

Thus, equating the expressions in Equations (19.5.54) and (19.5.55), yields

$$I_0(\omega_a - \omega_0) \left(\frac{\Delta t_{\text{int}}}{\Delta t_1} \right) = (I_0 + I_{\text{w}})\omega_b - (I_0)\omega_a \,. \qquad (19.5.57)$$

Solve Equation (19.5.57) for the angular velocity immediately after the collision,

$$\omega_b = \frac{I_0}{(I_0 + I_{\mathrm{w}})}\left((\omega_a - \omega_0)\left(\frac{\Delta t_{\mathrm{int}}}{\Delta t_1} \right) + \omega_a \right). \tag{19.5.58}$$

If there were no frictional torque, then the first term in the brackets would vanish, and the second term of Eq. (19.5.58) would be the only contribution to the final angular speed.

(e) The final angular acceleration α_2 is given by

$$\alpha_2 = \frac{0 - \omega_b}{\Delta t_2} = -\frac{I_0}{(I_0 + I_{\mathrm{w}})\Delta t_2}\left((\omega_a - \omega_0)\left(\frac{\Delta t_{\mathrm{int}}}{\Delta t_1} \right) + \omega_a \right). \tag{19.5.59}$$

Chapter 20 Rigid Body: Translation and Rotational Motion Kinematics for Fixed Axis Rotation

Chapter 20 Rigid Body: Translation and Rotational Motion Kinematics for Fixed Axis Rotation

> *Hence I feel no shame in asserting that this whole region engirdled by the moon, and the center of the earth, traverse this grand circle amid the rest of the planets in an annual revolution around the sun. Near the sun is the center of the universe. Moreover, since the sun remains stationary, whatever appears as a motion of the sun is really due rather to the motion of the earth.*[1]
>
> *Copernicus*

20.1 Introduction

The general motion of a rigid body of mass m consists of a translation of the center of mass with velocity $\vec{\mathbf{V}}_{cm}$ and a rotation about the center of mass with all elements of the rigid body rotating with the same angular velocity $\vec{\boldsymbol{\omega}}_{\text{cm}}$. We prove this result in Appendix A. Figure 20.1 shows the center of mass of a thrown rigid rod follows a parabolic trajectory while the rod rotates about the center of mass.

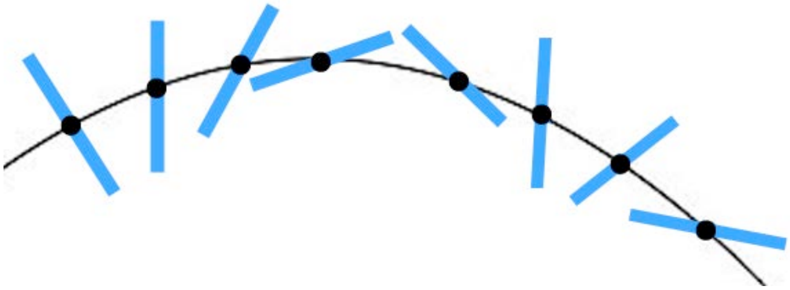

Figure 20.1 The center of mass of a thrown rigid rod follows a parabolic trajectory while the rod rotates about the center of mass.

20.2 Constrained Motion: Translation and Rotation

We shall encounter many examples of a rolling object whose motion is constrained. For example we will study the motion of an object rolling along a level or inclined surface and the motion of a yo-yo unwinding and winding along a string. We will examine the constraint conditions between the translational quantities that describe the motion of the center of mass, displacement, velocity and acceleration, and the rotational quantities that describe the motion about the center of mass, angular displacement, angular velocity and angular acceleration. We begin with a discussion about the rotation and translation of a rolling wheel.

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[1] Nicolaus Copernicus, *De revolutionibus orbium coelestium* (*On the Revolutions of the Celestial Spheres*), Book 1 Chapter 10.

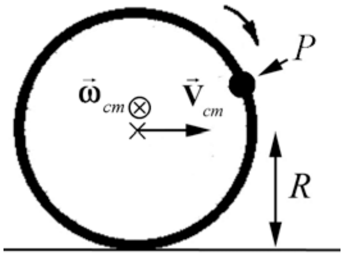

Figure 20.2 Rolling Wheel

Consider a wheel of radius R is rolling in a straight line (Figure 20.2). The center of mass of the wheel is moving in a straight line at a constant velocity $\vec{\mathbf{V}}_{cm}$. Let's analyze the motion of a point P on the rim of the wheel.

Let $\vec{\mathbf{v}}_P$ denote the velocity of a point P on the rim of the wheel with respect to reference frame O at rest with respect to the ground (Figure 20.3a). Let $\vec{\mathbf{v}}'_P$ denote the velocity of the point P on the rim with respect to the center of mass reference frame O_{cm} moving with velocity $\vec{\mathbf{V}}_{cm}$ with respect to at O (Figure 20.3b). (You should review the definition of the center of mass reference frame in Chapter 15.2.1.) We can use the law of addition of velocities (Eq.15.2.4) to relate these three velocities,

$$\vec{\mathbf{v}}_P = \vec{\mathbf{v}}'_P + \vec{\mathbf{V}}_{cm}\,. \tag{20.2.1}$$

Let's choose Cartesian coordinates for the translation motion and polar coordinates for the motion about the center of mass as shown in Figure 20.3.

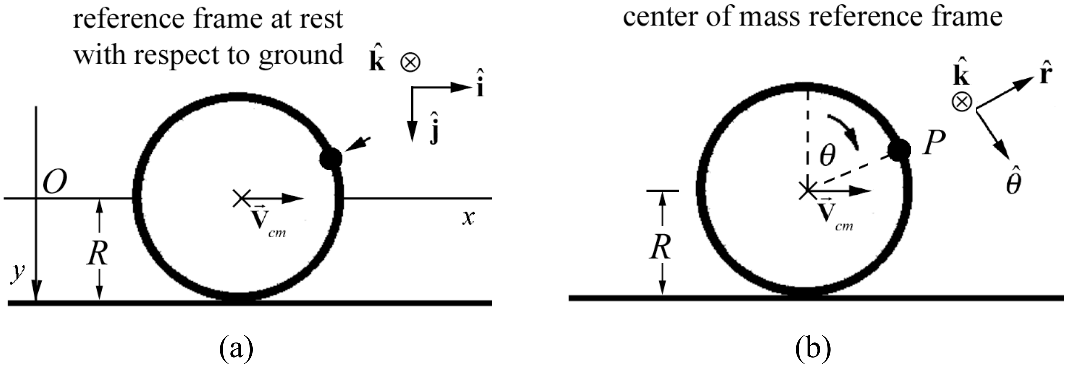

Figure 20.3 (a) reference frame fixed to ground, (b) center of mass reference frame

The center of mass velocity in the reference frame fixed to the ground is given by

$$\vec{\mathbf{V}}_{cm} = V_{\text{cm}}\,\hat{\mathbf{i}}\,. \tag{20.2.2}$$

where V_{cm} is the speed of the center of mass. The position of the center of mass in the reference frame fixed to the ground is given by

$$\vec{\mathbf{R}}_{\text{cm}}(t) = (X_{\text{cm},0} + V_{\text{cm}}t)\hat{\mathbf{i}} \ , \tag{20.2.3}$$

where $X_{\text{cm},0}$ is the initial x-component of the center of mass at $t = 0$. The angular velocity of the wheel in the center of mass reference frame is given by

$$\vec{\boldsymbol{\omega}}_{\text{cm}} = \omega_{\text{cm}}\hat{\mathbf{k}} . \tag{20.2.4}$$

where ω_{cm} is the angular speed. The point P on the rim is undergoing uniform circular motion with the velocity in the center of mass reference frame given by

$$\vec{\mathbf{v}}'_P = R\omega_{\text{cm}}\hat{\boldsymbol{\theta}} . \tag{20.2.5}$$

If we want to use the law of addition of velocities then we should express $\vec{\mathbf{v}}'_P = R\omega_{\text{cm}}\hat{\boldsymbol{\theta}}$ in Cartesian coordinates. Assume that at $t = 0$, $\theta(t = 0) = 0$ i.e. the point P is at the top of the wheel at $t = 0$. Then the unit vectors in polar coordinates satisfy (Figure 20.4)

$$\begin{aligned} \hat{\mathbf{r}} &= \sin\theta\hat{\mathbf{i}} - \cos\theta\hat{\mathbf{j}} \\ \hat{\boldsymbol{\theta}} &= \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}} \end{aligned} . \tag{20.2.6}$$

Therefore the velocity of the point P on the rim in the center of mass reference frame is given by

$$\vec{\mathbf{v}}'_P = R\omega_{\text{cm}}\hat{\boldsymbol{\theta}} = R\omega_{\text{cm}}(\cos\theta\hat{\mathbf{i}} - \sin\theta\hat{\mathbf{j}}) . \tag{20.2.7}$$

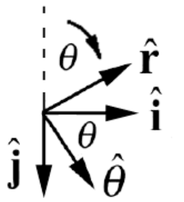

Figure 20.4 Unit vectors

Now substitute Eqs. (20.2.2) and (20.2.7) into Eq. (20.2.1) for the velocity of a point P on the rim in the reference frame fixed to the ground

$$\begin{aligned} \vec{\mathbf{v}}_P &= R\omega_{\text{cm}}(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}) + V_{\text{cm}}\ \hat{\mathbf{i}} \\ &= (V_{\text{cm}} + R\omega_{\text{cm}}\cos\theta)\hat{\mathbf{i}} + R\omega_{\text{cm}}\sin\theta\hat{\mathbf{j}} \end{aligned} . \tag{20.2.8}$$

The point P is in contact with the ground when $\theta = \pi$. At that instant the velocity of a point P on the rim in the reference frame fixed to the ground is

$$\vec{\mathbf{v}}_P(\theta = \pi) = (V_{\text{cm}} - R\omega_{\text{cm}})\hat{\mathbf{i}}. \qquad (20.2.9)$$

What velocity does the observer at rest on the ground measure for the point on the rim when that point is in contact with the ground? In order to understand the relationship between V_{cm} and ω_{cm}, we consider the displacement of the center of mass for a small time interval Δt (Figure 20.5).

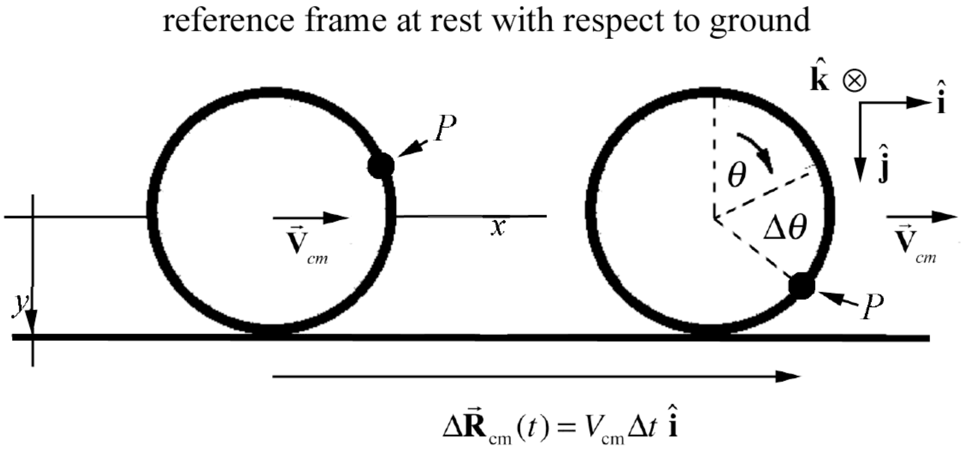

Figure 20.5 Displacement of center of mass in ground reference frame.

From Eq. (20.2.3) the x-component of the displacement of the center of mass is

$$\Delta X_{\text{cm}} = V_{\text{cm}}\Delta t. \qquad (20.2.10)$$

The point P on the rim in the center of mass reference frame is undergoing circular motion (Figure 20.6).

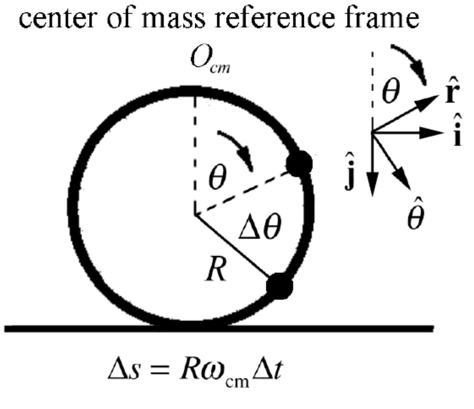

Figure 20.6: Small displacement of point on rim in center of mass reference frame.

In the center of mass reference frame, the magnitude of the tangential displacement is given by the arc length subtended by the angular displacement $\Delta\theta = \omega_{\text{cm}}\Delta t$,

$$\Delta s = R\Delta\theta = R\omega_{\text{cm}}\Delta t\,. \tag{20.2.11}$$

Case 1: if the x-component of the displacement of the center of mass is equal to the arc length subtended by $\Delta\theta$, then the wheel is rolling without slipping or skidding, ***rolling without slipping*** for short, along the surface with

$$\Delta X_{\text{cm}} = \Delta s\,. \tag{20.2.12}$$

Substitute Eq. (20.2.10) and Eq. (20.2.11) into Eq. (20.2.12) and divide through by Δt. Then the rolling without slipping condition becomes

$$V_{\text{cm}} = R\omega_{\text{cm}}, \qquad \text{(rolling without slipping)}. \tag{20.2.13}$$

Case 2: if the x-component of the displacement of the center of mass is greater than the arc length subtended by $\Delta\theta$, then the wheel is ***skidding*** along the surface with

$$\Delta X_{\text{cm}} > \Delta s\,. \tag{20.2.14}$$

Substitute Eqs. (20.2.10) and (20.2.11) into Eq. (20.2.14) and divide through by Δt, then

$$V_{\text{cm}} > R\omega_{\text{cm}}, \qquad \text{(skidding)}. \tag{20.2.15}$$

Case 3: if the x-component of the displacement of the center of mass is less than the arc length subtended by $\Delta\theta$, then the wheel is slipping along the surface with

$$\Delta X_{\text{cm}} < \Delta s\,. \tag{20.2.16}$$

Arguing as above the slipping condition becomes

$$V_{\text{cm}} < R\omega_{\text{cm}}, \qquad \text{(slipping)}. \tag{20.2.17}$$

20.2.1 Rolling without slipping

When a wheel is rolling without slipping, the velocity of a point P on the rim is zero when it is in contact with the ground. In Eq. (20.2.9) set $\theta = \pi$,

$$\vec{\mathbf{v}}_P(\theta = \pi) = (V_{\text{cm}} - R\omega_{\text{cm}})\hat{\mathbf{i}} = (R\omega_{\text{cm}} - R\omega_{\text{cm}})\hat{\mathbf{i}} = \vec{\mathbf{0}}\,. \tag{20.2.18}$$

This makes sense because the velocity of the point P on the rim in the center of mass reference frame when it is in contact with the ground points in the opposite direction as the translational motion of the center of mass of the wheel. The two velocities have the same magnitude so the vector sum is zero. The observer at rest on the ground sees the contact point on the rim at rest relative to the ground.

Thus any frictional force acting between the tire and the ground on the wheel is static friction because the two surfaces are instantaneously at rest with respect to each other. Recall that the direction of the static frictional force depends on the other forces acting on the wheel.

Example 20.1 Bicycle Wheel Rolling Without Slipping

Consider a bicycle wheel of radius R that is rolling in a straight line without slipping. The velocity of the center of mass in a reference frame fixed to the ground is given by velocity $\vec{\mathbf{V}}_{\text{cm}}$. A bead is fixed to a spoke a distance b from the center of the wheel (Figure 20.7). (a) Find the position, velocity, and acceleration of the bead as a function of time in the center of mass reference frame. (b) Find the position, velocity, and acceleration of the bead as a function of time as seen in a reference frame fixed to the ground.

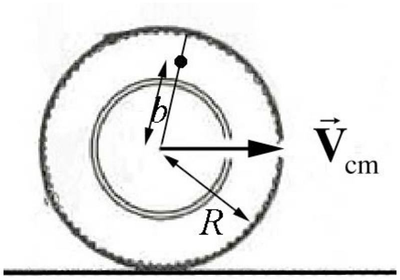

Figure 20.7 Example 20.1

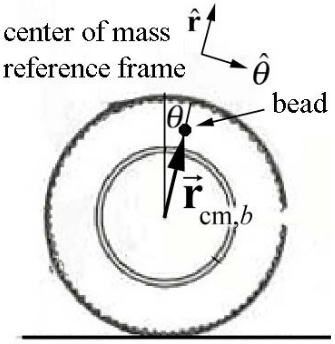

Figure 20.8 Coordinate system for bead in center of mass reference frame

Solution: a) Choose the center of mass reference frame with an origin at the center of the wheel, and moving with the wheel. Choose polar coordinates (Figure 20.8). The z-component of the angular velocity $\omega_{\text{cm}} = d\theta / dt > 0$. Then the bead is moving uniformly in a circle of radius $r = b$ with the position, velocity, and acceleration given by

$$\vec{\mathbf{r}}_b' = b\,\hat{\mathbf{r}}, \quad \vec{\mathbf{v}}_b' = b\omega_{\text{cm}}\,\hat{\theta}, \quad \vec{\mathbf{a}}_b' = -b\omega_{\text{cm}}^2\,\hat{\mathbf{r}} \ . \tag{20.2.19}$$

Because the wheel is rolling without slipping, the velocity of a point on the rim of the wheel has speed $v_P' = R\omega_{\text{cm}}$. This is equal to the speed of the center of mass of the wheel V_{cm}, thus

$$V_{cm} = R\omega_{\text{cm}} \ . \tag{20.2.20}$$

Note that at $t = 0$, the angle $\theta = \theta_0 = 0$. So the angle grows in time as

$$\theta(t) = \omega_{\text{cm}} t = (V_{\text{cm}} / R)t \,. \tag{20.2.21}$$

The velocity and acceleration of the bead with respect to the center of the wheel are then

$$\vec{\mathbf{v}}'_b = \frac{bV_{cm}}{R}\hat{\theta}, \quad \vec{\mathbf{a}}'_b = -\frac{bV_{\text{cm}}^2}{R^2}\hat{\mathbf{r}} \,. \tag{20.2.22}$$

b) Define a second reference frame fixed to the ground with choice of origin, Cartesian coordinates and unit vectors as shown in Figure 20.9.

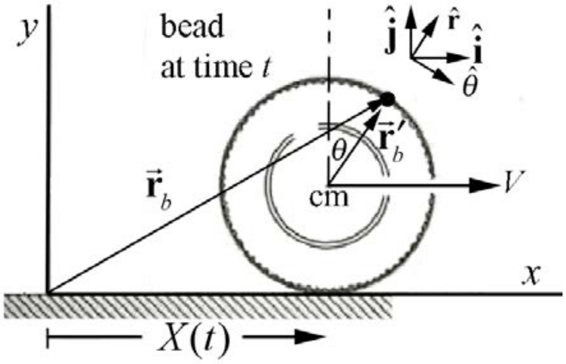

Figure 20.9 Coordinates of bead in reference frame fixed to ground

Then the position vector of the center of mass in the reference frame fixed to the ground is given by

$$\vec{\mathbf{R}}_{cm}(t) = X_{cm}\,\hat{\mathbf{i}} + R\,\hat{\mathbf{j}} = V_{cm}t\,\hat{\mathbf{i}} + R\,\hat{\mathbf{j}}. \tag{20.2.23}$$

The relative velocity of the two frames is the derivative

$$\vec{\mathbf{V}}_{\text{cm}} = \frac{d\vec{\mathbf{R}}_{\text{cm}}}{dt} = \frac{dX_{\text{cm}}}{dt}\hat{\mathbf{i}} = V_{\text{cm}}\,\hat{\mathbf{i}} \,. \tag{20.2.24}$$

Because the center of the wheel is moving at a uniform speed the relative acceleration of the two frames is zero,

$$\vec{\mathbf{A}}_{\text{cm}} = \frac{d\vec{\mathbf{V}}_{\text{cm}}}{dt} = \vec{\mathbf{0}} \,. \tag{20.2.25}$$

Define the position, velocity, and acceleration in this frame (with respect to the ground) by

$$\vec{\mathbf{r}}_b(t) = x_b(t)\,\hat{\mathbf{i}} + y_b(t)\,\hat{\mathbf{j}}, \quad \vec{\mathbf{v}}_b(t) = v_{b,x}(t)\,\hat{\mathbf{i}} + v_{b,y}(t)\,\hat{\mathbf{j}}, \quad \vec{\mathbf{a}}(t) = a_{b,x}(t)\,\hat{\mathbf{i}} + a_{b,y}(t)\,\hat{\mathbf{j}} \,. \tag{20.2.26}$$

Then the position vectors are related by

$$\vec{\mathbf{r}}_b(t) = \vec{\mathbf{R}}_{\text{cm}}(t) + \vec{\mathbf{r}}'_b(t)\,. \tag{20.2.27}$$

In order to add these vectors we need to decompose the position vector in the center of mass reference frame into Cartesian components,

$$\vec{\mathbf{r}}'_b(t) = b\,\hat{\mathbf{r}}(t) = b\sin\theta(t)\,\hat{\mathbf{i}} + b\cos\theta(t)\,\hat{\mathbf{j}}\,. \tag{20.2.28}$$

Then using the relation $\theta(t) = (V_{\text{cm}} / R)t$, Eq. (20.2.28) becomes

$$\begin{aligned}\vec{\mathbf{r}}_b(t) &= \vec{\mathbf{R}}_{\text{cm}}(t) + \vec{\mathbf{r}}'_b(t) = (V_{\text{cm}}t\,\hat{\mathbf{i}} + R\,\hat{\mathbf{j}}) + (b\sin\theta(t)\,\hat{\mathbf{i}} + b\cos\theta(t)\,\hat{\mathbf{j}}) \\ &= \left(V_{\text{cm}}t + b\sin((V_{\text{cm}} / R)t)\right)\hat{\mathbf{i}} + \left(R + b\cos((V_{\text{cm}} / R)t)\right)\hat{\mathbf{j}}\end{aligned}\,. \tag{20.2.29}$$

Thus the position components of the bead with respect to the reference frame fixed to the ground are given by

$$x_b(t) = V_{\text{cm}}t + b\sin((V_{\text{cm}} / R)t) \tag{20.2.30}$$

$$y_b(t) = R + b\cos((V_{\text{cm}} / R)t)\ . \tag{20.2.31}$$

A plot of the y-component vs. the x-component of the position of the bead in the reference frame fixed to the ground is shown in Figure 20.10 below using the values $V_{\text{cm}} = 5\ \text{m}\cdot\text{s}^{-1}$, $R = 0.25\ \text{m}$, and $b = 0.125\ \text{m}$. We can differentiate the position vector in the reference frame fixed to the ground to find the velocity of the bead

$$\vec{\mathbf{v}}_b(t) = \frac{d\vec{\mathbf{r}}_b}{dt}(t) = \frac{d}{dt}(V_{\text{cm}}t + b\sin((V_{\text{cm}} / R)t))\ \hat{\mathbf{i}} + \frac{d}{dt}(R + b\cos((V_{\text{cm}} / R)t)\,)\hat{\mathbf{j}}, \tag{20.2.32}$$

$$\vec{\mathbf{v}}_b(t) = (V_{\text{cm}} + (b / R)V\cos((V_{\text{cm}} / R)t))\,\hat{\mathbf{i}} - ((b / R)V_{\text{cm}}\sin((V_{\text{cm}} / R)t)\,)\hat{\mathbf{j}}\,. \tag{20.2.33}$$

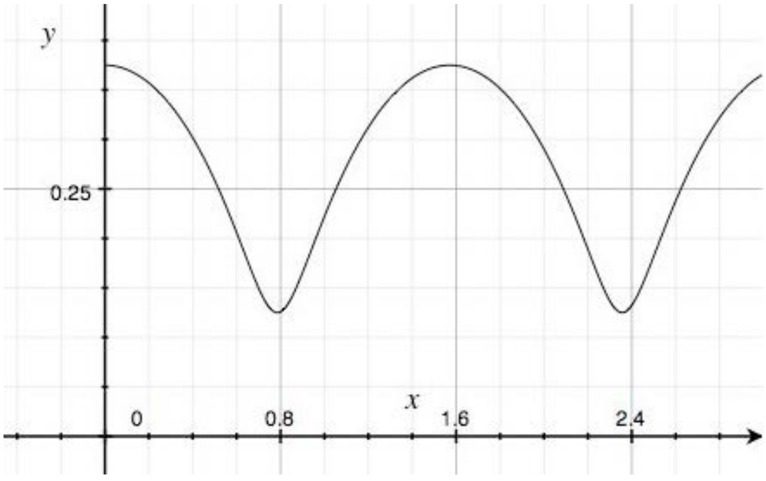

Figure 20.10 Plot of the y-component vs. the x-component of the position of the bead

Alternatively, we can decompose the velocity of the bead in the center of mass reference frame into Cartesian coordinates

$$\vec{\mathbf{v}}'_b(t) = (b / R)V_{\text{cm}}\,(\cos((V_{\text{cm}} / R)t)\,\hat{\mathbf{i}} - \sin((V_{\text{cm}} / R)t)\,\hat{\mathbf{j}})\,. \tag{20.2.34}$$

The law of addition of velocities is then

$$\vec{\mathbf{v}}_b(t) = \vec{\mathbf{V}}_{\text{cm}} + \vec{\mathbf{v}}'_b(t)\,, \tag{20.2.35}$$

$$\vec{\mathbf{v}}_b(t) = V_{\text{cm}}\,\hat{\mathbf{i}} + (b / R)V_{\text{cm}}\,(\cos((V_{\text{cm}} / R)t)\,\hat{\mathbf{i}} - \sin((V_{\text{cm}} / R)t)\,\hat{\mathbf{j}})\,, \tag{20.2.36}$$

$$\vec{\mathbf{v}}_b(t) = (V_{\text{cm}} + (b / R)V_{\text{cm}}\cos((V_{\text{cm}} / R)t))\,\hat{\mathbf{i}} - (b / R)\sin((V_{\text{cm}} / R)t)\,\hat{\mathbf{j}}\,, \tag{20.2.37}$$

in agreement with our previous result. The acceleration is the same in either frame so

$$\vec{\mathbf{a}}_b(t) = \vec{\mathbf{a}}'_b = -(b / R^2)V_{\text{cm}}^2\,\hat{\mathbf{r}} = -(b / R^2)V_{\text{cm}}^2(\sin((V_{\text{cm}} / R)t)\,\hat{\mathbf{i}} + \cos((V_{\text{cm}} / R)t)\,\hat{\mathbf{j}})\,. \tag{20.2.38}$$

When the bead is at the rim of the wheel, $b = R$, then the position of the bead in the reference frame fixed to the ground is given by

$$\vec{\mathbf{r}}_b(t) = (V_{\text{cm}}t + R\sin((V_{\text{cm}} / R)t))\,\hat{\mathbf{i}} + R(1 + \cos((V_{\text{cm}} / R)t)))\,\hat{\mathbf{j}}\,. \tag{20.2.39}$$

The path traced out by the bead in the reference frame fixed to the ground is called a ***cycloid***.

Example 20.2 Cylinder Rolling Without Slipping Down an Inclined Plane

A uniform cylinder of outer radius R and mass M with moment of inertia about the center of mass $I_{\text{cm}} = (1/2)M\,R^2$ starts from rest and rolls without slipping down an incline tilted at an angle β from the horizontal. The center of mass of the cylinder has dropped a vertical distance h when it reaches the bottom of the incline. Let g denote the gravitational constant. What is the relation between the component of the acceleration of the center of mass in the direction down the inclined plane and the component of the angular acceleration into the page of Figure 20.11?

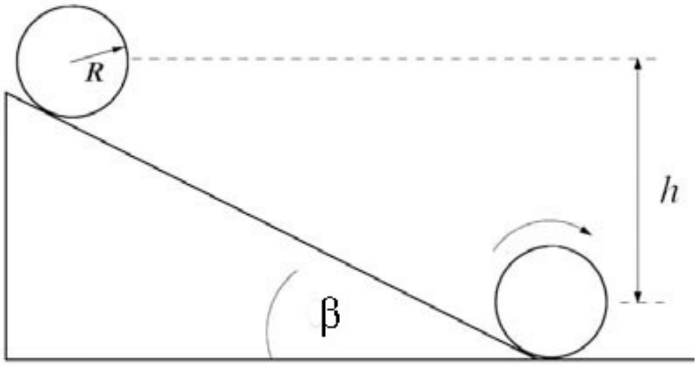

Figure 20.11 Example 20.2

Solution: We begin by choosing a coordinate system for the translational and rotational motion as shown in Figure 20.12.

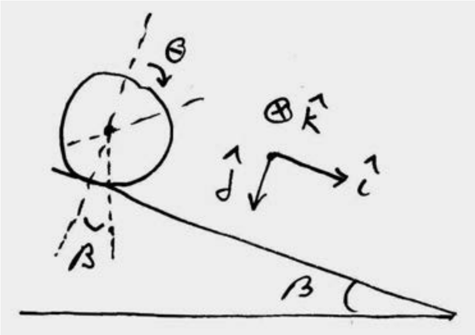

Figure 20.12 Coordinate system for rolling cylinder

For a time interval Δt, the displacement of the center of mass is given by $\Delta \vec{\mathbf{R}}_{cm}(t) = \Delta X_{cm}\ \hat{\mathbf{i}}$. The arc length due to the angular displacement of a point on the rim during the time interval Δt is given by $\Delta s = R\Delta\theta$. The rolling without slipping condition is

$$\Delta X_{cm} = R\Delta\theta \ .$$

If we divide both sides by Δt and take the limit as $\Delta t \to 0$ then the rolling without slipping condition show that the x-component of the center of mass velocity is equal to the magnitude of the tangential component of the velocity of a point on the rim

$$V_{\text{cm}} = \lim_{\Delta t \to 0} \frac{\Delta X_{\text{cm}}}{\Delta t} = \lim_{\Delta t \to 0} R\frac{\Delta\theta}{\Delta t} = R\omega_{\text{cm}} \ .$$

Similarly if we differentiate both sides of the above equation, we find a relation between the x-component of the center of mass acceleration is equal to the magnitude of the tangential component of the acceleration of a point on the rim

$$A_{\text{cm}} = \frac{dV_{\text{cm}}}{dt} = R\frac{d\omega_{\text{cm}}}{dt} = R\alpha_{\text{cm}} \ .$$

Example 20.3 Falling Yo-Yo

A Yo-Yo of mass m has an axle of radius b and a spool of radius R (Figure 20.13a). Its moment of inertia about the center of mass can be taken to be $I = (1/2)mR^2$ (the thickness of the string can be neglected). The Yo-Yo is released from rest. What is the relation between the angular acceleration about the center of mass and the linear acceleration of the center of mass?

Solution: Choose coordinates as shown in Figure 20.13b.

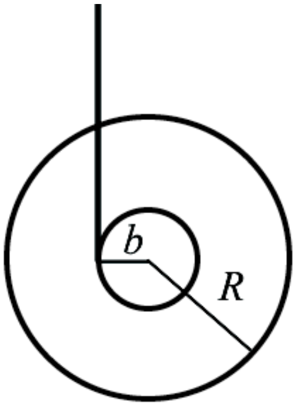

Figure 20.13a Example 20.3

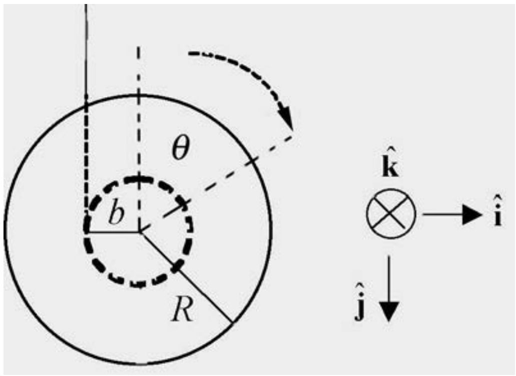

Figure 20.13b Coordinate system for Yo-Yo

Consider a point on the rim of the axle at a distance $r = b$ from the center of mass. As the yo-yo falls, the arc length $\Delta s = b\Delta\theta$ subtended by the rotation of this point is equal to length of string that has unraveled, an amount Δl. In a time interval Δt, $b\Delta\theta = \Delta l$. Therefore $b\Delta\theta / \Delta t = \Delta l / \Delta t$. Taking limits, noting that, $V_{\text{cm},y} = dl / dt$, we have that $b\omega_{\text{cm}} = V_{\text{cm},y}$. Differentiating a second time yields $b\alpha_{\text{cm}} = A_{\text{cm},y}$.

Example 20.4 Unwinding Drum

Drum A of mass m and radius R is suspended from a drum B also of mass m and radius R, which is free to rotate about its axis. The suspension is in the form of a massless metal tape wound around the outside of each drum, and free to unwind (Figure 20.14). Gravity acts with acceleration g downwards. Both drums are initially at rest. Find the initial acceleration of drum A, assuming that it moves straight down.

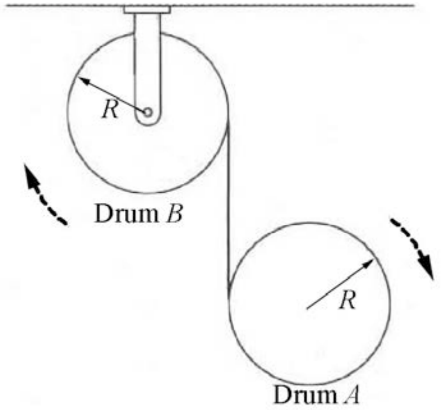

Figure 20.14 Example 20.4

Solution: The key to solving this problem is to determine the relation between the three kinematic quantities α_A, α_B, and a_A, the angular accelerations of the two drums and the linear acceleration of drum A. Choose the positive y-axis pointing downward with the origin at the center of drum B. After a time interval Δt, the center of drum A has undergone a displacement Δy. An amount of tape $\Delta l_A = R\Delta\theta_A$ has unraveled from drum A, and an amount of tape $\Delta l_B = R\Delta\theta_B$ has unraveled from drum B. Therefore the displacement of the center of drum A is equal to the total amount of tape that has unwound from the two drums, $\Delta y = \Delta l_A + \Delta l_B = R\Delta\theta_A + R\Delta\theta_B$. Dividing through by Δt and taking the limit as $\Delta t \to 0$ yields

$$\frac{dy}{dt} = R\frac{d\theta_A}{dt} + R\frac{d\theta_B}{dt}.$$

Differentiating a second time yields the desired relation between the angular accelerations of the two drums and the linear acceleration of drum A,

$$\frac{d^2y}{dt^2} = R\frac{d^2\theta_A}{dt^2} + R\frac{d^2\theta_B}{dt^2}$$
$$a_{A,y} = R\alpha_A + R\alpha_B.$$

20.3 Angular Momentum for a System of Particles Undergoing Translational and Rotational

We shall now show that the angular momentum of a body about a point S can be decomposed into two vector parts, the angular momentum of the center of mass (treated as a point particle) about the point S, and the angular momentum of the rotational motion about the center of mass.

Consider a system of N particles located at the points labeled $i = 1, 2, \cdots, N$. The angular momentum about the point S is the sum

$$\vec{\mathbf{L}}_S^{\text{total}} = \sum_{i=1}^{N} \vec{\mathbf{L}}_{S,i} = (\sum_{i=1}^{N} \vec{\mathbf{r}}_{S,i} \times m_i \vec{\mathbf{v}}_i), \tag{20.3.1}$$

where $\vec{\mathbf{r}}_{S,i}$ is the vector from the point S to the i^{th} particle (Figure 20.15) satisfying

$$\vec{\mathbf{r}}_{S,i} = \vec{\mathbf{r}}_{S,cm} + \vec{\mathbf{r}}_{\text{cm},i}, \tag{20.3.2}$$

$$\vec{\mathbf{v}}_{S,i} = \vec{\mathbf{V}}_{cm} + \vec{\mathbf{v}}_{\text{cm},i}, \tag{20.3.3}$$

where $\vec{\mathbf{v}}_{S,cm} = \vec{\mathbf{V}}_{cm}$. We can now substitute both Eqs. (20.3.2) and (20.3.3) into Eq. (20.3.1) yielding

$$\vec{\mathbf{L}}_S^{\text{total}} = \sum_{i=1}^{N} (\vec{\mathbf{r}}_{S,cm} + \vec{\mathbf{r}}_{\text{cm},i}) \times m_i (\vec{\mathbf{V}}_{cm} + \vec{\mathbf{v}}_{\text{cm},i}) \,. \tag{20.3.4}$$

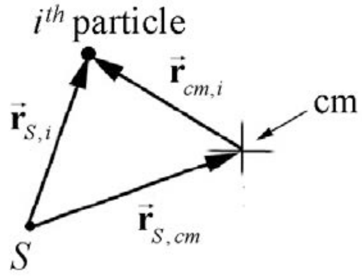

Figure 20.15 Vector Triangle

When we expand the expression in Equation (20.3.4), we have four terms,

$$\begin{aligned} \vec{\mathbf{L}}_S^{\text{total}} &= \sum_{i=1}^{N} (\vec{\mathbf{r}}_{S,cm} \times m_i \vec{\mathbf{v}}_{\text{cm},i}) + \sum_{i=1}^{N} (\vec{\mathbf{r}}_{S,cm} \times m_i \vec{\mathbf{V}}_{cm}) \\ &+ \sum_{i=1}^{N} (\vec{\mathbf{r}}_{\text{cm},i} \times m_i \vec{\mathbf{v}}_{\text{cm},i}) + \sum_{i=1}^{N} (\vec{\mathbf{r}}_{\text{cm},i} \times m_i \vec{\mathbf{V}}_{cm}). \end{aligned} \tag{20.3.5}$$

The vector $\vec{\mathbf{r}}_{S,cm}$ is a constant vector that depends only on the location of the center of mass and not on the location of the i^{th} particle. Therefore in the first term in the above equation, $\vec{\mathbf{r}}_{S,cm}$ can be taken outside the summation. Similarly, in the second term the velocity of the center of mass $\vec{\mathbf{V}}_{cm}$ is the same for each term in the summation, and may be taken outside the summation,

$$\begin{aligned} \vec{\mathbf{L}}_S^{\text{total}} &= \vec{\mathbf{r}}_{S,cm} \times \left(\sum_{i=1}^{N} m_i \vec{\mathbf{v}}_{\text{cm},i} \right) + \vec{\mathbf{r}}_{S,cm} \times \left(\sum_{i=1}^{N} m_i \right) \vec{\mathbf{V}}_{cm} \\ &+ \sum_{i=1}^{N} (\vec{\mathbf{r}}_{\text{cm},i} \times m_i \vec{\mathbf{v}}_{\text{cm},i}) + \left(\sum_{i=1}^{N} m_i \vec{\mathbf{r}}_{\text{cm},i} \right) \times \vec{\mathbf{V}}_{cm}. \end{aligned} \tag{20.3.6}$$

The first and third terms in Eq. (20.3.6) are both zero due to the fact that

$$\begin{aligned} &\sum_{i=1}^{N} m_i \vec{\mathbf{r}}_{\text{cm},i} = 0 \\ &\sum_{i=1}^{N} m_i \vec{\mathbf{v}}_{\text{cm},i} = 0. \end{aligned} \tag{20.3.7}$$

We first show that $\sum_{i=1}^{N} m_i \vec{\mathbf{r}}_{\text{cm},i}$ is zero. We begin by using Eq. (20.3.2),

$$\begin{aligned}\sum_{i=1}^{N}(m_i\vec{\mathbf{r}}_{\mathrm{cm},i}) &= \sum_{i=1}^{N}(m_i(\vec{\mathbf{r}}_i-\vec{\mathbf{r}}_{S,cm}))\\ &= \sum_{i=1}^{N}m_i\vec{\mathbf{r}}_i-\left(\sum_{i=1}^{N}(m_i)\right)\vec{\mathbf{r}}_{S,cm} = \sum_{i=1}^{N}m_i\vec{\mathbf{r}}_i - m^{total}\vec{\mathbf{r}}_{S,cm}.\end{aligned} \tag{20.3.8}$$

Substitute the definition of the center of mass (Eq. 10.5.3) into Eq. (20.3.8) yielding

$$\sum_{i=1}^{N}(m_i\vec{\mathbf{r}}_{\mathrm{cm},i}) = \sum_{i=1}^{N}m_i\vec{\mathbf{r}}_i - m^{total}\frac{1}{m^{total}}\sum_{i=1}^{N}m_i\vec{\mathbf{r}}_i = \vec{0}. \tag{20.3.9}$$

The vanishing of $\sum_{i=1}^{N}m_i\vec{\mathbf{v}}_{\mathrm{cm},i}=0$ follows directly from the definition of the center of mass frame, that the momentum in the center of mass is zero. Equivalently the derivative of Eq. (20.3.9) is zero. We could also simply calculate and find that

$$\begin{aligned}\sum_i m_i\vec{\mathbf{v}}_{\mathrm{cm},i} &= \sum_i m_i(\vec{\mathbf{v}}_i-\vec{\mathbf{V}}_{\mathrm{cm}})\\ &= \sum_i m_i\vec{\mathbf{v}}_i-\vec{\mathbf{V}}_{\mathrm{cm}}\sum_i m_i\\ &= m^{\mathrm{total}}\vec{\mathbf{V}}_{\mathrm{cm}}-\vec{\mathbf{V}}_{\mathrm{cm}}m^{\mathrm{total}}\\ &= \vec{\mathbf{0}}.\end{aligned} \tag{20.3.10}$$

We can now simplify Eq. (20.3.6) for the angular momentum about the point S using the fact that, $m_T=\sum_{i=1}^{N}m_i$, and $\vec{\mathbf{p}}_{\mathrm{sys}}=m_T\vec{\mathbf{V}}_{cm}$ (in reference frame O):

$$\vec{\mathbf{L}}_S^{\mathrm{total}} = \vec{\mathbf{r}}_{S,cm}\times\vec{\mathbf{p}}^{\mathrm{sys}}+\sum_{i=1}^{N}(\vec{\mathbf{r}}_{cm,i}\times m_i\vec{\mathbf{v}}_{cm,i}). \tag{20.3.11}$$

Consider the first term in Equation (20.3.11), $\vec{\mathbf{r}}_{S,\mathrm{cm}}\times\vec{\mathbf{p}}_{\mathrm{sys}}$; the vector $\vec{\mathbf{r}}_{S,\mathrm{cm}}$ is the vector from the point S to the center of mass. If we treat the system as a point-like particle of mass m_T located at the center of mass, then the momentum of this point-like particle is $\vec{\mathbf{p}}_{\mathrm{sys}}=m_T\vec{\mathbf{V}}_{cm}$. Thus the first term is the angular momentum about the point S of this “point-like particle”, which is called the ***orbital angular momentum*** about S,

$$\vec{\mathbf{L}}_S^{\mathrm{orbital}} = \vec{\mathbf{r}}_{S,\mathrm{cm}}\times\vec{\mathbf{p}}_{\mathrm{sys}}. \tag{20.3.12}$$

for the system of particles.

Consider the second term in Equation (20.3.11), $\sum_{i=1}^{N}(\vec{\mathbf{r}}_{cm,i} \times m_i \vec{\mathbf{v}}_{cm,i})$; the quantity inside the summation is the angular momentum of the i^{th} particle with respect to the origin in the center of mass reference frame O_{cm} (recall the origin in the center of mass reference frame is the center of mass of the system),

$$\vec{\mathbf{L}}_{cm,i} = \vec{\mathbf{r}}_{cm,i} \times m_i \vec{\mathbf{v}}_{cm,i} . \tag{20.3.13}$$

Hence the total angular momentum of the system with respect to the center of mass in the center of mass reference frame is given by

$$\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = \sum_{i=1}^{N} \vec{\mathbf{L}}_{cm,i} = \sum_{i=1}^{N} (\vec{\mathbf{r}}_{cm,i} \times m_i \vec{\mathbf{v}}_{cm,i}) . \tag{20.3.14}$$

a vector quantity we call the ***spin angular momentum***. Thus we see that the total angular momentum about the point S is the sum of these two terms,

$$\vec{\mathbf{L}}_{S}^{\text{total}} = \vec{\mathbf{L}}_{S}^{\text{orbital}} + \vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} . \tag{20.3.15}$$

This decomposition of angular momentum into a piece associated with the translational motion of the center of mass and a second piece associated with the rotational motion about the center of mass in the center of mass reference frame is the key conceptual foundation for what follows.

Example 20.5 Earth's Motion Around the Sun

The earth, of mass $m_{\text{e}} = 5.97 \times 10^{24}\ \text{kg}$ and (mean) radius $R_{\text{e}} = 6.38 \times 10^{6}\ \text{m}$, moves in a nearly circular orbit of radius $r_{\text{s,e}} = 1.50 \times 10^{11}\ \text{m}$ around the sun with a period $T_{\text{orbit}} = 365.25\ \text{days}$, and spins about its axis in a period $T_{\text{spin}} = 23\ \text{hr}\ 56\ \text{min}$, the axis inclined to the normal to the plane of its orbit around the sun by $23.5°$ (in Figure 20.16, the relative size of the earth and sun, and the radius and shape of the orbit are not representative of the actual quantities).

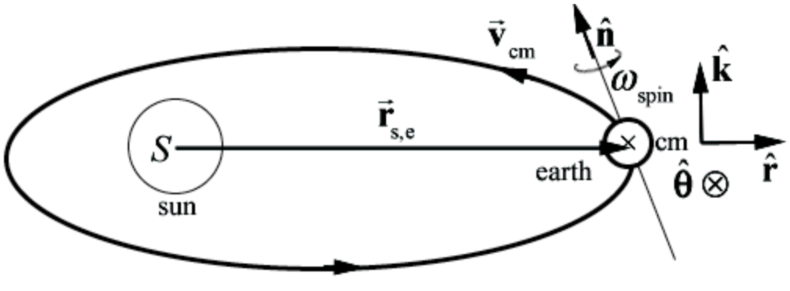

Figure 20.16 Example 20.5

If we approximate the earth as a uniform sphere, then the moment of inertia of the earth about its center of mass is

$$I_{\mathrm{cm}} = \frac{2}{5} m_{\mathrm{e}} R_{\mathrm{e}}^2 . \tag{20.3.16}$$

If we choose the point S to be at the center of the sun, and assume the orbit is circular, then the orbital angular momentum is

$$\vec{\mathbf{L}}_S^{\mathrm{orbital}} = \vec{\mathbf{r}}_{S,\mathrm{cm}} \times \vec{\mathbf{p}}_{\mathrm{sys}} = r_{\mathrm{s,e}}\, \hat{\mathbf{r}} \times m_{\mathrm{e}} v_{\mathrm{cm}}\, \hat{\boldsymbol{\theta}} = r_{\mathrm{s,e}} m_{\mathrm{e}} v_{\mathrm{cm}}\, \hat{\mathbf{k}} . \tag{20.3.17}$$

The velocity of the center of mass of the earth about the sun is related to the orbital angular velocity by

$$v_{\mathrm{cm}} = r_{\mathrm{s,e}}\, \omega_{\mathrm{orbit}} , \tag{20.3.18}$$

where the orbital angular speed is

$$\begin{aligned} \omega_{\mathrm{orbit}} &= \frac{2\pi}{T_{\mathrm{orbit}}} = \frac{2\pi}{(365.25\ \mathrm{d})(8.640 \times 10^4\ \mathrm{s} \cdot \mathrm{d}^{-1})} \\ &= 1.991 \times 10^{-7}\ \mathrm{rad} \cdot \mathrm{s}^{-1}. \end{aligned} \tag{20.3.19}$$

The orbital angular momentum about S is then

$$\begin{aligned} \vec{\mathbf{L}}_S^{\mathrm{orbital}} &= m_{\mathrm{e}} r_{\mathrm{s,e}}^2\, \omega_{\mathrm{orbit}}\, \hat{\mathbf{k}} \\ &= (5.97 \times 10^{24}\ \mathrm{kg})(1.50 \times 10^{11}\ \mathrm{m})^2 (1.991 \times 10^{-7}\ \mathrm{rad} \cdot \mathrm{s}^{-1})\, \hat{\mathbf{k}} \\ &= (2.68 \times 10^{40}\ \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-1})\, \hat{\mathbf{k}}. \end{aligned} \tag{20.3.20}$$

The spin angular momentum is given by

$$\vec{\mathbf{L}}_{\mathrm{cm}}^{\mathrm{spin}} = I_{\mathrm{cm}}\, \vec{\boldsymbol{\omega}}_{\mathrm{spin}} = \frac{2}{5} m_{\mathrm{e}} R_{\mathrm{e}}^2\, \omega_{\mathrm{spin}}\, \hat{\mathbf{n}} , \tag{20.3.21}$$

where $\hat{\mathbf{n}}$ is a unit normal pointing along the axis of rotation of the earth and

$$\omega_{\mathrm{spin}} = \frac{2\pi}{T_{\mathrm{spin}}} = \frac{2\pi}{8.616 \times 10^4\ \mathrm{s}} = 7.293 \times 10^{-5}\ \mathrm{rad} \cdot \mathrm{s}^{-1} . \tag{20.3.22}$$

The spin angular momentum is then

$$\begin{aligned} \vec{\mathbf{L}}_{\mathrm{cm}}^{\mathrm{spin}} &= \frac{2}{5} (5.97 \times 10^{24}\ \mathrm{kg})(6.38 \times 10^6\ \mathrm{m})^2 (7.293 \times 10^{-5}\ \mathrm{rad} \cdot \mathrm{s}^{-1})\, \hat{\mathbf{n}} \\ &= (7.10 \times 10^{33}\ \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-1})\, \hat{\mathbf{n}}. \end{aligned} \tag{20.3.23}$$

The ratio of the magnitudes of the orbital angular momentum about S to the spin angular momentum is greater than a million,

$$\frac{L_S^{\text{orbital}}}{L_{\text{cm}}^{\text{spin}}}=\frac{m_{\text{e}}\,r_{\text{s,e}}^2\,\omega_{\text{orbit}}}{(2/5)m_{\text{e}}R_{\text{e}}^2\,\omega_{\text{spin}}}=\frac{5}{2}\frac{r_{\text{s,e}}^2}{R_{\text{e}}^2}\frac{T_{\text{spin}}}{T_{\text{orbit}}}=3.77\times10^6\,, \tag{20.3.24}$$

as this ratio is proportional to the square of the ratio of the distance to the sun to the radius of the earth. The angular momentum about S is then

$$\vec{\mathbf{L}}_S^{\text{total}}=m_{\text{e}}\,r_{\text{s,e}}^2\,\omega_{\text{orbit}}\,\hat{\mathbf{k}}+\frac{2}{5}m_{\text{e}}R_{\text{e}}^2\omega_{\text{spin}}\,\hat{\mathbf{n}}\,. \tag{20.3.25}$$

The orbit and spin periods are known to far more precision than the average values used for the earth's orbit radius and mean radius. Two different values have been used for one "day;" in converting the orbit period from days to seconds, the value for the *solar day*, $T_{\text{solar}}=86,400\,\text{s}$ was used. In converting the earth's spin angular frequency, the *sidereal day*, $T_{\text{sidereal}}=T_{\text{spin}}=86,160\,\text{s}$ was used. The two periods, the solar day from noon to noon and the sidereal day from the difference between the times that a fixed star is at the same place in the sky, do differ in the third significant figure.

20.4 Kinetic Energy of a System of Particles

Consider a system of particles. The i^{th} particle has mass m_i and velocity $\vec{\mathbf{v}}_i$ with respect to a reference frame O. The kinetic energy of the system of particles is given by

$$\begin{aligned}K&=\sum_i\frac{1}{2}m_i\,v_i^2=\frac{1}{2}\sum_i m_i\,\vec{\mathbf{v}}_i\cdot\vec{\mathbf{v}}_i\\&=\frac{1}{2}\sum_i m_i\,(\vec{\mathbf{v}}_{\text{cm},i}+\vec{\mathbf{V}}_{\text{cm}})\cdot(\vec{\mathbf{v}}_{\text{cm},i}+\vec{\mathbf{V}}_{\text{cm}}).\end{aligned} \tag{20.4.1}$$

where Equation 15.2.6 has been used to express $\vec{\mathbf{v}}_i$ in terms of $\vec{\mathbf{v}}_{cm,i}$ and $\vec{\mathbf{V}}_{\text{cm}}$. Expanding the last dot product in Equation (20.4.1),

$$\begin{aligned}K&=\frac{1}{2}\sum_i m_i\,(\vec{\mathbf{v}}_{\text{cm},i}\cdot\vec{\mathbf{v}}_{cm,i}+\vec{\mathbf{V}}_{\text{cm}}\cdot\vec{\mathbf{V}}_{\text{cm}}+2\,\vec{\mathbf{v}}_{\text{cm},i}\cdot\vec{\mathbf{V}}_{\text{cm}})\\&=\frac{1}{2}\sum_i m_i(\vec{\mathbf{v}}_{\text{cm},i}\cdot\vec{\mathbf{v}}_{\text{cm},i})+\frac{1}{2}\sum_i m_i(\vec{\mathbf{V}}_{\text{cm}}\cdot\vec{\mathbf{V}}_{\text{cm}})+\sum_i m_i\vec{\mathbf{v}}_{\text{cm},i}\cdot\vec{\mathbf{V}}_{\text{cm}}\\&=\sum_i\frac{1}{2}m_i v_{\text{cm},i}^2+\frac{1}{2}\sum_i m_i\,V_{\text{cm}}^2+\left(\sum_i m\,\vec{\mathbf{v}}_{\text{cm},i}\right)\cdot\vec{\mathbf{V}}_{\text{cm}}.\end{aligned} \tag{20.4.2}$$

The last term in the third equation in (20.4.2) vanishes as we showed in Eq. (20.3.7). Then Equation (20.4.2) reduces to

$$\begin{aligned} K &= \sum_i \frac{1}{2} m_i v_{\mathrm{cm},i}^2 + \frac{1}{2} \sum_i m_i V_{\mathrm{cm}}^2 \\ &= \sum_i \frac{1}{2} m_i v_{\mathrm{cm},i}^2 + \frac{1}{2} m^{\mathrm{total}} V_{\mathrm{cm}}^2 . \end{aligned} \tag{20.4.3}$$

We interpret the first term as the sum of the individual kinetic energies of the particles of the system in the center of mass reference frame O_{cm} and the second term as the kinetic energy of the center of mass motion in reference frame O.

At this point, it's important to note that no assumption was made regarding the mass elements being constituents of a rigid body. Equation (20.4.3) is valid for a rigid body, a gas, a firecracker (but K is certainly not the same before and after detonation), and the sixteen pool balls after the break, or any collection of objects for which the center of mass can be determined.

20.5 Rotational Kinetic Energy for a Rigid Body Undergoing Fixed Axis Rotation

The rotational kinetic energy for the rigid body, using $\vec{\mathbf{v}}_{\mathrm{cm},i} = (r_{\mathrm{cm},\,i})_{\perp} \omega_{\mathrm{cm}} \hat{\boldsymbol{\theta}}$, simplifies to

$$K_{\mathrm{rot}} = \frac{1}{2} I_{\mathrm{cm}} \omega_{\mathrm{cm}}^2 . \tag{20.5.1}$$

Therefore the total kinetic energy of a translating and rotating rigid body is

$$K_{\mathrm{total}} = K_{\mathrm{trans}} + K_{\mathrm{rot}} = \frac{1}{2} m V_{\mathrm{cm}}^2 + \frac{1}{2} I_{\mathrm{cm}} \omega_{\mathrm{cm}}^2 . \tag{20.5.2}$$

Appendix 20A Chasles's Theorem: Rotation and Translation of a Rigid Body

We now return to our description of the translating and rotating rod that we first considered when we began our discussion of rigid bodies. We shall now show that the motion of any rigid body consists of a translation of the center of mass and rotation about the center of mass.

We shall demonstrate this for a rigid body by dividing up the rigid body into point-like constituents. Consider two point-like constituents with masses m_1 and m_2. Choose a coordinate system with a choice of origin such that body 1 has position $\vec{\mathbf{r}}_1$ and body 2 has position $\vec{\mathbf{r}}_2$ (Figure 20A.1). The relative position vector is given by

$$\vec{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2 . \qquad (20.A.1)$$

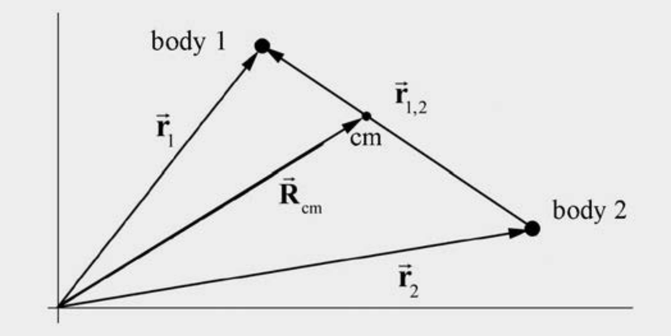

Figure 20A.1 Two-body coordinate system.

Recall we defined the center of mass vector, $\vec{\mathbf{R}}_{cm}$, of the two-body system as

$$\vec{\mathbf{R}}_{cm} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{m_1 + m_2} . \qquad (20.A.2)$$

In Figure 20A.2 we show the center of mass coordinate system.

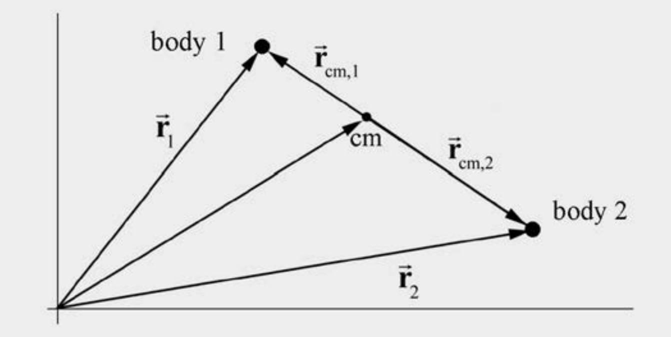

Figure 20A.2 Position coordinates with respect to center of mass

The position vector of the object 1 with respect to the center of mass is given by

$$\vec{\mathbf{r}}_{\text{cm},1} = \vec{\mathbf{r}}_1 - \vec{\mathbf{R}}_{\text{cm}} = \vec{\mathbf{r}}_1 - \frac{m_1 \vec{\mathbf{r}}_1 + m_2 \vec{\mathbf{r}}_2}{m_1 + m_2} = \frac{m_2}{m_1 + m_2}(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2) = \frac{\mu}{m_1}\vec{\mathbf{r}}_{1,2}, \qquad (20.A.3)$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \qquad (20.A.4)$$

is the reduced mass. In addition, the relative position vector between the two objects is independent of the choice of reference frame,

$$\vec{\mathbf{r}}_{12} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2 = (\vec{\mathbf{r}}_{\text{cm},1} + \vec{\mathbf{R}}_{\text{cm}}) - (\vec{\mathbf{r}}_{\text{cm},2} + \vec{\mathbf{R}}_{\text{cm}}) = \vec{\mathbf{r}}_{\text{cm},1} - \vec{\mathbf{r}}_{\text{cm},2} = \vec{\mathbf{r}}_{\text{cm},1,2}. \qquad (20.A.5)$$

Because the center of mass is at the origin in the center of mass reference frame,

$$\frac{m_1 \vec{\mathbf{r}}_{\text{cm},1} + m_2 \vec{\mathbf{r}}_{\text{cm},2}}{m_1 + m_2} = \vec{\mathbf{0}}. \qquad (20.A.6)$$

Therefore

$$m_1 \vec{\mathbf{r}}_{\text{cm},1} = -m_2 \vec{\mathbf{r}}_{\text{cm},2} \qquad (20.A.7)$$

$$m_1 \left|\vec{\mathbf{r}}_{\text{cm},1}\right| = m_2 \left|\vec{\mathbf{r}}_{\text{cm},2}\right|. \qquad (20.A.8)$$

The displacement of object 1 about the center of mass is given by taking the derivative of Eq. (20.A.3),

$$d\vec{\mathbf{r}}_{\text{cm},1} = \frac{\mu}{m_1} d\vec{\mathbf{r}}_{1,2}. \qquad (20.A.9)$$

A similar calculation for the position of object 2 with respect to the center of mass yields for the position and displacement with respect to the center of mass

$$\vec{\mathbf{r}}_{\text{cm},2} = \vec{\mathbf{r}}_2 - \vec{\mathbf{R}}_{\text{cm}} = -\frac{\mu}{m_2}\vec{\mathbf{r}}_{1,2}, \qquad (20.A.10)$$

$$d\vec{\mathbf{r}}_{\text{cm},2} = -\frac{\mu}{m_2} d\vec{\mathbf{r}}_{1,2}. \qquad (20.A.11)$$

Let $i = 1,2$. An arbitrary displacement of the i^{th} object is given respectively by

$$d\vec{\mathbf{r}}_i = d\vec{\mathbf{r}}_{\text{cm},i} + d\vec{\mathbf{R}}_{\text{cm}}, \qquad (20.A.12)$$

which is the sum of a displacement about the center of mass $d\vec{\mathbf{r}}_{\text{cm},i}$ and a displacement of the center of mass $d\vec{\mathbf{R}}_{\text{cm}}$. The displacement of objects 1 and 2 are constrained by the condition that the distance between the objects must remain constant since the body is rigid. In particular, the distance between objects 1 and 2 is given by

$$\left|\vec{\mathbf{r}}_{1,2}\right|^2 = (\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2)\cdot(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2). \tag{20.A.13}$$

Because this distance is constant we can differentiate Eq. (20.A.13), yielding the ***rigid body condition*** that

$$0 = 2(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2)\cdot(d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2) = 2\vec{\mathbf{r}}_{1,2}\cdot d\vec{\mathbf{r}}_{1,2} \tag{20.A.14}$$

20A.1. Translation of the Center of Mass

The condition (Eq. (20.A.14)) can be satisfied if the relative displacement vector between the two objects is zero,

$$d\vec{\mathbf{r}}_{1,2} = d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2 = \vec{\mathbf{0}}. \tag{20.A.15}$$

This implies, using, Eq. (20.A.9) and Eq. (20.A.11), that the displacement with respect to the center of mass is zero,

$$d\vec{\mathbf{r}}_{\text{cm},1} = d\vec{\mathbf{r}}_{\text{cm},2} = \vec{\mathbf{0}}. \tag{20.A.16}$$

Thus by Eq. (20.A.12), the displacement of each object is equal to the displacement of the center of mass,

$$d\vec{\mathbf{r}}_i = d\vec{\mathbf{R}}_{\text{cm}}, \tag{20.A.17}$$

which means that the body is undergoing pure translation.

20A.2 Rotation about the Center of Mass

Now suppose that $d\vec{\mathbf{r}}_{1,2} = d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2 \neq \vec{\mathbf{0}}$. The rigid body condition can be expressed in terms of the center of mass coordinates. Using Eq. (20.A.9), the rigid body condition (Eq. (20.A.14)) becomes

$$0 = 2\frac{\mu}{m_1}\vec{\mathbf{r}}_{1,2}\cdot d\vec{\mathbf{r}}_{\text{cm},1}. \tag{20.A.18}$$

Because the relative position vector between the two objects is independent of the choice of reference frame (Eq. (20.A.5)), the rigid body condition Eq. (20.A.14) in the center of mass reference frame is then given by

$$0 = 2\vec{\mathbf{r}}_{\text{cm},1,2}\cdot d\vec{\mathbf{r}}_{\text{cm},1}. \tag{20.A.19}$$

This condition is satisfied if the relative displacement is perpendicular to the line passing through the center of mass,

$$\vec{\mathbf{r}}_{\text{cm},1,2} \perp d\vec{\mathbf{r}}_{\text{cm},1} \,. \tag{20.A.20}$$

By a similar argument, $\vec{\mathbf{r}}_{\text{cm},1,2} \perp d\vec{\mathbf{r}}_{\text{cm},2}$. In order for these displacements to correspond to a rotation about the center of mass, the displacements must have the same angular displacement.

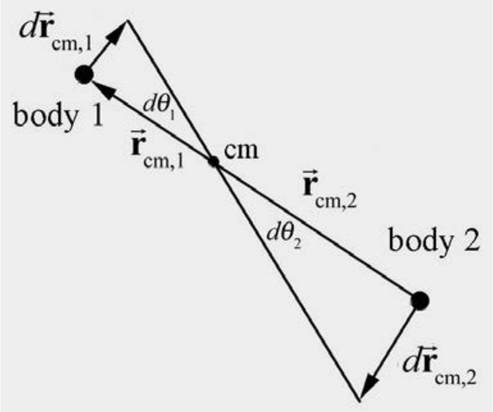

Figure 20A.3 Infinitesimal angular displacements in the center of mass reference frame

In Figure 20A.3, the infinitesimal angular displacement of each object is given by

$$d\theta_1 = \frac{\left|d\vec{\mathbf{r}}_{\text{cm},1}\right|}{\left|\vec{\mathbf{r}}_{\text{cm},1}\right|} \,, \tag{20.A.21}$$

$$d\theta_2 = \frac{\left|d\vec{\mathbf{r}}_{\text{cm},2}\right|}{\left|\vec{\mathbf{r}}_{\text{cm},2}\right|} \,. \tag{20.A.22}$$

From Eq. (20.A.9) and Eq. (20.A.11), we can rewrite Eqs. (20.A.21) and (20.A.22) as

$$d\theta_1 = \frac{\mu}{m_1}\frac{\left|d\vec{\mathbf{r}}_{1,2}\right|}{\left|\vec{\mathbf{r}}_{\text{cm},1}\right|} \,, \tag{20.A.23}$$

$$d\theta_2 = \frac{\mu}{m_2}\frac{\left|d\vec{\mathbf{r}}_{1,2}\right|}{\left|\vec{\mathbf{r}}_{\text{cm},2}\right|} \,. \tag{20.A.24}$$

Recall that in the center of mass reference frame $m_1\left|\vec{\mathbf{r}}_{\text{cm},1}\right| = m_2\left|\vec{\mathbf{r}}_{\text{cm},2}\right|$ (Eq. (20.A.8)) and hence the angular displacements are equal,

$$d\theta_1 = d\theta_2 = d\theta \,. \qquad (20.A.25)$$

Therefore the displacement of the i^{th} object $d\vec{\mathbf{r}}_i$ differs from the displacement of the center of mass $d\vec{\mathbf{R}}_{\text{cm}}$ by a vector that corresponds to an infinitesimal rotation in the center of mass reference frame

$$d\vec{\mathbf{r}}_i = d\vec{\mathbf{R}}_{\text{cm}} + d\vec{\mathbf{r}}_{\text{cm},i} \,. \qquad (20.A.26)$$

We have shown that the displacement of a rigid body is the vector sum of the displacement of the center of mass (translation of the center of mass) and an infinitesimal rotation about the center of mass.

Chapter 21 Rigid Body Dynamics: Rotation and Translation about a Fixed Axis

Chapter 21 Rigid Body Dynamics: Rotation and Translation about a Fixed Axis

> *Accordingly, we find Euler and D'Alembert devoting their talent and their patience to the establishment of the laws of rotation of the solid bodies. Lagrange has incorporated his own analysis of the problem with his general treatment of mechanics, and since his time M. Poinsôt has brought the subject under the power of a more searching analysis than that of the calculus, in which ideas take the place of symbols, and intelligent propositions supersede equations.* [1]
>
> *James Clerk Maxwell*

21.1 Introduction

We shall analyze the motion of systems of particles and rigid bodies that are undergoing translational and rotational motion about a fixed direction. Because the body is translating, the axis of rotation is no longer fixed in space. We shall describe the motion by a translation of the center of mass and a rotation about the center of mass. By choosing a reference frame moving with the center of mass, we can analyze the rotational motion separately and discover that the torque about the center of mass is equal to the change in the angular momentum about the center of mass. For a rigid body undergoing fixed axis rotation about the center of mass, our rotational equation of motion is similar to one we have already encountered for fixed axis rotation, $\vec{\boldsymbol{\tau}}_{\text{cm}}^{\text{ext}} = d\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} / dt$.

21.2 Translational Equation of Motion

We shall think about the system of particles as follows. We treat the whole system as a single point-like particle of mass m_T located at the center of mass moving with the velocity of the center of mass $\vec{\mathbf{V}}_{cm}$. The external force acting on the system acts at the center of mass and from our earlier result (Eq. 10.4.9) we have that

$$\vec{\mathbf{F}}^{\text{ext}} = \frac{d\vec{\mathbf{p}}_{\text{sys}}}{dt} = \frac{d}{dt}(m_T \vec{\mathbf{V}}_{cm}). \tag{21.2.1}$$

21.3 Translational and Rotational Equations of Motion

For a system of particles, the torque about a point S can be written as

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[1] J. C. Maxwell on Louis Poinsôt (1777-1859) in 'On a Dynamical Top' (1857). In W. D. Niven (ed.), The Scientific Papers of James Clerk Maxwell (1890), Vol. 1, 248.

$$\vec{\boldsymbol{\tau}}_S^{\text{ext}} = \sum_{i=1}^{N} (\vec{\mathbf{r}}_i \times \vec{\mathbf{F}}_i) . \tag{21.3.1}$$

where we have assumed that all internal torques cancel in pairs. Let choose the point S to be the origin of the reference frame O, then $\vec{\mathbf{r}}_{S,\text{cm}} = \vec{\mathbf{R}}_{cm}$ (Figure 21.1). (You may want to recall the main properties of the center of mass reference frame by reviewing Chapter 15.2.1.)

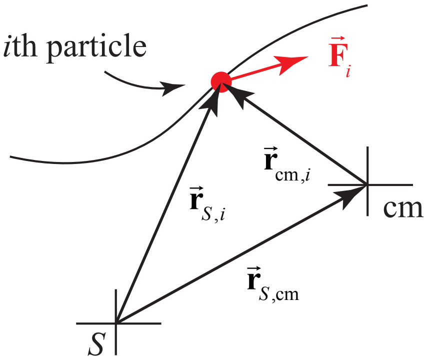


Figure 21.1 Torque diagram for center of mass reference frame

We can now apply $\vec{\mathbf{r}}_{S,i} = \vec{\mathbf{r}}_{S,\text{cm}} + \vec{\mathbf{r}}_{\text{cm},i}$ to Eq. (21.3.1) yielding

$$\vec{\boldsymbol{\tau}}_S^{\text{ext}} = \sum_{i=1}^{N} (\vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_i) = \sum_{i=1}^{N} ((\vec{\mathbf{r}}_{\text{cm},i} + \vec{\mathbf{r}}_{\text{cm},i}) \times \vec{\mathbf{F}}_i) = \sum_{i=1}^{N} (\vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{F}}_i) + \sum_{i=1}^{N} (\vec{\mathbf{r}}_{\text{cm},i} \times \vec{\mathbf{F}}_i) . \tag{21.3.2}$$

The term

$$\vec{\boldsymbol{\tau}}_{S,cm}^{\text{ext}} = \vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{F}}^{\text{ext}} \tag{21.3.3}$$

in Eq. (21.3.2) corresponds to the external torque about the point S where all the external forces act at the center of mass (Figure 21.2).

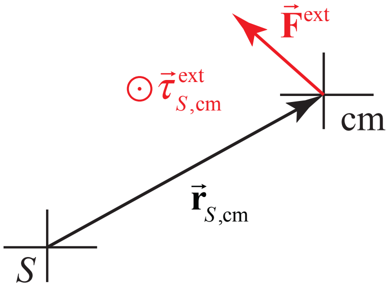


Figure 21.2 Torque diagram for "point-like" particle located at center of mass

The term,

$$\vec{\boldsymbol{\tau}}_{\text{cm}}^{\text{ext}} = \sum_{i=1}^{N} (\vec{\mathbf{r}}_{\text{cm},i} \times \vec{\mathbf{F}}_i) . \tag{21.3.4}$$

is the sum of the torques on the individual particles in the center of mass reference frame. If we assume that all internal torques cancel in pairs, then

$$\vec{\boldsymbol{\tau}}_{\text{cm}}^{\text{ext}} = \sum_{i=1}^{N}(\vec{\mathbf{r}}_{\text{cm},i} \times \vec{\mathbf{F}}_{i}^{\text{ext}}) . \qquad (21.3.5)$$

We conclude that the external torque about the point S can be decomposed into two pieces,

$$\vec{\boldsymbol{\tau}}_{S}^{\text{ext}} = \vec{\boldsymbol{\tau}}_{S,cm}^{\text{ext}} + \vec{\boldsymbol{\tau}}_{\text{cm}}^{\text{ext}} . \qquad (21.3.6)$$

We showed in Chapter 20.3 that

$$\vec{\mathbf{L}}_{S}^{\text{sys}} = \vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{p}}^{\text{sys}} + \sum_{i=1}^{N}(\vec{\mathbf{r}}_{cm,i} \times m_i \vec{\mathbf{v}}_{cm,i}) , \qquad (21.3.7)$$

where the first term in Eq. (21.3.7) is the orbital angular momentum of the center of mass about the point S

$$\vec{\mathbf{L}}_{S}^{\text{orbital}} = \vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{p}}^{\text{sys}} , \qquad (21.3.8)$$

and the second term in Eq. (21.3.7) is the spin angular momentum about the center of mass (independent of the point S)

$$\vec{\mathbf{L}}_{S}^{\text{spin}} = \sum_{i=1}^{N}(\vec{\mathbf{r}}_{cm,i} \times m_i \vec{\mathbf{v}}_{cm,i}) . \qquad (21.3.9)$$

The angular momentum about the point S can therefore be decomposed into two terms

$$\vec{\mathbf{L}}_{S}^{\text{sys}} = \vec{\mathbf{L}}_{S}^{\text{orbital}} + \vec{\mathbf{L}}_{S}^{\text{spin}} . \qquad (21.3.10)$$

Recall that that we have previously shown that it is always true that

$$\vec{\boldsymbol{\tau}}_{S}^{\text{ext}} = \frac{d\vec{\mathbf{L}}_{S}^{\text{sys}}}{dt} . \qquad (21.3.11)$$

Therefore we can therefore substitute Eq. (21.3.6) on the LHS of Eq. (21.3.11) and substitute Eq. (21.3.10) on the RHS of Eq. (21.3.11) yielding as

$$\vec{\boldsymbol{\tau}}_{S,cm}^{\text{ext}} + \vec{\boldsymbol{\tau}}_{\text{cm}}^{\text{ext}} = \frac{d\vec{\mathbf{L}}_{S}^{\text{orbital}}}{dt} + \frac{d\vec{\mathbf{L}}_{S}^{\text{spin}}}{dt} . \qquad (21.3.12)$$

We shall now show that Eq. (21.3.12) can also be decomposed in two separate conditions. We begin by analyzing the first term on the RHS of Eq. (21.3.12). We differentiate Eq. (21.3.8) and find that

$$\frac{d\vec{\mathbf{L}}_S^{\text{orbital}}}{dt} = \frac{d}{dt}(\vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{p}}^{\text{sys}}) . \tag{21.3.13}$$

We apply the vector identity

$$\frac{d}{dt}(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \frac{d\vec{\mathbf{A}}}{dt} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \frac{d\vec{\mathbf{B}}}{dt} , \tag{21.3.14}$$

to Eq. (21.3.13) yielding

$$\frac{d\vec{\mathbf{L}}_S^{\text{orbital}}}{dt} = \frac{d\vec{\mathbf{r}}_{S,\text{cm}}}{dt} \times \vec{\mathbf{p}}_{\text{sys}} + \vec{\mathbf{r}}_{S,\text{cm}} \times \frac{d\vec{\mathbf{p}}_{\text{sys}}}{dt} . \tag{21.3.15}$$

The first term in Eq. (21.3.21) is zero because

$$\frac{d\vec{\mathbf{r}}_{S,\text{cm}}}{dt} \times \vec{\mathbf{p}}_{\text{sys}} = \vec{\mathbf{V}}_{cm} \times m^{\text{total}} \vec{\mathbf{V}}_{cm} = \vec{\mathbf{0}} . \tag{21.3.16}$$

Therefore the time derivative of the orbital angular momentum about a point S, Eq. (21.3.15), becomes

$$\frac{d\vec{\mathbf{L}}_S^{\text{orbital}}}{dt} = \vec{\mathbf{r}}_{S,\text{cm}} \times \frac{d\vec{\mathbf{p}}_{\text{sys}}}{dt} . \tag{21.3.17}$$

In Eq. (21.3.17), the time derivative of the momentum of the system is the external force,

$$\vec{\mathbf{F}}^{\text{ext}} = \frac{d\vec{\mathbf{p}}_{\text{sys}}}{dt} . \tag{21.3.18}$$

The expression in Eq. (21.3.17) then becomes the first of our relations

$$\frac{d\vec{\mathbf{L}}_S^{\text{orbital}}}{dt} = \vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{F}}^{\text{ext}} = \vec{\boldsymbol{\tau}}_{S,cm}^{\text{ext}} . \tag{21.3.19}$$

Thus the time derivative of the orbital angular momentum about the point S is equal to the external torque about the point S where all the external forces act at the center of mass, (we treat the system as a point-like particle located at the center of mass).

We now consider the second term on the RHS of Eq. (21.3.12), the time derivative of the spin angular momentum about the center of mass. We differentiate Eq. (21.3.9),

$$\frac{d\vec{\mathbf{L}}_S^{\text{spin}}}{dt} = \frac{d}{dt}\sum_{i=1}^{N}(\vec{\mathbf{r}}_{cm,i} \times m_i \vec{\mathbf{v}}_{cm,i}) . \tag{21.3.20}$$

We again use the product rule for taking the time derivatives of a vector product (Eq. (21.3.14)). Then Eq. (21.3.20) the becomes

$$\frac{d\vec{\mathbf{L}}_S^{\text{spin}}}{dt} = \sum_{i=1}^{N}\left(\frac{d\vec{\mathbf{r}}_{cm,i}}{dt} \times m_i \vec{\mathbf{v}}_{cm,i}\right) + \sum_{i=1}^{N}\left(\vec{\mathbf{r}}_{cm,i} \times \frac{d}{dt}(m_i \vec{\mathbf{v}}_{cm,i})\right) . \tag{21.3.21}$$

The first term in Eq. (21.3.21) is zero because

$$\sum_{i=1}^{N}\left(\frac{d\vec{\mathbf{r}}_{cm,i}}{dt} \times m_i \vec{\mathbf{v}}_{cm,i}\right) = \sum_{i=1}^{N}(\vec{\mathbf{v}}_{cm,i} \times m_i \vec{\mathbf{v}}_{cm,i}) = \vec{\mathbf{0}} . \tag{21.3.22}$$

Therefore the time derivative of the spin angular momentum about the center of mass, Eq. (21.3.21), becomes

$$\frac{d\vec{\mathbf{L}}_S^{\text{spin}}}{dt} = \sum_{i=1}^{N}\left(\vec{\mathbf{r}}_{cm,i} \times \frac{d}{dt}(m_i \vec{\mathbf{v}}_{cm,i})\right) . \tag{21.3.23}$$

The force, acting on an element of mass m_i, is

$$\vec{\mathbf{F}}_i = \frac{d}{dt}(m_i \vec{\mathbf{v}}_{cm,i}) . \tag{21.3.24}$$

The expression in Eq. (21.3.23) then becomes

$$\frac{d\vec{\mathbf{L}}_S^{\text{spin}}}{dt} = \sum_{i=1}^{N}(\vec{\mathbf{r}}_{cm,i} \times \vec{\mathbf{F}}_i) . \tag{21.3.25}$$

The term, $\sum_{i=1}^{N}(\vec{\mathbf{r}}_{\text{cm},i} \times \vec{\mathbf{F}}_i)$, is the sum of the torques on the individual particles in the center of mass reference frame. If we again assume that all internal torques cancel in pairs, Eq. (21.3.25) may be expressed as

$$\frac{d\vec{\mathbf{L}}_S^{\text{spin}}}{dt} = \sum_{i=1}^{N}(\vec{\mathbf{r}}_{cm,i} \times \vec{\mathbf{F}}_i^{\text{ext}}) = \sum_{i=1}^{N}\vec{\boldsymbol{\tau}}_{\text{cm},i}^{\text{ext}} = \vec{\boldsymbol{\tau}}_{\text{cm}}^{\text{ext}} , \tag{21.3.26}$$

which is the second of our two relations.

21.3.1 Summary

For a system of particles, there are two conditions that always hold (Eqs. (21.3.19) and (21.3.26)) when we calculate the torque about a point S; we treat the system as a point-like particle located at the center of mass of the system. All the external forces $\vec{\mathbf{F}}^{\text{ext}}$ act at the center of mass. We calculate the orbital angular momentum of the center of mass and determine its time derivative and then apply

$$\vec{\boldsymbol{\tau}}^{\text{ext}}_{S,\text{cm}} = \vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{F}}^{\text{ext}} = \frac{d\vec{\mathbf{L}}^{\text{orbital}}_{S}}{dt} . \tag{21.3.27}$$

In addition, we calculate the torque about the center of mass due to all the forces acting on the particles in the center of mass reference frame. We calculate the time derivative of the angular momentum of the system with respect to the center of mass in the center of mass reference frame and then apply

$$\vec{\boldsymbol{\tau}}^{\text{ext}}_{\text{cm}} = \sum_{i=1}^{N} (\vec{\mathbf{r}}_{cm,i} \times \vec{\mathbf{F}}^{\text{ext}}_{i}) = \frac{d\vec{\mathbf{L}}^{\text{spin}}_{\text{cm}}}{dt} . \tag{21.3.28}$$

21.4 Translation and Rotation of a Rigid Body Undergoing Fixed Axis Rotation

For the special case of rigid body of mass m, we showed that with respect to a reference frame in which the center of mass of the rigid body is moving with velocity $\vec{\mathbf{V}}_{cm}$, all elements of the rigid body are rotating about the center of mass with the same angular velocity $\vec{\boldsymbol{\omega}}_{\text{cm}}$. For the rigid body of mass m and momentum $\vec{\mathbf{p}} = m\vec{\mathbf{V}}_{cm}$, the translational equation of motion is still given by Eq. (21.2.1), which we repeat in the form

$$\vec{\mathbf{F}}^{\text{ext}} = m\vec{\mathbf{A}}_{cm} . \tag{21.4.1}$$

For fixed axis rotation, choose the z-axis as the axis of rotation that passes through the center of mass of the rigid body. We have already seen in our discussion of angular momentum of a rigid body that the angular momentum does not necessary point in the same direction as the angular velocity. However we can take the z-component of Eq. (21.3.28)

$$\tau^{\text{ext}}_{cm,z} = \frac{dL^{\text{spin}}_{cm,z}}{dt} . \tag{21.4.2}$$

For a rigid body rotating about the center of mass with $\vec{\boldsymbol{\omega}}_{\text{cm}} = \omega_{\text{cm},z}\hat{\mathbf{k}}$, the z-component of angular momentum about the center of mass is

$$L^{\text{spin}}_{cm,z} = I_{\text{cm}}\omega_{\text{cm},z} . \tag{21.4.3}$$

The z-component of the rotational equation of motion about the center of mass is

$$\tau^{\text{ext}}_{cm,z} = I_{\text{cm}} \frac{d\omega_{\text{cm},z}}{dt} = I_{\text{cm}} \alpha_{\text{cm},z} . \tag{21.4.4}$$

21.5 Work-Energy Theorem

For a rigid body, we can also consider the work-energy theorem separately for the translational motion and the rotational motion. Once again treat the rigid body as a point-like particle moving with velocity $\vec{\mathbf{V}}_{cm}$ in reference frame O. We can use the same technique that we used when treating point particles to show that the work done by the external forces is equal to the change in kinetic energy

$$\begin{aligned} W^{\text{ext}}_{\text{trans}} &= \int_i^f \vec{\mathbf{F}}^{\text{ext}} \cdot d\vec{\mathbf{r}} = \int_i^f \frac{d(m\vec{\mathbf{V}}_{cm})}{dt} \cdot d\vec{\mathbf{R}}_{cm} = m\int_i^f \frac{d(\vec{\mathbf{V}}_{cm})}{dt} \cdot \vec{\mathbf{V}}_{cm}\, dt \\ &= \frac{1}{2} m \int_i^f d(\vec{\mathbf{V}}_{cm} \cdot \vec{\mathbf{V}}_{cm}) = \frac{1}{2} m V^2_{\text{cm},\,f} - \frac{1}{2} m V^2_{\text{cm},\,i} = \Delta K_{\text{trans}}. \end{aligned} \tag{21.5.1}$$

For the rotational motion we go to the center of mass reference frame and we determine the rotational work done i.e. the integral of the z-component of the torque about the center of mass with respect to $d\theta$ as we did for fixed axis rotational work. Then

$$\begin{aligned} &\int_i^f (\boldsymbol{\tau}^{\text{ext}}_{\text{cm}})_z \, d\theta = \int_i^f I_{\text{cm}} \frac{d\omega_{\text{cm},z}}{dt} d\theta = \int_i^f I_{\text{cm}}\, d\omega_{\text{cm},z} \frac{d\theta}{dt} = \int_i^f I_{\text{cm}}\, d\omega_{\text{cm},z} \omega_{\text{cm},z} \\ &= \frac{1}{2} I_{\text{cm}} \omega^2_{\text{cm},\,f} - \frac{1}{2} I_{\text{cm}} \omega^2_{\text{cm},\,i} = \Delta K_{\text{rot}} \end{aligned} . \tag{21.5.2}$$

In Eq. (21.5.2) we expressed our result in terms of the angular speed ω_{cm} because it appears as a square. Therefore we can combine these two separate results, Eqs. (21.5.1) and (21.5.2), and determine the work-energy theorem for a rotating and translating rigid body that undergoes fixed axis rotation about the center of mass.

$$\begin{aligned} W &= \left(\frac{1}{2} m V^2_{\text{cm,f}} + \frac{1}{2} I_{\text{cm}} \omega^2_{\text{cm},f} \right) - \left(\frac{1}{2} m V^2_{\text{cm,f}} + \frac{1}{2} I_{\text{cm}} \omega^2_{\text{cm},i} \right) \\ &= \Delta K_{\text{trans}} + \Delta K_{\text{rot}} = \Delta K. \end{aligned} \tag{21.5.3}$$

Equations (21.4.1), (21.4.4), and (21.5.3) are principles that we shall employ to analyze the motion of a rigid bodies undergoing translation and fixed axis rotation about the center of mass.

21.6 Worked Examples

Example 21.1 Angular Impulse

Two point-like objects are located at the points A and B, of respective masses $M_A = 2M$, and $M_B = M$, as shown in the figure below. The two objects are initially oriented along the y-axis and connected by a rod of negligible mass of length D, forming a rigid body. A force of magnitude $F = |\vec{\mathbf{F}}|$ along the x direction is applied to the object at B at $t = 0$ for a short time interval Δt, (Figure 21.3). Neglect gravity. Give all your answers in terms of M and D as needed. What is the magnitude of the angular velocity of the system after the collision?

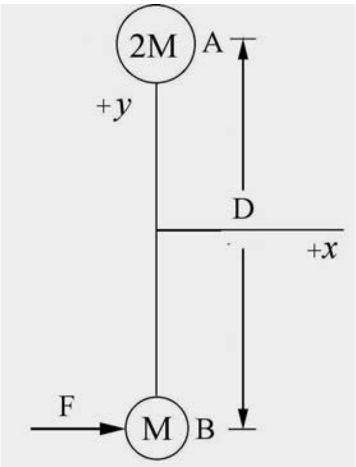
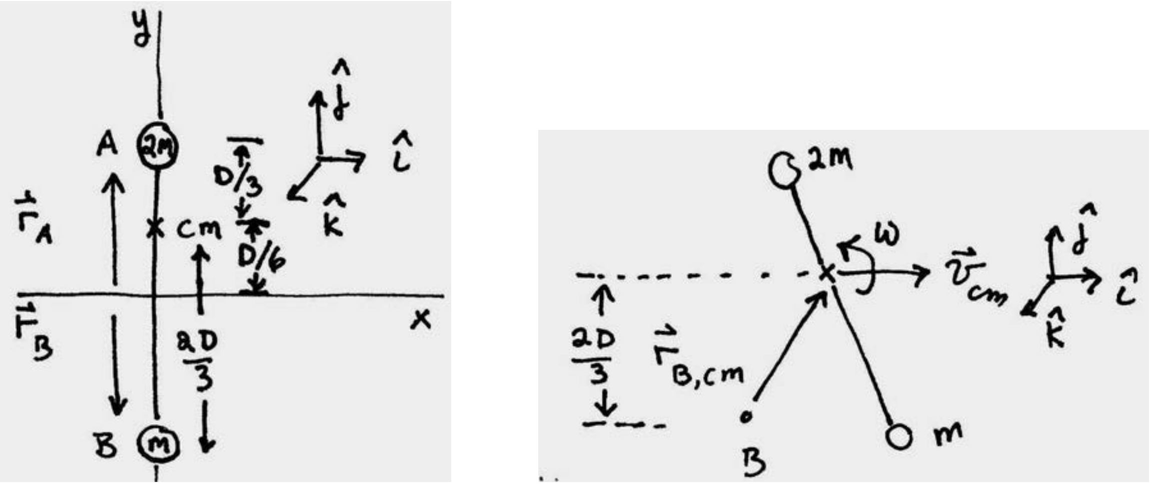


Figure 21.3 Example 21.1

Solutions: An impulse of magnitude $F\,\Delta t$ is applied in the $+x$ direction, and the center of mass of the system will move in this direction. The two masses will rotate about the center of mass, counterclockwise in the figure. Before the force is applied we can calculate the position of the center of mass (Figure 21.4a),

$$\vec{\mathbf{R}}_{\text{cm}} = \frac{M_A\vec{\mathbf{r}}_A + M_B\vec{\mathbf{r}}_B}{M_A + M_B} = \frac{2M(D/2)\hat{\mathbf{j}} + M(D/2)(-\hat{\mathbf{j}})}{3M} = (D/6)\hat{\mathbf{j}}\,. \qquad (21.6.1)$$

The center of mass is a distance $(2/3)\text{D}$ from the object at B and is a distance $(1/3)\text{D}$ from the object at A.



(a) (b)

Figure 21.4 (a) Center of mass of system, (b) Angular momentum about point B

Because $F\Delta t\hat{\mathbf{i}} = 3M\vec{\mathbf{V}}_{\text{cm}}$, the magnitude of the velocity of the center of mass is then $F\Delta t / 3\text{M}$ and the direction is in the positive $\hat{\mathbf{i}}$-direction. Because the force is applied at the point B, there is no torque about the point B, hence the angular momentum is constant about the point B. The initial angular momentum about the point B is zero. The angular momentum about the point B (Figure 21.4b) after the impulse is applied is the sum of two terms,

$$\begin{aligned} \vec{\mathbf{0}} &= \vec{\mathbf{L}}_{B,f} = \vec{\mathbf{r}}_{B,f} \times 3M\vec{\mathbf{V}}_{\text{cm}} + \vec{\mathbf{L}}_{\text{cm}} = (2D/3)\hat{\mathbf{j}} \times F\Delta t\,\hat{\mathbf{i}} + \vec{\mathbf{L}}_{\text{cm}} \\ \vec{\mathbf{0}} &= (2DF\Delta t/3)(-\hat{\mathbf{k}}) + \vec{\mathbf{L}}_{\text{cm}}. \end{aligned} \tag{21.6.2}$$

The angular momentum about the center of mass is given by

$$.\vec{\mathbf{L}}_{cm} = I_{cm}\omega\,\hat{\mathbf{k}} = (2M(D/3)^2 + M(2D/3)^2)\omega\,\hat{\mathbf{k}} = (2/3)MD^2\omega\,\hat{\mathbf{k}}. \tag{21.6.3}$$

Thus the angular about the point B after the impulse is applied is

$$\vec{\mathbf{0}} = (2DF\Delta t/3)(-\hat{\mathbf{k}}) + (2/3)MD^2\omega\,\hat{\mathbf{k}} \tag{21.6.4}$$

We can solve this Eq. (21.6.4) for the angular speed

$$\omega = \frac{F\Delta t}{MD}\ . \tag{21.6.5}$$

Example 21.2 Person on a railroad car moving in a circle

A person of mass M is standing on a railroad car, which is rounding an unbanked turn of radius R at a speed v. His center of mass is at a height of L above the car midway between his feet, which are separated by a distance of d. The man is facing the direction of motion (Figure 21.5). What is the magnitude of the normal force on each foot?

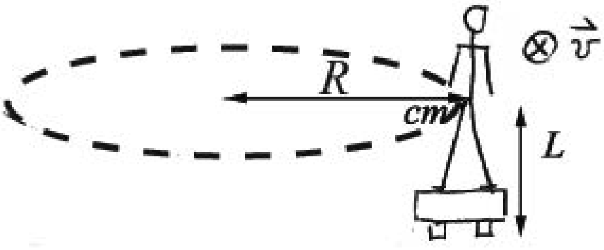


Figure 21.5 Example 21.2

Solution: We begin by choosing a cylindrical coordinate system and drawing a free-body force diagram, shown in Figure 21.6.

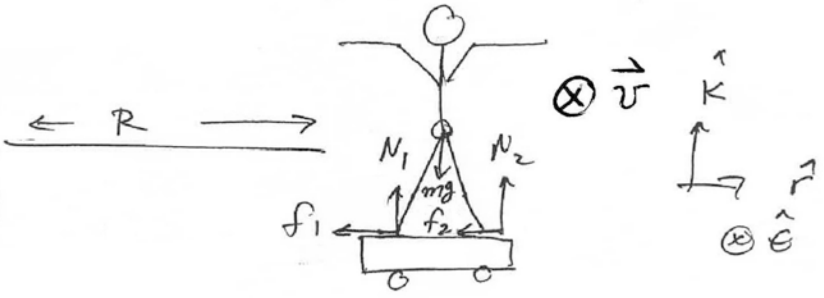


Figure 21.6 Coordinate system for Example 21.2

We decompose the contact force between the inner foot closer to the center of the circular motion and the ground into a tangential component corresponding to static friction $\vec{\mathbf{f}}_1$ and a perpendicular component, $\vec{\mathbf{N}}_1$. In a similar fashion we decompose the contact force between the outer foot further from the center of the circular motion and the ground into a tangential component corresponding to static friction $\vec{\mathbf{f}}_2$ and a perpendicular component, $\vec{\mathbf{N}}_2$. We do not assume that the static friction has its maximum magnitude nor do we assume that $\vec{\mathbf{f}}_1 = \vec{\mathbf{f}}_2$ or $\vec{\mathbf{N}}_1 = \vec{\mathbf{N}}_2$. The gravitational force acts at the center of mass.

We shall use our two dynamical equations of motion, Eq. (21.4.1) for translational motion and Eq. (21.4.4) for rotational motion about the center of mass noting that we are considering the special case that $\vec{\boldsymbol{\alpha}}_{cm} = 0$ because the object is not rotating about the center of mass. In order to apply Eq. (21.4.1), we treat the person as a point-like particle located at the center of mass and all the external forces act at this point. The radial component of Newton's Second Law (Eq. (21.4.1) is given by

$$\hat{\mathbf{r}} : -f_1 - f_2 = -m\frac{v^2}{R}. \qquad (21.6.6)$$

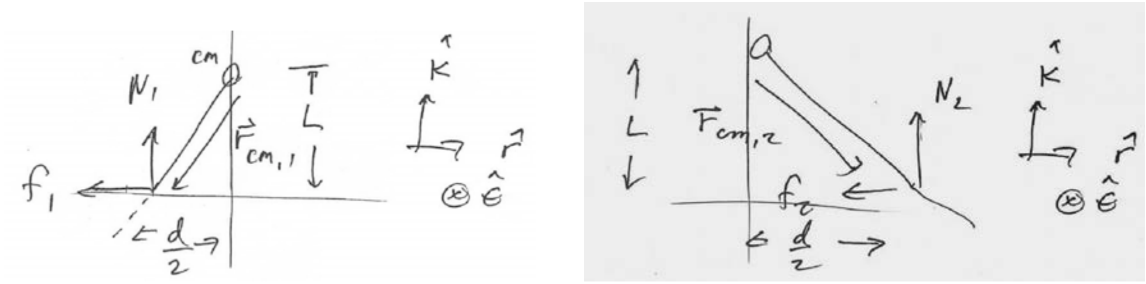
The vertical component of Newton's Second Law is given by

$$\hat{\mathbf{k}} : N_1 + N_2 - mg = 0 . \tag{21.6.7}$$

The rotational equation of motion (Eq. (21.4.4) is

$$\vec{\boldsymbol{\tau}}_{\text{cm}}^{\text{total}} = 0 . \tag{21.6.8}$$

We begin our calculation of the torques about the center of mass by noting that the gravitational force does not contribute to the torque because it is acting at the center of mass. We draw a torque diagram in Figure 21.7a showing the location of the point of application of the forces, the point we are computing the torque about (which in this case is the center of mass), and the vector $\vec{\mathbf{r}}_{\text{cm},1}$ from the point we are computing the torque about to the point of application of the forces.



(a) (b)

Figure 21.7 Torque diagram for (a) inner foot, (b) outer foot

The torque on the inner foot is given by

$$\vec{\boldsymbol{\tau}}_{\text{cm},1} = \vec{\mathbf{r}}_{\text{cm},1} \times (\vec{\mathbf{f}}_1 + \vec{\mathbf{N}}_1) = \left(-\frac{d}{2}\hat{\mathbf{r}} - L\hat{\mathbf{k}} \right) \times (-f_1\hat{\mathbf{r}} + N_1\hat{\mathbf{k}}) = \left(\frac{d}{2}N_1 + Lf_1 \right)\hat{\boldsymbol{\theta}} . \tag{21.6.9}$$

We draw a similar torque diagram (Figure 21.7b) for the forces applied to the outer foot. The torque on the outer foot is given by

$$\vec{\boldsymbol{\tau}}_{\text{cm},2} = \vec{\mathbf{r}}_{\text{cm},2} \times (\vec{\mathbf{f}}_2 + \vec{\mathbf{N}}_2) = \left(+\frac{d}{2}\hat{\mathbf{r}} - L\hat{\mathbf{k}} \right) \times (-f_2\hat{\mathbf{r}} + N_2\hat{\mathbf{k}}) = \left(-\frac{d}{2}N_2 + Lf_2 \right)\hat{\boldsymbol{\theta}} . \tag{21.6.10}$$

Notice that the forces $\vec{\mathbf{f}}_1$, $\vec{\mathbf{N}}_1$, and $\vec{\mathbf{f}}_2$ all contribute torques about the center of mass in the positive $\hat{\boldsymbol{\theta}}$-direction while $\vec{\mathbf{N}}_2$ contributes a torque about the center of mass in the negative $\hat{\boldsymbol{\theta}}$-direction. According to Eq. (21.6.8) the sum of these torques about the center of mass must be zero. Therefore

$$\begin{aligned}\vec{\boldsymbol{\tau}}_{\text{cm}}^{\text{ext}} = \vec{\boldsymbol{\tau}}_{\text{cm},1} + \vec{\boldsymbol{\tau}}_{\text{cm},2} &= \left(\frac{d}{2}N_1 + Lf_1\right)\hat{\boldsymbol{\theta}} + \left(-\frac{d}{2}N_2 + Lf_2\right)\hat{\boldsymbol{\theta}} \\ &= \left(\frac{d}{2}(N_1 - N_2) + L(f_1 + f_2)\right)\hat{\boldsymbol{\theta}} = \vec{\mathbf{0}}.\end{aligned} \tag{21.6.11}$$

Notice that the magnitudes of the two frictional forces appear together as a sum in Eqs. (21.6.11) and (21.6.6). We now can solve Eq. (21.6.6) for $f_1 + f_2$ and substitute the result into Eq. (21.6.11) yielding the condition that

$$\frac{d}{2}(N_1 - N_2) + Lm\frac{v^2}{R} = 0\,. \tag{21.6.12}$$

We can rewrite this Eq. as

$$N_2 - N_1 = \frac{2Lmv^2}{Rd}\,. \tag{21.6.13}$$

We also rewrite Eq. (21.6.7) in the form

$$N_2 + N_1 = mg\,. \tag{21.6.14}$$

We now can solve for N_2 by adding together Eqs. (21.6.13) and (21.6.14), and then divide by two,

$$N_2 = \frac{1}{2}\left(Mg + \frac{2Lmv^2}{Rd}\right). \tag{21.6.15}$$

We now can solve for N_1 by subtracting Eqs. (21.6.13) from (21.6.14), and then divide by two,

$$N_1 = \frac{1}{2}\left(mg - \frac{2Lmv^2}{Rd}\right). \tag{21.6.16}$$

Check the result: we see that the normal force acting on the outer foot is greater in magnitude than the normal force acting on the inner foot. We expect this result because as we increase the speed v, we find that at a maximum speed v_{max}, the normal force on the inner foot goes to zero and we start to rotate in the positive $\hat{\boldsymbol{\theta}}$-direction, tipping outward. We can find this maximum speed by setting $N_1 = 0$ in Eq. (21.6.16) resulting in

$$v_{\text{max}} = \sqrt{\frac{gRd}{2L}}\,. \tag{21.6.17}$$

Example 21.3 Torque, Rotation and Translation: Yo-Yo

A Yo-Yo of mass m has an axle of radius b and a spool of radius R. Its moment of inertia about the center can be taken to be $I_{cm} = (1/2)mR^2$ and the thickness of the string can be neglected (Figure 21.8). The Yo-Yo is released from rest. You will need to assume that the center of mass of the Yo-Yo descends vertically, and that the string is vertical as it unwinds. (a) What is the tension in the cord as the Yo-Yo descends? (b) What is the magnitude of the angular acceleration as the yo-yo descends and the magnitude of the linear acceleration? (c) Find the magnitude of the angular velocity of the Yo-Yo when it reaches the bottom of the string, when a length l of the string has unwound.

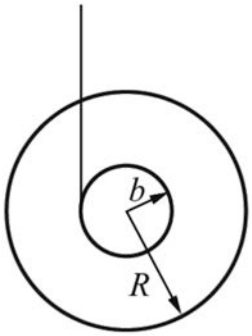


Figure 21.8 Example 21.3

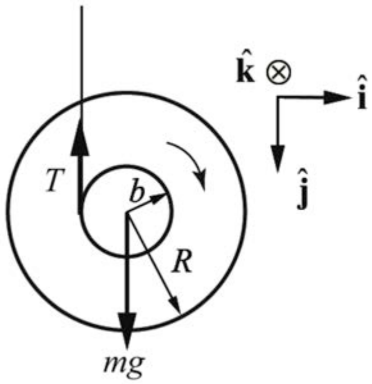


Figure 21.9 Torque diagram for Yo-Yo

Solutions: a) as the Yo-Yo descends it rotates clockwise in Figure 21.9. The torque about the center of mass of the Yo-Yo is due to the tension and increases the magnitude of the angular velocity. The direction of the torque is into the page in Figure 21.9 (positive z-direction). Use the right-hand rule to check this, or use the vector product definition of torque,

$$\vec{\tau}_{\text{cm}} = \vec{\mathbf{r}}_{\text{cm},T} \times \vec{\mathbf{T}}. \tag{21.6.18}$$

About the center of mass, $\vec{\mathbf{r}}_{\text{cm},T} = -b\,\hat{\mathbf{i}}$ and $\vec{\mathbf{T}} = -T\,\hat{\mathbf{j}}$, so the torque is

$$\vec{\tau}_{\text{cm}} = \vec{\mathbf{r}}_{\text{cm},T} \times \vec{\mathbf{T}} = (-b\,\hat{\mathbf{i}}) \times (-T\,\hat{\mathbf{j}}) = bT\,\hat{\mathbf{k}}. \tag{21.6.19}$$

Apply Newton's Second Law in the $\hat{\mathbf{j}}$-direction,

$$mg - T = ma_y. \tag{21.6.20}$$

Apply the rotational equation of motion for the Yo-Yo,

$$bT = I_{\mathrm{cm}}\alpha_z , \tag{21.6.21}$$

where α_z is the z-component of the angular acceleration. The z-component of the angular acceleration and the y-component of the linear acceleration are related by the constraint condition

$$a_y = b\alpha_z , \tag{21.6.22}$$

where b is the axle radius of the Yo-Yo. Substitute Eq. (21.6.22) into (21.6.20) yielding

$$mg - T = mb\alpha_z . \tag{21.6.23}$$

Now solve Eq. (21.6.21) for α_z and substitute the result into Eq.(21.6.23),

$$mg - T = \frac{mb^2 T}{I_{\mathrm{cm}}} . \tag{21.6.24}$$

Solve Eq. (21.6.24) for the tension T,

$$T = \frac{mg}{\left(1 + \dfrac{mb^2}{I_{\mathrm{cm}}}\right)} = \frac{mg}{\left(1 + \dfrac{mb^2}{(1/2)mR^2}\right)} = \frac{mg}{\left(1 + \dfrac{2b^2}{R^2}\right)} . \tag{21.6.25}$$

b) Substitute Eq. (21.6.25) into Eq. (21.6.21) to determine the z-component of the angular acceleration,

$$\alpha_z = \frac{bT}{I_{\mathrm{cm}}} = \frac{2bg}{(R^2 + 2b^2)} . \tag{21.6.26}$$

Using the constraint condition Eq. (21.6.22), we determine the y-component of linear acceleration

$$a_y = b\alpha_z = \frac{2b^2 g}{(R^2 + 2b^2)} = \frac{g}{1 + R^2 / 2b^2} . \tag{21.6.27}$$

Note that both quantities $a_z > 0$ and $\alpha_z > 0$, so Eqs. (21.6.26) and (21.6.27) are the magnitudes of the respective quantities. For a typical Yo-Yo, the acceleration is much less than that of an object in free fall.

c) Use conservation of energy to determine the magnitude of the angular velocity of the Yo-Yo when it reaches the bottom of the string. As in Figure 21.9, choose the downward vertical direction as the positive $\hat{\mathbf{j}}$-direction and let $y = 0$ designate the location of the center of mass of the Yo-Yo when the string is completely wound. Choose $U(y=0)=0$ for the zero reference potential energy. This choice of direction and reference means that the gravitational potential energy will be negative and decreasing while the Yo-Yo descends. For this case, the gravitational potential energy is

$$U=-mg\,y. \tag{21.6.28}$$

The Yo-Yo is not yet moving downward or rotating, and the center of mass is located at $y=0$ so the mechanical energy in the initial state, when the Yo-Yo is completely wound, is zero

$$E_i=U(y=0)=0. \tag{21.6.29}$$

Denote the linear speed of the Yo-Yo as v_f and its angular speed as ω_f (at the point $y=l$). The constraint condition between v_f and ω_f is given by

$$v_f=b\omega_f, \tag{21.6.30}$$

consistent with Eq. (21.6.22). The kinetic energy is the sum of translational and rotational kinetic energy, where we have used $I_{\text{cm}}=(1/2)mR^2$, and so mechanical energy in the final state, when the Yo-Yo is completely unwound, is

$$\begin{aligned}E_f=K_f+U_f&=\frac{1}{2}mv_f^2+\frac{1}{2}I_{\text{cm}}\omega_f^2-mgl\\&=\frac{1}{2}mb^2\omega_f^2+\frac{1}{4}mR^2\omega_f^2-mgl.\end{aligned} \tag{21.6.31}$$

There are no external forces doing work on the system (neglect air resistance), so

$$0=E_f=E_i. \tag{21.6.32}$$

Thus

$$\left(\frac{1}{2}mb^2+\frac{1}{4}mR^2\right)\omega_f^2=mgl. \tag{21.6.33}$$

Solving for ω_f,

$$\omega_f=\sqrt{\frac{4gl}{(2b^2+R^2)}}. \tag{21.6.34}$$

We may also use kinematics to determine the final angular velocity by solving for the time interval Δt that it takes for the Yo-Yo to travel a distance l at the constant acceleration found in Eq. (21.6.27)),

$$\Delta t = \sqrt{2l / a_y} = \sqrt{\frac{l(R^2 + 2b^2)}{b^2 g}} \qquad (21.6.35)$$

The final angular velocity of the Yo-Yo is then (using Eq. (21.6.26) for the z-component of the angular acceleration),

$$\omega_f = \alpha_z \, \Delta t = \sqrt{\frac{4gl}{(R^2 + 2b^2)}}\,, \qquad (21.6.36)$$

in agreement with Eq. (21.6.34).

Example 21.4 Cylinder Rolling Down Inclined Plane

A uniform cylinder of outer radius R and mass M with moment of inertia about the center of mass, $I_{\text{cm}} = (1/2)M\,R^2$, starts from rest and rolls without slipping down an incline tilted at an angle β from the horizontal. The center of mass of the cylinder has dropped a vertical distance h when it reaches the bottom of the incline Figure 21.10. Let g denote the gravitational constant. The coefficient of static friction between the cylinder and the surface is μ_{s}. What is the magnitude of the velocity of the center of mass of the cylinder when it reaches the bottom of the incline?

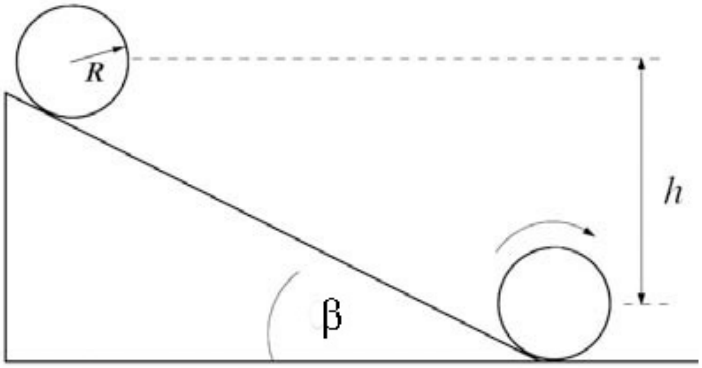


Figure 21.10 Example 21.4

Solution: We shall solve this problem three different ways.

1. Apply the torque condition about the center of mass and the force law for the center of mass motion.
2. Apply the energy methods.
3. Use torque about a fixed point that lies along the line of contact between the cylinder and the surface,

First Approach: Rotation about center of mass and translation of center of mass

We shall apply the torque condition (Eq. (21.4.4)) about the center of mass and the force law (Eq. (21.4.1)) for the center of mass motion. We will first find the acceleration and hence the speed at the bottom of the incline using kinematics. The forces are shown in Figure 21.11.

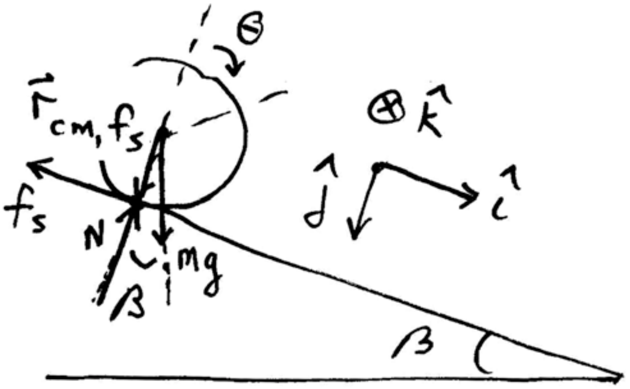


Figure 21.11 Torque diagram about center of mass

Choose $x = 0$ at the point where the cylinder just starts to roll. Newton's Second Law, applied in the x- and y-directions in turn, yields

$$Mg\sin\beta - f_s = Ma_x , \tag{21.6.37}$$

$$-N + Mg\cos\beta = 0 . \tag{21.6.38}$$

Choose the center of the cylinder to compute the torque about (Figure 21.10). Then, the only force exerting a torque about the center of mass is the friction force, therefore the rotational equation of motion is

$$f_s\, R = I_{\text{cm}}\alpha_z . \tag{21.6.39}$$

Use $I_{\text{cm}} = (1/2)M\,R^2$ and the kinematic constraint for the no-slipping condition $\alpha_z = a_x / R$ in Eq. (21.6.39) to solve for the magnitude of the static friction force yielding

$$f_s = (1/2)Ma_x \; . \tag{21.6.40}$$

Substituting Eq. (21.6.40) into Eq. (21.6.37) yields

$$Mg\sin\theta - (1/2)Ma_x = Ma_x \; , \tag{21.6.41}$$

which we can solve for the acceleration

$$a_x = \frac{2}{3}g\sin\beta . \tag{21.6.42}$$

In the time t_f it takes to reach the bottom, the displacement of the cylinder is $x_f = h / \sin\beta$. The x-component of the velocity v_x at the bottom is $v_{x,f} = a_x t_f$. Thus $x_f = (1/2) a_x t_f^{\,2}$. After eliminating t_f, we have $x_f = v_{x,f}^{\,2} / 2a_x$, so the x-component of the velocity when the cylinder reaches the bottom of the inclined plane is

$$v_{x,f} = \sqrt{2a_x x_f} = \sqrt{2((2/3)g\sin\beta)(h/\sin\beta)} = \sqrt{(4/3)gh}\,. \qquad (21.6.43)$$

Note that if we substitute Eq. (21.6.42) into Eq. (21.6.40) the magnitude of the frictional force is

$$f_s = (1/3)Mg\sin\beta\,. \qquad (21.6.44)$$

In order for the cylinder to roll without slipping

$$f_s \le \mu_s Mg\cos\beta\,. \qquad (21.6.45)$$

Combining Eq. (21.6.44) and Eq. (21.6.45) we have the condition that

$$(1/3)Mg\sin\beta \le \mu_s Mg\cos\beta \qquad (21.6.46)$$

Thus in order to roll without slipping, the coefficient of static friction must satisfy

$$\mu_s \ge \frac{1}{3}\tan\beta\,. \qquad (21.6.47)$$

Second Approach: Energy Methods

We shall use the fact that the energy of the cylinder-earth system is constant since the static friction force does no work.

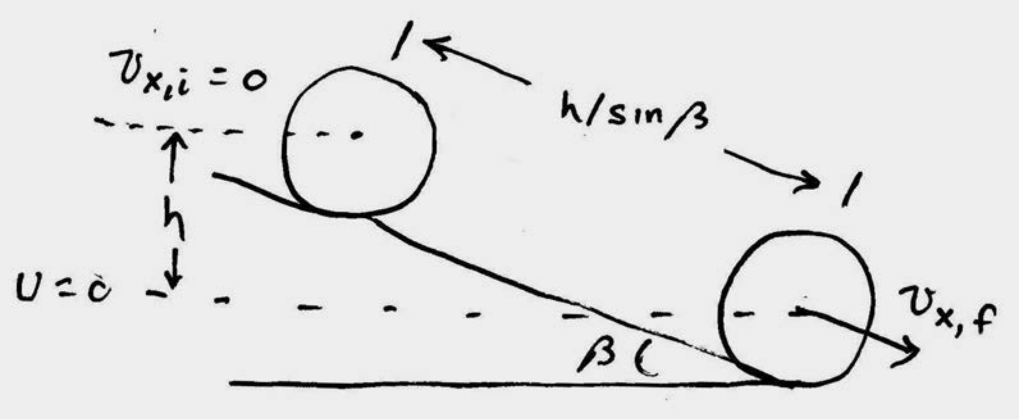


Figure 21.12 Energy diagram for cylinder

Choose a zero reference point for potential energy at the center of mass when the cylinder reaches the bottom of the incline plane (Figure 21.12). Then the initial potential energy is

$$U_i = Mgh. \tag{21.6.48}$$

For the given moment of inertia, the final kinetic energy is

$$\begin{aligned}
K_\text{f} &= \frac{1}{2} M v_{x,f}{}^2 + \frac{1}{2} I_\text{cm} \omega_{z,f}{}^2 \\
&= \frac{1}{2} M v_{x,f}{}^2 + \frac{1}{2}(1/2)MR^2(v_{x,f}/R)^2 \\
&= \frac{3}{4} M v_{x,f}{}^2 .
\end{aligned} \tag{21.6.49}$$

Setting the final kinetic energy equal to the initial gravitational potential energy leads to

$$Mgh = \frac{3}{4} M v_{x,f}{}^2 . \tag{21.6.50}$$

The magnitude of the velocity of the center of mass of the cylinder when it reaches the bottom of the incline is

$$v_{x,f} = \sqrt{(4/3)gh}, \tag{21.6.51}$$

in agreement with Eq. (21.6.43).

Third Approach: Torque about a fixed point that lies along the line of contact between the cylinder and the surface

Choose a fixed point P that lies along the line of contact between the cylinder and the surface. Then the torque diagram is shown in Figure 21.13.

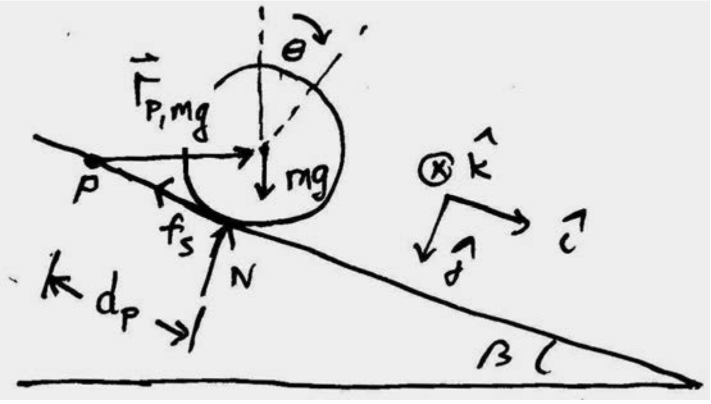


Figure 21.13 Torque about a point along the line of contact

The gravitational force $M\vec{\mathbf{g}} = Mg\sin\beta\,\hat{\mathbf{i}} + Mg\cos\beta\,\hat{\mathbf{j}}$ acts at the center of mass. The vector from the point P to the center of mass is given by $\vec{\mathbf{r}}_{P,mg} = d_P\,\hat{\mathbf{i}} - R\,\hat{\mathbf{j}}$, so the torque due to the gravitational force about the point P is given by

$$\begin{aligned}\vec{\boldsymbol{\tau}}_{P,Mg} &= \vec{\mathbf{r}}_{P,Mg} \times M\vec{\mathbf{g}} = (d_P\,\hat{\mathbf{i}} - R\,\hat{\mathbf{j}}) \times (Mg\sin\beta\,\hat{\mathbf{i}} + Mg\cos\beta\,\hat{\mathbf{j}}) \\ &= (d_P Mg\cos\beta + RMg\sin\beta)\hat{\mathbf{k}}.\end{aligned} \qquad (21.6.52)$$

The normal force acts at the point of contact between the cylinder and the surface and is given by $\vec{\mathbf{N}} = -N\,\hat{\mathbf{j}}$. The vector from the point P to the point of contact between the cylinder and the surface is $\vec{\mathbf{r}}_{P,N} = d_P\,\hat{\mathbf{i}}$. Therefore the torque due to the normal force about the point P is given by

$$\vec{\boldsymbol{\tau}}_{P,N} = \vec{\mathbf{r}}_{P,N} \times \vec{\mathbf{N}} = (d_P\,\hat{\mathbf{i}}) \times (-N\,\hat{\mathbf{j}}) = -d_P N\,\hat{\mathbf{k}}\,. \qquad (21.6.53)$$

Substituting Eq. (21.6.38) for the normal force into Eq. (21.6.53) yields

$$\vec{\boldsymbol{\tau}}_{P,N} = -d_P Mg\cos\beta\hat{\mathbf{k}}\,. \qquad (21.6.54)$$

Therefore the sum of the torques about the point P is

$$\vec{\boldsymbol{\tau}}_P = \vec{\boldsymbol{\tau}}_{P,Mg} + \vec{\boldsymbol{\tau}}_{P,N} = (d_P Mg\cos\beta + RMg\sin\beta)\hat{\mathbf{k}} - d_P Mg\cos\beta\hat{\mathbf{k}} = Rmg\sin\beta\hat{\mathbf{k}}\,. \qquad (21.6.55)$$

The angular momentum about the point P is given by

$$\begin{aligned}\vec{\mathbf{L}}_P &= \vec{\mathbf{L}}_{\text{cm}} + \vec{\mathbf{r}}_{P,\text{cm}} \times M\vec{\mathbf{V}}_{cm} = I_{\text{cm}}\omega_z\hat{\mathbf{k}} + (d_P\,\hat{\mathbf{i}} - R\,\hat{\mathbf{j}}) \times (Mv_x)\,\hat{\mathbf{i}} \\ &= (I_{\text{cm}}\omega_z + RMv_x)\,\hat{\mathbf{k}}\end{aligned}\,. \qquad (21.6.56)$$

The time derivative of the angular momentum about the point P is then

$$\frac{d\vec{\mathbf{L}}_P}{dt} = (I_{\text{cm}}\alpha_z + RMa_x)\,\hat{\mathbf{k}}\,. \qquad (21.6.57)$$

Therefore the torque law about the point P, becomes

$$RMg\sin\beta\hat{\mathbf{k}} = (I_{\text{cm}}\alpha_z + RMa_x)\hat{\mathbf{k}}\;\;. \qquad (21.6.58)$$

Using the fact that $I_{\text{cm}} = (1/2)MR^2$ and $\alpha_x = a_x / R$, the z-component of Eq. (21.6.58) is then

$$RMg\sin\beta = (1/2)MRa_x + Rma_x = (3/2)MRa_x\,. \qquad (21.6.59)$$

We can now solve Eq. (21.6.59) for the x-component of the acceleration

$$a_x = (2/3)g\sin\beta\,, \qquad (21.6.60)$$

in agreement with Eq. (21.6.42).

Example 21.5 Bowling Ball

A bowling ball of mass m and radius R is initially thrown down an alley with an initial speed v_i, and it slides without rolling but due to friction it begins to roll (Figure 21.14). The moment of inertia of the ball about its center of mass is $I_{\text{cm}} = (2/5)mR^2$. Using conservation of angular momentum about a point (you need to find that point), find the speed v_f and the angular speed ω_f of the bowling ball when it just starts to roll without slipping?

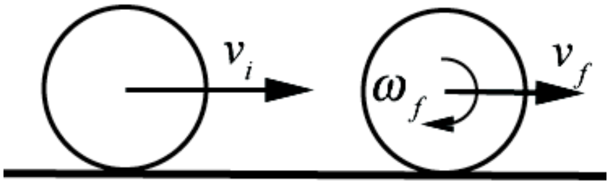


Figure 21.14 Example 21.5

Solution: We begin introducing coordinates for the angular and linear motion. Choose an angular coordinate θ increasing in the clockwise direction. Choose the positive $\hat{\mathbf{k}}$ unit vector pointing into the page in Figure 21.15.

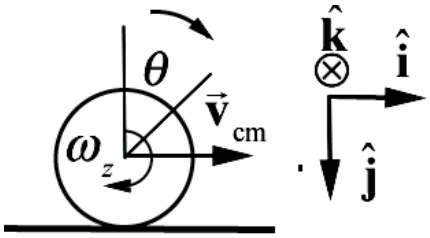


Figure 21.15 Coordinate system for ball

Then the angular velocity vector is $\vec{\boldsymbol{\omega}} = \omega_z\hat{\mathbf{k}} = d\theta/dt\,\hat{\mathbf{k}}$, and the angular acceleration vector is $\vec{\boldsymbol{\alpha}} = \alpha_z\hat{\mathbf{k}} = d^2\theta/dt^2\,\hat{\mathbf{k}}$. Choose the positive $\hat{\mathbf{i}}$ unit vector pointing to the right in Figure 21.15. Then the velocity of the center of mass is given by $\vec{\mathbf{v}}_{\text{cm}} = v_{\text{cm},x}\hat{\mathbf{i}} = dx_{\text{cm}}/dt\,\hat{\mathbf{i}}$, and the acceleration of the center of mass is given by $\vec{\mathbf{a}}_{\text{cm}} = a_{\text{cm},x}\hat{\mathbf{i}} = d^2x_{\text{cm}} / dt^2\,\hat{\mathbf{i}}$. The free-body force diagram is shown in Figure 21.16.

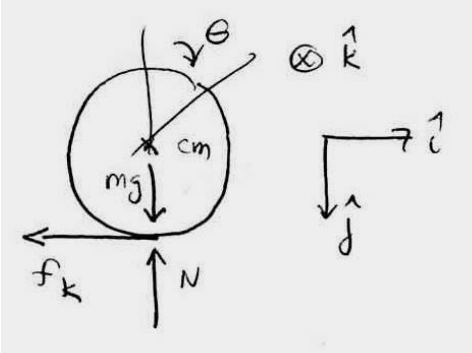


Figure 21.16 Free-body force diagram for ball

At $t = 0$, when the ball is released, $\vec{\mathbf{v}}_{cm,0} = v_0\hat{\mathbf{i}}$ and $\vec{\boldsymbol{\omega}}_0 = \vec{\mathbf{0}}$, so the ball is skidding and hence the frictional force on the ball due to the sliding of the ball on the surface is kinetic friction, hence acts in the negative $\hat{\mathbf{i}}$-direction. Because there is kinetic friction and non-conservative work, mechanical energy is not constant. The rotational equation of motion is $\vec{\boldsymbol{\tau}}_S = d\vec{\mathbf{L}}_S / dt$. In order for angular momentum about some point to remain constant throughout the motion, the torque about that point must also be zero throughout the motion. As the ball moves down the alley, the contact point will move, but the frictional force will always be directed along the line of contact between the bowling bowl and the surface. Choose any fixed point S along the line of contact then

$$\vec{\boldsymbol{\tau}}_{S,f_k} = \vec{\mathbf{r}}_{S,f_k} \times \vec{\mathbf{f}}_k = \vec{\mathbf{0}} \tag{21.6.61}$$

because $\vec{\mathbf{r}}_{S,f_k}$ and $\vec{\mathbf{f}}_k$ are anti-parallel. The gravitation force acts at the center of mass hence the torque due to gravity about S is

$$\vec{\boldsymbol{\tau}}_{S,mg} = \vec{\mathbf{r}}_{S,mg} \times m\vec{\mathbf{g}} = dmg\hat{\mathbf{k}}\,, \tag{21.6.62}$$

where d is the distance from S to the contact point between the ball and the ground. The torque due to the normal force about S is

$$\vec{\boldsymbol{\tau}}_{S,N} = \vec{\mathbf{r}}_{S,N} \times m\vec{\mathbf{g}} = -dN\hat{\mathbf{k}}\,, \tag{21.6.63}$$

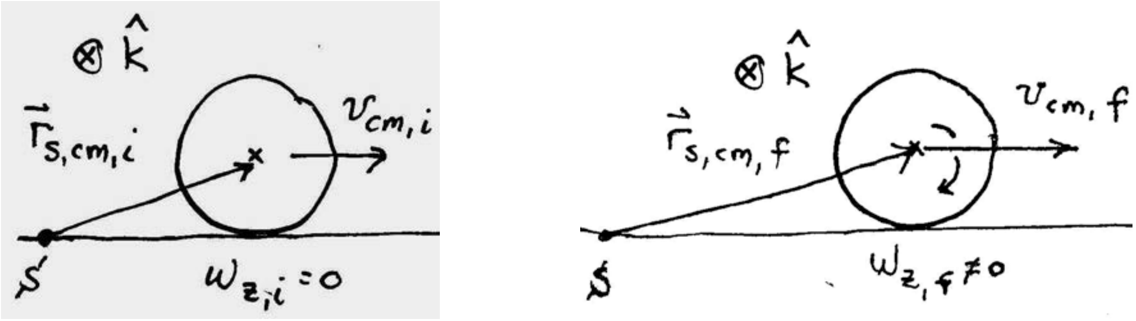
with the same moment arm d. Because the ball is not accelerating in the $\hat{\mathbf{j}}$-direction, from Newton's Second Law, we note that $mg - N = 0$. Therefore

$$\vec{\boldsymbol{\tau}}_{S,N} + \vec{\boldsymbol{\tau}}_{S,mg} = d(mg - N)\hat{\mathbf{k}} = \vec{\mathbf{0}}\,. \tag{21.6.64}$$

There is no torque about any fixed point S along the line of contact between the bowling bowl and the surface; therefore the angular momentum about that point S is constant,

$$\vec{\mathbf{L}}_{S,i} = \vec{\mathbf{L}}_{S,f} \,. \tag{21.6.65}$$

Choose one fixed point S along the line of contact (Figure 21.17).



(a) (b)

Figure 21.17 Angular momentum about S: (a) initial, (b) final

The initial angular momentum about S is only due to the translation of the center of mass (Figure 21.17a),

$$\vec{\mathbf{L}}_{S,i} = \vec{\mathbf{r}}_{S,\text{cm},i} \times m\vec{\mathbf{v}}_{\text{cm},i} = m R v_{\text{cm},i}\hat{\mathbf{k}} \,. \tag{21.6.66}$$

In Figure 21.17b, the ball is rolling without slipping. The final angular momentum about S has both a translational and rotational contribution

$$\vec{\mathbf{L}}_{S,f} = \vec{\mathbf{r}}_{S,\text{cm},f} \times m\vec{\mathbf{v}}_{\text{cm},f} + I_{cm}\vec{\boldsymbol{\omega}}_{f} = m R v_{\text{cm},f}\hat{\mathbf{k}} + I_{\text{cm}}\omega_{z,f}\hat{\mathbf{k}} \,. \tag{21.6.67}$$

When the ball is rolling without slipping, $v_{\text{cm},f} = R\omega_{z,f}$ and also $I_{\text{cm}} = (2/5)mR^2$. Therefore the final angular momentum about S is

$$\vec{\mathbf{L}}_{S,f} = (mR + (2/5)mR)v_{\text{cm},f}\hat{\mathbf{k}} = (7/5)mRv_{\text{cm},f}\hat{\mathbf{k}} \,. \tag{21.6.68}$$

Equating the z-components in Eqs. (21.6.66) and (21.6.68) yields

$$mRv_{\text{cm},i} = (7/5)mRv_{\text{cm},f} \,, \tag{21.6.69}$$

which we can solve for

$$v_{\text{cm},f} = (5/7)v_{cm,i} \,. \tag{21.6.70}$$

The final angular velocity vector is

$$\vec{\boldsymbol{\omega}} = \omega_{z,f}\hat{\mathbf{k}} = \frac{v_{\text{cm},f}}{R}\hat{\mathbf{k}} = \frac{5v_{\text{cm},i}}{7R}\hat{\mathbf{k}} \ . \tag{21.6.71}$$

We could also solve this problem by analyzing the translational motion and the rotational motion about the center of mass. Gravity exerts no torque about the center of mass, and the normal component of the contact force has a zero moment arm; the only force that exerts a torque is the frictional force, with a moment arm of R (the force vector and the radius vector are perpendicular). The frictional force should be in the negative x-direction. From the right-hand rule, the direction of the torque is into the page, and hence in the positive z-direction. Equating the z-component of the torque to the rate of change of angular momentum about the center of mass yields

$$\tau_{cm} = R f_k = I_{cm}\alpha_z \, , \tag{21.6.72}$$

where f_k is the magnitude of the kinetic frictional force and α_z is the z-component of the angular acceleration of the bowling ball. Note that Eq. (21.6.72) results in a positive z-component of the angular acceleration, which is consistent with the ball tending to rotate as indicated Figure 21.15. The frictional force is also the only force in the horizontal direction, and will cause an acceleration of the center of mass,

$$a_{cm,x} = -f_k \,/\, m. \tag{21.6.73}$$

Note that the x-component of the acceleration will be negative, as expected. Now we need to consider the kinematics. The bowling ball will increase its z-component of the angular velocity as given in Eq. (21.6.72) and decrease its x-component of the velocity as given in Eq. (21.6.73),

$$\begin{aligned} \omega_z(t) &= \alpha_z\, t = \frac{Rf_k}{I_{\text{cm}}}t \\ v_{\text{cm},x}(t) &= v_{\text{cm},i} - \frac{f_k}{m}t. \end{aligned} \tag{21.6.74}$$

As soon as the ball stops slipping, the kinetic friction no longer acts, static friction is zero, and the ball moves with constant angular and linear velocity. Denote the time when this happens as t_f. At this time the rolling without slipping condition, $\omega_z(t_f) = v_{\text{cm},x}(t_f)\,/\,R$, holds and the relations in Eq. (21.6.74) become

$$\begin{aligned} R^2\frac{f_k}{I_{cm}}t_f &= v_{\text{cm},x,f} \\ v_{\text{cm},x,i} - \frac{f_k}{m}t_f &= v_{\text{cm},x,f}. \end{aligned} \tag{21.6.75}$$

We can now solve the first equation in Eq. (21.6.75) for t_f and find that

$$t_f = \frac{I_{\text{cm}}}{f_k R^2} v_{\text{cm},x,f} \,. \tag{21.6.76}$$

We now substitute Eq. (21.6.76) into the second equation in Eq. (21.6.75) and find that

$$\begin{aligned} v_{\text{cm},x,f} &= v_{\text{cm},x,i} - \frac{f_k}{m} \frac{I_{\text{cm}}}{f_k R^2} v_{\text{cm},x,f} \\ v_{\text{cm},x,f} &= v_{\text{cm},x,i} - \frac{I_{\text{cm}}}{m R^2} v_{\text{cm},x,f} \,. \end{aligned} \tag{21.6.77}$$

The second equation in (21.6.77) is easily solved for

$$v_{\text{cm},x,f} = \frac{v_0}{1 + I_{\text{cm}} / mR^2} = \frac{5}{7} v_{\text{cm},x,i} \,, \tag{21.6.78}$$

agreeing with Eq. (21.6.70) where we have used $I_{\text{cm}} = (2/5) m R^2$ for a uniform sphere.

Example 21.6 Rotation and Translation Object and Stick Collision

A long narrow uniform stick of length l and mass m lies motionless on ice (assume the ice provides a frictionless surface). The center of mass of the stick is the same as the geometric center (at the midpoint of the stick). The moment of inertia of the stick about its center of mass is I_{cm}. A puck (with putty on one side) has the same mass m as the stick. The puck slides without spinning on the ice with a velocity of $\vec{\mathbf{v}}_i$ toward the stick, hits one end of the stick, and attaches to it (Figure 21.18). You may assume that the radius of the puck is much less than the length of the stick so that the moment of inertia of the puck about its center of mass is negligible compared to I_{cm}. (a) How far from the midpoint of the stick is the center of mass of the stick-puck combination after the collision? (b) What is the linear velocity $\vec{\mathbf{v}}_{\text{cm},f}$ of the stick plus puck after the collision? (c) Is mechanical energy conserved during the collision? Explain your reasoning. (d) What is the angular velocity $\vec{\boldsymbol{\omega}}_{\text{cm},f}$ of the stick plus puck after the collision? (e) How far does the stick's center of mass move during one rotation of the stick?

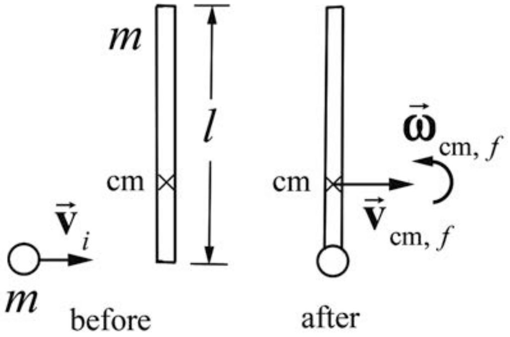


Figure 21.18 Example 21.6

Solution: In this problem we will calculate the center of mass of the puck-stick system after the collision. There are no external forces or torques acting on this system so the momentum of the center of mass is constant before and after the collision and the angular momentum about the center of mass of the puck-stick system is constant before and after the collision. We shall use these relations to compute the final angular velocity of the puck-stick about the center of mass.

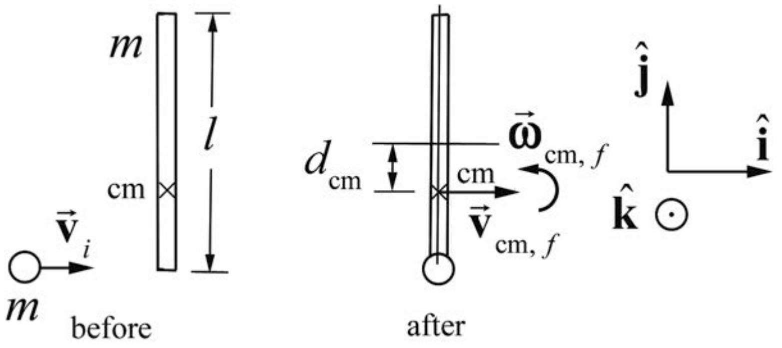


Figure 21.19 Center of mass of the system

a) With respect to the center of the stick, the center of mass of the stick-puck combination is

$$d_{\text{cm}} = \frac{m_{\text{stick}} d_{\text{stick}} + m_{\text{puck}} d_{\text{puck}}}{m_{\text{stick}} + m_{\text{puck}}} = \frac{m(l/2)}{m+m} = \frac{l}{4}. \tag{21.6.79}$$

where we are neglecting the radius of the puck (Figure 21.19).

b) During the collision, the only net forces on the system (the stick-puck combination) are the internal forces between the stick and the puck (transmitted through the putty). Hence, the linear momentum is constant. Initially only the puck had linear momentum $\vec{\mathbf{p}}_i = m\vec{\mathbf{v}}_i = mv_i\hat{\mathbf{i}}$. After the collision, the center of mass of the system is moving with velocity $\vec{\mathbf{v}}_{\text{cm},f} = v_{\text{cm},f}\hat{\mathbf{i}}$. Equating initial and final linear momenta,

$$m v_i = (2m) v_{\text{cm},f} \Rightarrow \ v_{\text{cm},f} = \frac{v_i}{2} \ . \qquad (21.6.80)$$

The direction of the velocity is the same as the initial direction of the puck's velocity.

c) The forces that deform the putty do negative work (the putty is compressed somewhat), and so mechanical energy is not conserved; the collision is totally inelastic.

d) Choose the center of mass of the stick-puck combination, as found in part a), as the point S about which to find angular momentum. This choice means that after the collision there is no angular momentum due to the translation of the center of mass. Before the collision, the angular momentum was entirely due to the motion of the puck,

$$\vec{\mathbf{L}}_{S,i} = \vec{\mathbf{r}}_{\text{puck}} \times \vec{\mathbf{p}}_i = (l/4)(m v_i)\hat{\mathbf{k}} \,, \qquad (21.6.81)$$

where $\hat{\mathbf{k}}$ is directed out of the page in Figure 21.19. After the collision, the angular momentum is

$$\vec{\mathbf{L}}_{S,f} = I_{\text{cm},f}\,\omega_{cm,f}\,\hat{\mathbf{k}} \,, \qquad (21.6.82)$$

where $I_{\text{cm},f}$ is the moment of inertia about the center of mass of the stick-puck combination. This moment of inertia of the stick about the new center of mass is found from the parallel axis theorem and the moment of inertia of the puck, which is $m(l/4)^2$. Therefore

$$I_{\text{cm},f} = I_{\text{cm, stick}} + I_{\text{cm, puck}} = (I_{\text{cm}} + m(l/4)^2) + m(l/4)^2 = I_{\text{cm}} + \frac{m l^2}{8} . \qquad (21.6.83)$$

Inserting this expression into Eq. (21.6.82), equating the expressions for $\vec{\mathbf{L}}_{S,i}$ and $\vec{\mathbf{L}}_{S,f}$ and solving for $\omega_{\text{cm},f}$ yields

$$\omega_{\text{cm},f} = \frac{m(l/4)}{I_{\text{cm}} + m l^2/8} v_i \,. \qquad (21.6.84)$$

If the stick is uniform, $I_{\text{cm}} = m l^2/12$ and Eq. (21.6.84) reduces to

$$\omega_{\text{cm},f} = \frac{6}{5}\frac{v_i}{l} \,. \qquad (21.6.85)$$

It may be tempting to try to calculate angular momentum about the contact point C, where the putty hits the stick. If this is done, there is no initial angular momentum, and after the collision the angular momentum will be the sum of two parts, the angular momentum of the center of mass of the stick and the angular moment about the center of the stick,

$$\vec{\mathbf{L}}_{C,f} = \vec{\mathbf{r}}_{\text{cm}} \times \vec{\mathbf{p}}_{\text{cm}} + I_{\text{cm}} \vec{\boldsymbol{\omega}}_{\text{cm},f} \,. \tag{21.6.86}$$

There are two crucial things to note: First, the speed of the center of mass is not the speed found in part b); the rotation must be included, so that $v_{\text{cm}} = v_i / 2 - \omega_{\text{cm},f}(l/4)$. Second, the direction of $\vec{\mathbf{r}}_{\text{cm}} \times \vec{\mathbf{p}}_{\text{cm}}$ with respect to the contact point C is, from the right-hand rule, *into* the page, or the $-\hat{\mathbf{k}}$-direction, opposite the direction of $\vec{\boldsymbol{\omega}}_{\text{cm},f}$. This is to be expected, as the sum in Eq. (21.6.86) must be zero. Adding the $\hat{\mathbf{k}}$-components (the only components) in Eq. (21.6.86),

$$-(l/2)m(v_i / 2 - \omega_{\text{cm},f}(l/4)) + I_{\text{cm}}\omega_{\text{cm},f} = 0 \,. \tag{21.6.87}$$

Solving Eq. (21.6.87) for $\omega_{\text{cm},f}$ yields Eq. (21.6.84).

This alternative derivation should serve two purposes. One is that it doesn't matter which point we use to find angular momentum. The second is that use of foresight, in this case choosing the center of mass of the system so that the final velocity does not contribute to the angular momentum, can prevent extra calculation. It's often a matter of trial and error ("learning by misadventure") to find the "best" way to solve a problem.

e) The time of one rotation will be the same for all observers, independent of choice of origin. This fact is crucial in solving problems, in that the angular velocity will be the same (this was used in the alternate derivation for part d) above). The time for one rotation is the period $T = 2\pi / \omega_f$ and the distance the center of mass moves is

$$\begin{aligned} x_{\text{cm}} &= v_{\text{cm}} T = 2\pi \frac{v_{\text{cm}}}{\omega_{\text{cm},f}} \\ &= 2\pi \frac{v_i / 2}{\left(\dfrac{m(l/4)}{I_{\text{cm}} + m l^2 / 8} \right) v_i} \\ &= 2\pi \frac{I_{\text{cm}} + m l^2 / 8}{m(l/2)}. \end{aligned} \tag{21.6.88}$$

Using $I_{\text{cm}} = ml^2 / 12$ for a uniform stick gives

$$x_{\text{cm}} = \frac{5}{6} \pi l \,. \tag{21.6.89}$$

Chapter 22 Three Dimensional Rotations and Gyroscopes

Chapter 22 Three Dimensional Rotations and Gyroscopes

> *Hypothesis: The earth, having once received a rotational movement around an axis, which agrees with its axis on the figure or only differs from it slightly, will always conserve this uniform movement, and its axis of rotation will always remain the same and will be directed toward the same points of the sky, unless the earth should be subjected to external forces which might cause some change either in the speed of rotational movement or in the position of the axis of rotation.*[1]
>
> *Leonhard Euler*

22.1 Introduction to Three Dimensional Rotations

Most of the examples and applications we have considered concerned the rotation of rigid bodies about a fixed axis. However, there are many examples of rigid bodies that rotate about an axis that is changing its direction. A turning bicycle wheel, a gyroscope, the earth's precession about its axis, a spinning top, and a coin rolling on a table are all examples of this type of motion. These motions can be very complex and difficult to analyze. However, for each of these motions we know that if there a non-zero torque about a point S, then the angular momentum about S must change in time, according to the rotational equation of motion,

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt}. \tag{22.1.1}$$

We also know that the angular momentum about S of a rotating body is the sum of the orbital angular momentum about S and the spin angular momentum about the center of mass.

$$\vec{\mathbf{L}}_S = \vec{\mathbf{L}}_S^{\text{orbital}} + \vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}. \tag{22.1.2}$$

For fixed axis rotation the spin angular momentum about the center of mass is just

$$\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = I_{\text{cm}}\vec{\boldsymbol{\omega}}_{\text{cm}}. \tag{22.1.3}$$

where $\vec{\boldsymbol{\omega}}_{\text{cm}}$ is the angular velocity about the center of mass and is directed along the fixed axis of rotation.

22.1.1 Angular Velocity for Three Dimensional Rotations

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[1] L. Euler, Recherches sur la precession des equinoxes et sur la nutation de l'axe de la terre (Research concerning the precession of the equinoxes and of the nutation of the earth's axis), *Memoires de l'academie des sciences de Berlin* 5, 1751, pp. 289-325

When the axis of rotation is no longer fixed, the angular velocity will no longer point in a fixed direction.

> *For an object that is rotating with angular coordinates* $(\theta_x, \theta_y, \theta_z)$ *about each respective Cartesian axis, the* ***angular velocity*** *of an object that is rotating about each axis is defined to be*

$$\begin{aligned}\vec{\boldsymbol{\omega}} &= \frac{d\theta_x}{dt}\hat{\mathbf{i}} + \frac{d\theta_y}{dt}\hat{\mathbf{j}} + \frac{d\theta_z}{dt}\hat{\mathbf{k}} \\ &= \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}\end{aligned} \tag{22.1.4}$$

This definition is the result of a property of very small (infinitesimal) angular rotations in which the order of rotations does matter. For example, consider an object that undergoes a rotation about the x-axis, $\vec{\boldsymbol{\omega}}_x = \omega_x\hat{\mathbf{i}}$, and then a second rotation about the y-axis, $\vec{\boldsymbol{\omega}}_y = \omega_y\hat{\mathbf{j}}$. Now consider a different sequence of rotations. The object first undergoes a rotation about the y-axis, $\vec{\boldsymbol{\omega}}_y = \omega_y\hat{\mathbf{j}}$, and then undergoes a second rotation about the x-axis, $\vec{\boldsymbol{\omega}}_x = \omega_x\hat{\mathbf{i}}$. In both cases the object will end up in the same position indicated that $\vec{\boldsymbol{\omega}}_x + \vec{\boldsymbol{\omega}}_y = \vec{\boldsymbol{\omega}}_y + \vec{\boldsymbol{\omega}}_x$, a necessary condition that must be satisfied in order for a physical quantity to be a vector quantity.

Example 22.1 Angular Velocity of a Rolling Bicycle Wheel

A bicycle wheel of mass m and radius R rolls without slipping about the z-axis. An axle of length b passes through its center. The bicycle wheel undergoes two simultaneous rotations. The wheel circles around the z-axis with angular speed Ω and associated angular velocity $\vec{\boldsymbol{\Omega}} = \Omega_z\hat{\mathbf{k}}$ (Figure 22.1). Because the wheel is rotating without slipping, it is spinning about its center of mass with angular speed ω_{spin} and associated angular velocity $\vec{\boldsymbol{\omega}}_{\text{spin}} = -\omega_{\text{spin}}\hat{\mathbf{r}}$.

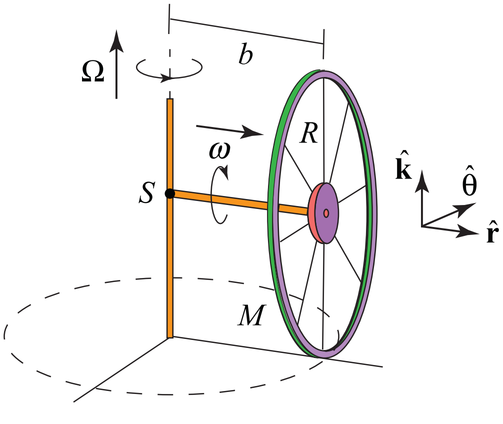


Figure 22.1 Example 22.1

The angular velocity of the wheel is the sum of these two vector contributions

$$\vec{\boldsymbol{\omega}} = \Omega\,\hat{\mathbf{k}} - \omega_{\text{spin}}\hat{\mathbf{r}}\ . \tag{22.1.5}$$

Because the wheel is rolling without slipping, $v_{\text{cm}} = b\Omega = \omega_{\text{spin}} R$ and so $\omega_{\text{spin}} = b\Omega / R$. The angular velocity is then

$$\vec{\boldsymbol{\omega}} = \Omega\,(\hat{\mathbf{k}} - (b / R)\hat{\mathbf{r}})\ . \tag{22.1.6}$$

The orbital angular momentum about the point S where the axle meets the axis of rotation (Figure 22.1), is then

$$\vec{\mathbf{L}}_S^{\text{orbital}} = bmv_{\text{cm}}\hat{\mathbf{k}} = mb^2\Omega\,\hat{\mathbf{k}}\ . \tag{22.1.7}$$

The spin angular momentum about the center of mass is more complicated. The wheel is rotating about both the z-axis and the radial axis. Therefore

$$\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = I_z\Omega\,\hat{\mathbf{k}} + I_r\omega_{\text{spin}}(-\hat{\mathbf{r}})\ . \tag{22.1.8}$$

Therefore the angular momentum about S is the sum of these two contributions

$$\begin{aligned}\vec{\mathbf{L}}_S &= mb^2\Omega\,\hat{\mathbf{k}} + I_z\Omega\,\hat{\mathbf{k}} + I_r\omega_{\text{spin}}(-\hat{\mathbf{r}}) \\ &= (mb^2\Omega + I_z\Omega)\,\hat{\mathbf{k}} - I_r(b\,\Omega / R)\hat{\mathbf{r}}.\end{aligned} \tag{22.1.9}$$

Comparing Eqs. (22.1.6) and (22.1.9), we note that the angular momentum about S is not proportional to the angular velocity.

22.2 Gyroscope

A toy gyroscope of mass m consists of a spinning flywheel mounted in a suspension frame that allows the flywheel's axle to point in any direction. One end of the axle is supported on a pylon a distance d from the center of mass of the gyroscope.

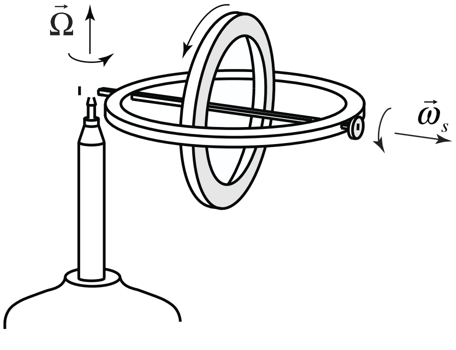


Figure 22.2a Toy Gyroscope

Choose polar coordinates so that the axle of the gyroscope flywheel is aligned along the r-axis and the vertical axis is the z-axis (Figure 22.2 shows a schematic representation of the gyroscope).

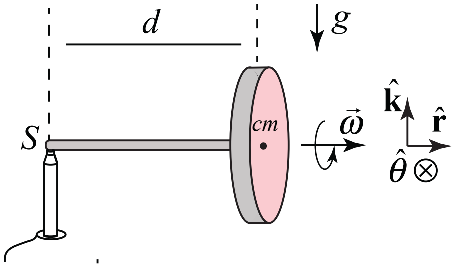


Figure 22.2 A toy gyroscope.

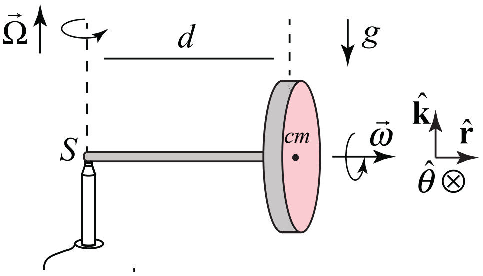


Figure 22.3 Angular rotations

The flywheel is spinning about its axis with a ***spin angular velocity***,

$$\vec{\boldsymbol{\omega}}_s = \omega_s \hat{\mathbf{r}}\,, \tag{22.2.1}$$

where ω_s is the radial component and $\omega_s > 0$ for the case illustrated in Figure 22.2.

When we release the gyroscope it undergoes a very surprising motion. Instead of falling downward, the center of mass rotates about a vertical axis that passes through the contact point S of the axle with the pylon with a ***precessional angular velocity***

$$\vec{\boldsymbol{\Omega}} = \Omega_z \hat{\mathbf{k}} = \frac{d\theta}{dt}\hat{\mathbf{k}}\,, \tag{22.2.2}$$

where $\Omega_z = d\theta / dt$ is the z-component and $\Omega_z > 0$ for the case illustrated in Figure 22.3. Therefore the angular velocity of the flywheel is the sum of these two contributions

$$\vec{\boldsymbol{\omega}} = \vec{\boldsymbol{\omega}}_s + \vec{\boldsymbol{\Omega}} = \omega_s \hat{\mathbf{r}} + \Omega_z \hat{\mathbf{k}}\,. \tag{22.2.3}$$

We shall study the special case where the magnitude of the precession component $|\Omega_z|$ of the angular velocity is much less than the magnitude of the spin component $|\omega_s|$ of the spin angular velocity, $|\Omega_z| << |\omega_s|$, so that the magnitude of the angular velocity $|\vec{\boldsymbol{\omega}}| \simeq |\omega_{\text{s}}|$ and Ω_z and ω_s are nearly constant. These assumptions are collectively called the ***gyroscopic approximation***.

The force diagram for the gyroscope is shown in Figure 22.4. The gravitational force acts at the center of the mass and is directed downward, $\vec{\mathbf{F}}^g = -mg\,\hat{\mathbf{k}}$. There is also a contact force, $\vec{\mathbf{F}}^c$, between the end of the axle and the pylon. It may seem that the contact force, $\vec{\mathbf{F}}^c$, has only an upward component, $\vec{\mathbf{F}}^v = F_z\,\hat{\mathbf{k}}$, but as we shall soon see there must also be a radial inward component to the contact force, $\vec{\mathbf{F}}^r = F_r\,\hat{\mathbf{r}}$, with $F_r < 0$, because the center of mass undergoes circular motion.

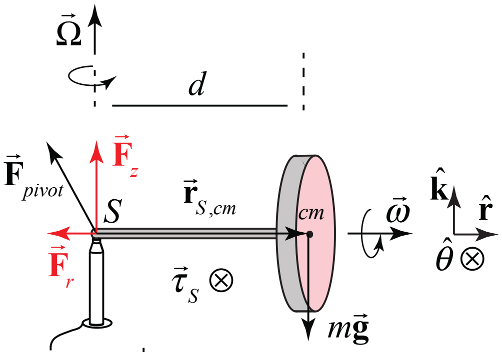


Figure 22.4 Force and torque diagram for the gyroscope

The reason that the gyroscope does not fall down is that the vertical component of the contact force exactly balances the gravitational force

$$F_z - mg = 0 \ . \tag{22.2.4}$$

What about the torque about the contact point S? The contact force acts at S so it does not contribute to the torque about S; only the gravitational force contributes to the torque about S (Figure 22.5b). The direction of the torque about S is given by

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{F}}_{\text{gravity}} = d\,\hat{\mathbf{r}} \times mg(-\hat{\mathbf{k}}) = d\,mg\hat{\boldsymbol{\theta}}\,, \tag{22.2.5}$$

and is in the positive $\hat{\boldsymbol{\theta}}$-direction. However we know that if there a non-zero torque about S, then the angular momentum about S must change in time, according to

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt} \,. \tag{22.2.6}$$

The angular momentum about the point S of the gyroscope is given by

$$\vec{\mathbf{L}}_S = \vec{\mathbf{L}}_S^{\text{orbital}} + \vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} \,. \tag{22.2.7}$$

The orbital angular momentum about the point S is

$$\vec{\mathbf{L}}_S^{\text{orbital}} = \vec{\mathbf{r}}_{S,cm} \times m\vec{\mathbf{v}}_{cm} = d\,\hat{\mathbf{r}} \times md\Omega_z\,\hat{\boldsymbol{\theta}} = md^2\Omega_z\,\hat{\mathbf{k}} \,. \tag{22.2.8}$$

The magnitude of the orbital angular momentum about S is nearly constant and the direction does not change. Therefore

$$\frac{d}{dt}\vec{\mathbf{L}}_S^{\text{orbital}} = \vec{\mathbf{0}}. \tag{22.2.9}$$

The spin angular momentum includes two terms. Recall that the flywheel undergoes two separate rotations about different axes. It is spinning about the flywheel axis with spin angular velocity $\vec{\boldsymbol{\omega}}_s$. As the flywheel precesses around the pivot point, the flywheel rotates about the z-axis with precessional angular velocity $\vec{\boldsymbol{\Omega}}$ (Figure 22.5). The spin angular momentum therefore is given by

$$\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = I_r \omega_s \hat{\mathbf{r}} + I_z \Omega_z \ \hat{\mathbf{k}}\,, \tag{22.2.10}$$

where I_r is the moment of inertia with respect to the flywheel axis and I_z is the moment of inertia with respect to the z-axis. If we assume the axle is massless and the flywheel is uniform with radius R, then $I_r = (1/2)mR^2$. By the perpendicular axis theorem $I_r = I_z + I_y = 2I_z$, hence $I_z = (1/4)mR^2$.

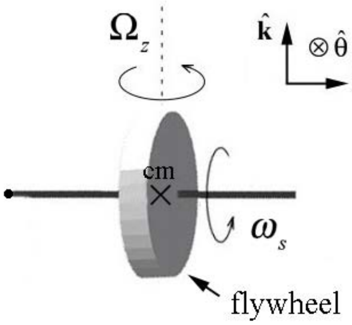


Figure 22.5: Rotations about center of mass of flywheel

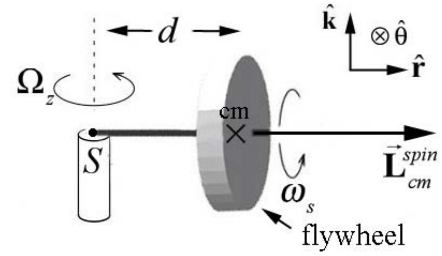


Figure 22.6 Spin angular momentum.

Recall that the gyroscopic approximation holds when $\left|\Omega_z\right| << \left|\omega_s\right|$, which implies that $I_z\Omega_z << I_r\omega_s$, and therefore we can ignore the contribution to the spin angular momentum from the rotation about the vertical axis, and so

$$\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} \simeq I_{\text{cm}}\omega_s\hat{\mathbf{r}}\,. \tag{22.2.11}$$

(The contribution to the spin angular momentum due to the rotation about the z-axis, $I_z\Omega_z \ \hat{\mathbf{k}}$, is nearly constant in both magnitude and direction so it does not change in time, $d(I_z\Omega_z \ \hat{\mathbf{k}})/dt \simeq \vec{\mathbf{0}}$.) Therefore the angular momentum about S is approximately

$$\vec{\mathbf{L}}_S \simeq \vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = I_{\text{cm}}\omega_s\hat{\mathbf{r}}\,. \tag{22.2.12}$$

Our initial expectation that the gyroscope should fall downward due to the torque that the gravitational force exerts about the contact point S leads to a violation of the torque law. If the center of mass did start to fall then the change in the spin angular momentum, $\Delta \vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}$, would point in the negative z-direction and that would contradict the vector aspect of Eq. (22.2.6). Instead of falling down, the angular momentum about the center of mass, $\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}$, must change direction such that the direction of $\Delta \vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}$ is in the same direction as torque about S (Eq. (22.2.5)), the positive $\hat{\boldsymbol{\theta}}$-direction.

Recall that in our study of circular motion, we have already encountered several examples in which the direction of a constant magnitude vector changes. We considered a point object of mass m moving in a circle of radius r. When we choose a coordinate system with an origin at the center of the circle, the position vector $\vec{\mathbf{r}}$ is directed radially outward. As the mass moves in a circle, the position vector has a constant magnitude but changes in direction. The velocity vector is given by

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}(r\,\hat{\mathbf{r}}) = r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} = r\omega_z\hat{\boldsymbol{\theta}} \qquad (22.2.13)$$

and has direction that is perpendicular to the position vector (tangent to the circle), (Figure 22.7a)).

Figure 22.7 (a) Rotating position and velocity vector; (b) velocity and acceleration vector for uniform circular motion

For uniform circular motion, the magnitude of the velocity is constant but the direction constantly changes and we found that the acceleration is given by (Figure 22.7b)

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d}{dt}(v_\theta\hat{\boldsymbol{\theta}}) = v_\theta\frac{d\theta}{dt}(-\hat{\mathbf{r}}) = r\omega_z\omega_z(-\hat{\mathbf{r}}) = -r\omega_z^2\hat{\mathbf{r}}\,. \qquad (22.2.14)$$

Note that we used the facts that

$$\begin{aligned}\frac{d\hat{\mathbf{r}}}{dt} &= \frac{d\theta}{dt}\hat{\boldsymbol{\theta}}, \\ \frac{d\hat{\boldsymbol{\theta}}}{dt} &= -\frac{d\theta}{dt}\hat{\mathbf{r}}\end{aligned}\,, \qquad (22.2.15)$$

in Eqs. (22.2.13) and (22.2.14). We can apply the same reasoning to how the spin angular changes in time (Figure 22.8).

The time derivative of the spin angular momentum is given by

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \frac{d\vec{\mathbf{L}}^{\text{spin}}_{\text{cm},\omega_s}}{dt} = \left|\vec{\mathbf{L}}^{\text{spin}}_{\text{cm},\omega_s}\right| \frac{d\theta}{dt}\,\hat{\boldsymbol{\theta}} = \left|\vec{\mathbf{L}}^{\text{spin}}_{\text{cm},\omega_s}\right| \Omega_z\,\hat{\boldsymbol{\theta}} = I_r \omega_s \Omega_z \hat{\boldsymbol{\theta}}\,. \qquad (22.2.16)$$

where $\Omega_z = d\theta / dt$ is the z-component and $\Omega_z > 0$. The center of mass of the flywheel rotates about a vertical axis that passes through the contact point S of the axle with the pylon with a precessional angular velocity

$$\vec{\boldsymbol{\Omega}} = \Omega_z \hat{\mathbf{k}} = \frac{d\theta}{dt}\hat{\mathbf{k}}\,, \qquad (22.2.17)$$

Substitute Eqs. (22.2.16) and (22.2.5) into Eq. (22.2.6) yielding

$$d\,mg\hat{\boldsymbol{\theta}} = \left|\vec{\mathbf{L}}^{\text{spin}}_{\text{cm}}\right| \Omega_z \hat{\boldsymbol{\theta}}\,. \qquad (22.2.18)$$

Solving Equation (22.2.18) for the z-component of the precessional angular velocity of the gyroscope yields

$$\Omega_z = \frac{d\,mg}{\left|\vec{\mathbf{L}}^{\text{spin}}_{\text{cm}}\right|} = \frac{d\,mg}{I_{\text{cm}}\,\omega_{\text{s}}}\,. \qquad (22.2.19)$$

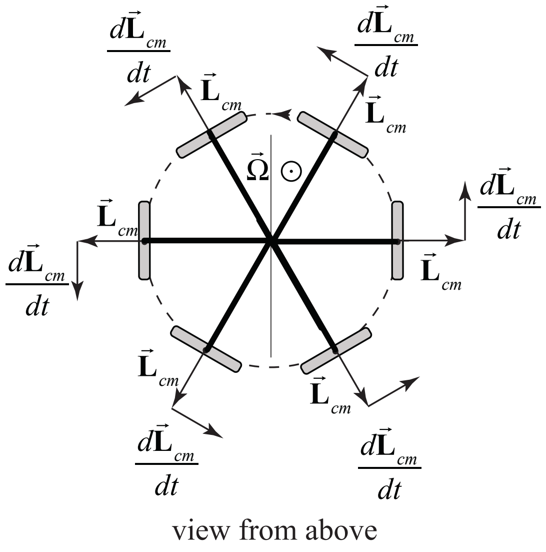
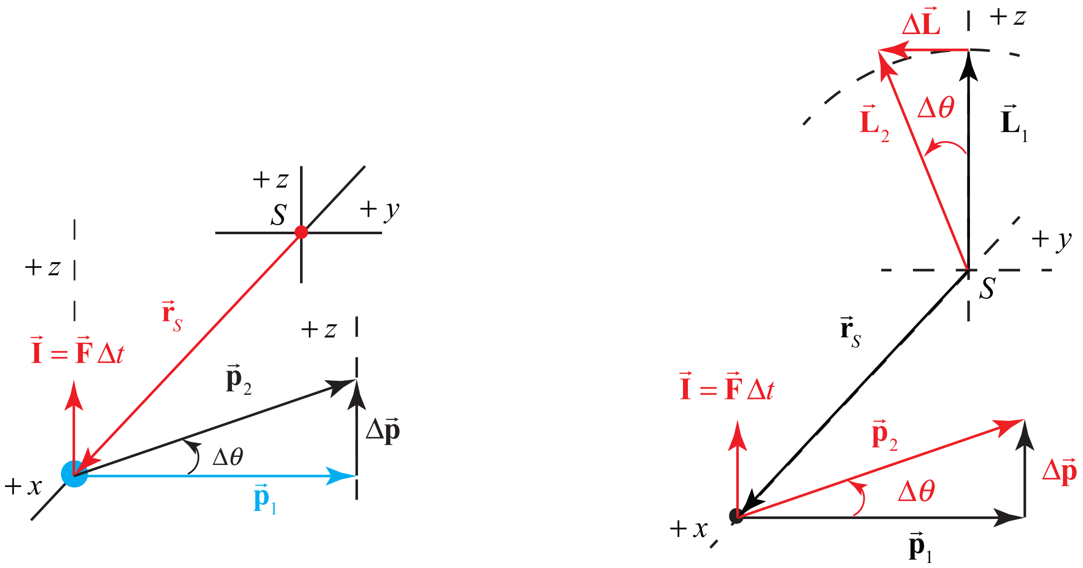


Figure 22.8 Time changing direction of the spin angular momentum

22.3 Why Does a Gyroscope Precess?

Why does a gyroscope precess? We now understand that the torque is causing the spin angular momentum to change but the motion still seems mysterious. We shall try to understand why the angular momentum changes direction by first examining the role of force and impulse on a single rotating particle and then generalize to a rotating disk.

22.3.1 Deflection of a Particle by a Small Impulse

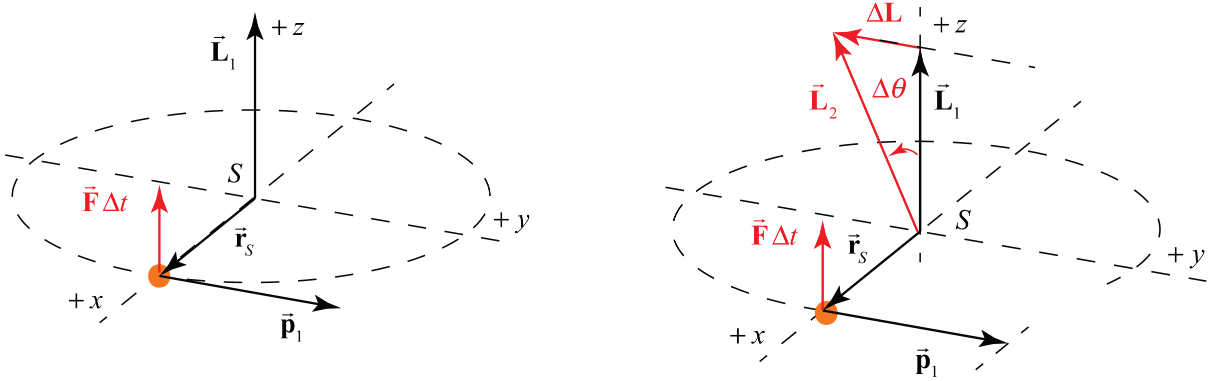


(a) (b)

Figure 22.9 (a) Deflection of a particle by a small impulse, (b) change in angular momentum about origin

We begin by first considering how a particle with momentum $\vec{\mathbf{p}}_1$ undergoes a deflection due to a small impulse (Figure 22.9a). If the impulse $\left|\vec{\mathbf{I}}\right| << \left|\vec{\mathbf{p}}_1\right|$, the primary effect is to rotate the momentum $\vec{\mathbf{p}}_1$ about the x-axis by a small angle θ, with $\vec{\mathbf{p}}_2 = \vec{\mathbf{p}}_1 + \Delta\vec{\mathbf{p}}$. The application of $\vec{\mathbf{I}}$ causes a change in the angular momentum $\vec{\mathbf{L}}_{O,1}$ about the origin S, according to the torque equation, $\Delta\vec{\mathbf{L}}_S = \vec{\boldsymbol{\tau}}_{\text{ave},\,S}\Delta t = (\vec{\mathbf{r}}_S \times \vec{\mathbf{F}}_{\text{ave}})\Delta t$. Because $\vec{\mathbf{I}} = \Delta\vec{\mathbf{p}} = \vec{\mathbf{F}}_{\text{ave}}\Delta t$, we have that $\Delta\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{I}}$. As a result, $\Delta\vec{\mathbf{L}}_S$ rotates about the x-axis by a small angle θ, to a new angular momentum $\vec{\mathbf{L}}_{S,2} = \vec{\mathbf{L}}_{S,1} + \Delta\vec{\mathbf{L}}_S$. Note that although $\vec{\mathbf{L}}_S$ is in the z-direction, $\Delta\vec{\mathbf{L}}_S$ is in the negative y-direction (Figure 22.9b).

22.3.2 Effect of Small Impulse on Tethered Object



(a) (b)

Figure 22.10a Small impulse on object undergoing circular motion, (b) change in angular momentum

Now consider an object that is attached to a string and is rotating about a fixed point S with momentum $\vec{\mathbf{p}}_1$. The object is given an impulse $\vec{\mathbf{I}}$ perpendicular to $\vec{\mathbf{r}}_S$ and to $\vec{\mathbf{p}}_1$. Neglect gravity. As a result $\Delta\vec{\mathbf{L}}_S$ rotates about the x-axis by a small angle θ (Figure 22.10a). Note that although $\vec{\mathbf{I}}$ is in the z-direction, $\Delta\vec{\mathbf{L}}_S$ is in the negative y- direction (Figure 22.10b). Note that although $\vec{\mathbf{I}}$ is in the z-direction, the plane in which the ball moves also rotates about the x-axis by the same angle (Figure 22.11).

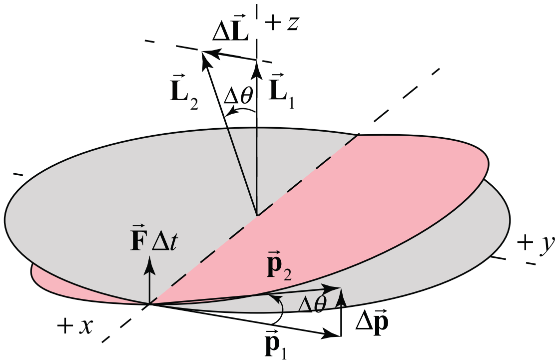


Figure 22.11 Plane of object rotates about x-axis

Example 22.2 Effect of Large Impulse on Tethered Object

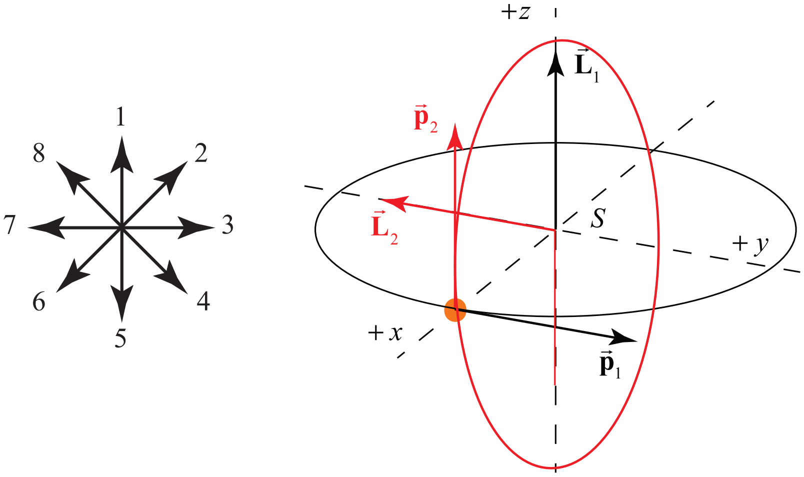


Figure 22.12 Example 22.2

What impulse, $\vec{\mathbf{I}}$, must be given to the ball in order to rotate its orbit by 90 degrees as shown without changing its speed (Figure 21.12)?

Solution: h. The impulse $\vec{\mathbf{I}}$ must halt the momentum $\vec{\mathbf{p}}_1$ and provide a momentum $\vec{\mathbf{p}}_2$ of equal magnitude along the z-direction such that $\vec{\mathbf{I}} = \Delta\vec{\mathbf{p}}$.

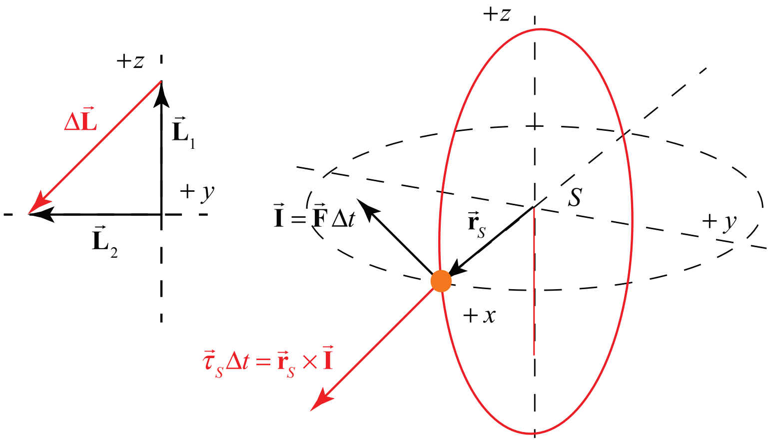


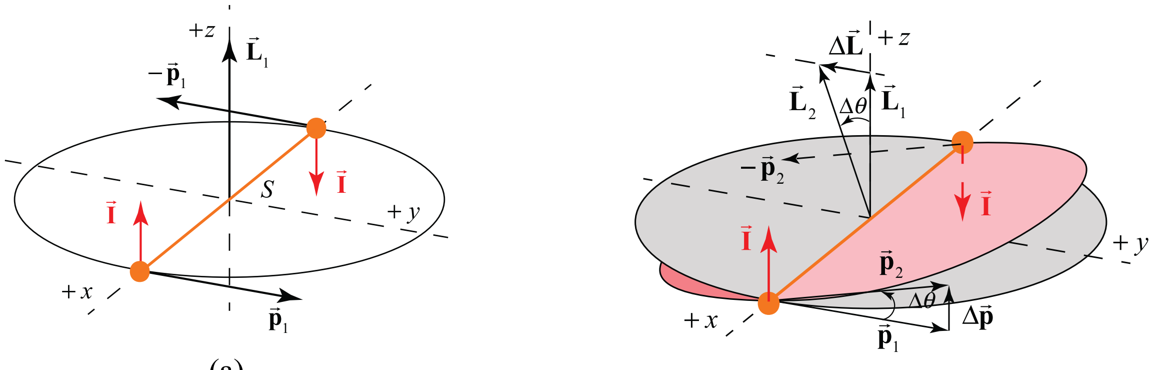
Figure 22.13 Impulse and torque about S

The angular impulse about S must be equal to the change in angular momentum about S

$$\vec{\boldsymbol{\tau}}_S \Delta t = \vec{\mathbf{r}}_S \times \vec{\mathbf{I}} = (\vec{\mathbf{r}}_S \times \Delta\vec{\mathbf{p}}) = \Delta\vec{\mathbf{L}}_S \qquad (22.3.1)$$

The change in angular momentum, $\Delta\vec{\mathbf{L}}_S$, due to the torque about S, cancels the z-component of $\vec{\mathbf{L}}_S$ and adds a component of the same magnitude in the negative y-direction (Figure 22.13).

22.3.3 Effect of Small Impulse Couple on Baton



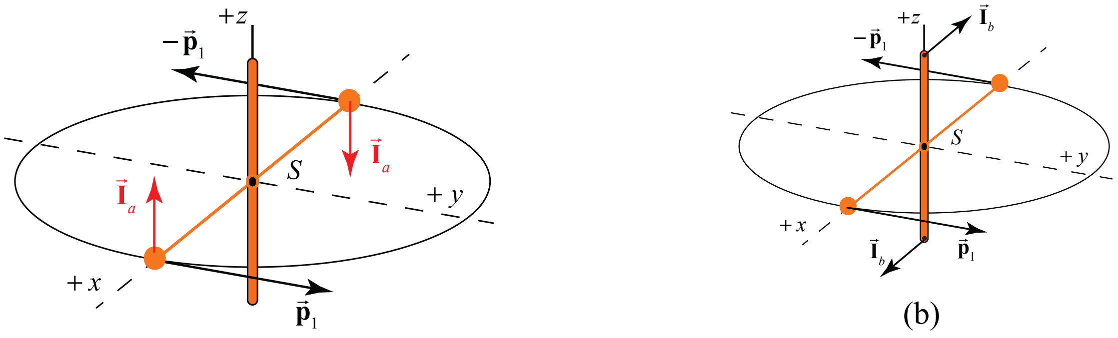
(a)

(b)

Figure 22.14 (a) and (b)

Now consider two equal masses at the ends of a massless rod, which spins about its center. We apply an impulse couple to insure no motion of the center of mass. Again note that the impulse couple is applied in the z-direction (Figure 22.14a). The resulting torque about S lies along the negative y-direction and the plane of rotation tilts about the x-axis (Figure 22.14b).

22.3.4 Effect of Small Impulse Couple on Massless Shaft of Baton



(b)

(a)

Figure 22.15 Apply impulse couple to (a) objects and (b) shaft

Instead of applying the impulse couple $\vec{\mathbf{I}}_a$ to the masses (Figure 21.15a), one could apply the same impulse couple $\vec{\mathbf{I}}_b = \vec{\mathbf{I}}_a$ to the vertical massless shaft that is connected to the baton (Figure 22.15b) to achieve the same result.

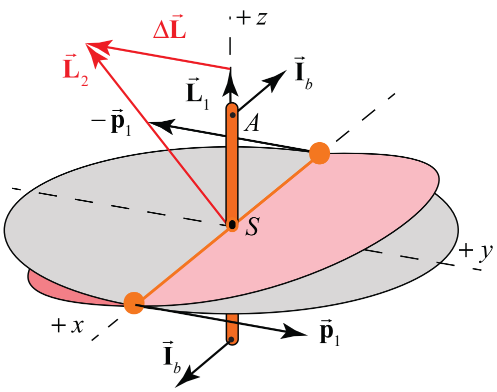


Figure 22.16 Twisting shaft causes shaft and plane to rotate about x-axis

Twisting the shaft around the y-axis causes the shaft and the plane in which the baton moves to rotate about the x-axis.

22.3.5 Effect of a Small Impulse Couple on a Rotating Disk

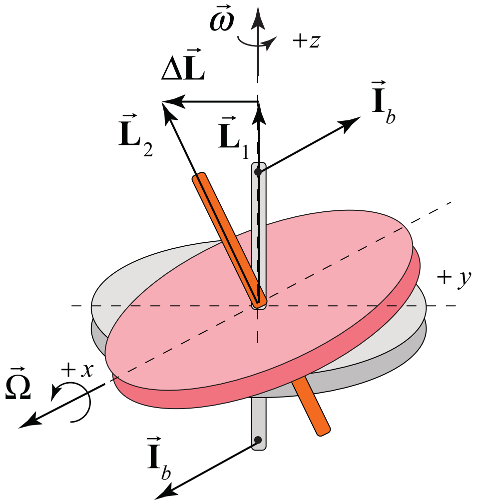


Figure 22.17 Impulse couple causes a disk to rotate about the x-axis.

Now let's consider a rotating disk. The plane of a rotating disk and its shaft behave just like the plane of the rotating baton and its shaft when one attempts to twist the shaft about the y-axis. The plane of the disk rotates about the x-axis (Figure 22.17). This unexpected result is due to the large pre-existing angular momentum about S, $\vec{\mathbf{L}}_1$, due to the spinning disk. It does not matter where along the shaft the impulse couple is applied, as long as it creates the same torque about S.

22.3.6 Effect of a Force Couple on a Rotating Disk

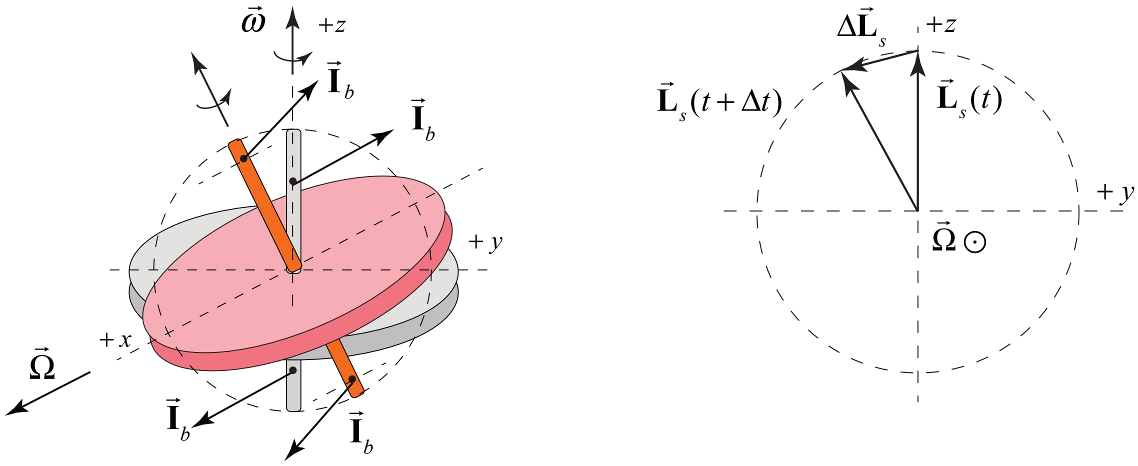


Figure 22.18 A series of small impulse couples causes the tip of the shaft to execute circular motion about the x-axis

A series of small impulse couples, or equivalently a continuous force couple (with force $\vec{\mathbf{F}}$), causes the tip of the shaft to execute circular motion about the x-axis (Figure 22.18). The magnitude of the angular momentum about S changes according to $\left|d\vec{\mathbf{L}}_S\right| = \left|\vec{\mathbf{L}}_S\right|\Omega\, dt = I\omega\,\Omega\, dt$. Recall that torque and changing angular momentum about S are related by $\vec{\boldsymbol{\tau}}_S = d\vec{\mathbf{L}}_S / dt$. Therefore $\left|\vec{\boldsymbol{\tau}}_S\right| = \left|\vec{\mathbf{L}}_S\right|\Omega = I\omega\,\Omega$. The precession rate of the shaft is the ratio of the magnitude of the torque to the angular momentum $\Omega = \left|\vec{\boldsymbol{\tau}}_S\right| / \left|\vec{\mathbf{L}}_S\right| = \left|\vec{\boldsymbol{\tau}}_S\right| / I\omega$.

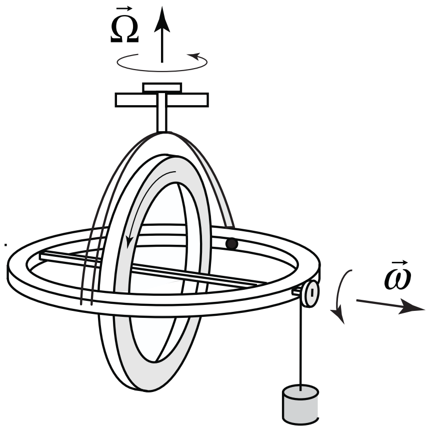


Figure 22.19 Precessing gyroscope with hanging object

Thus we can explain the motion of a precessing gyroscope in which the torque about the center of mass is provided by the force of gravity on the hanging object (Figure 22.19).

22.3.7 Effect of a Small Impulse Couple on a Non-Rotating Disc

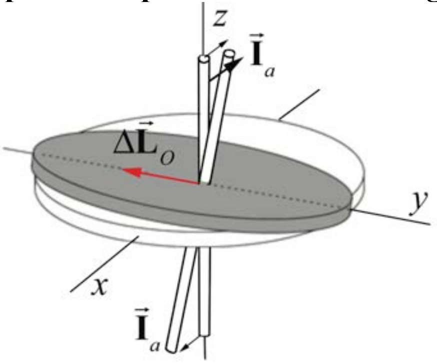


Figure 22.20 Impulse couple on non-rotating disk causes shaft to rotate about negative y -axis.

If the disk is not rotating to begin with, $\Delta\vec{\mathbf{L}}_S$ is also the final $\vec{\mathbf{L}}_S$. The shaft moves in the direction it is pushed (Figure 22.20).

22.4 Worked Examples

Example 22.3 Tilted Toy Gyroscope

A wheel is at one end of an axle of length d. The axle is pivoted at an angle ϕ with respect to the vertical. The wheel is set into motion so that it executes uniform precession; that is, the wheel's center of mass moves with uniform circular motion with z -component of precessional angular velocity Ω_z. The wheel has mass m and moment of inertia I_{cm} about its center of mass. Its spin angular velocity $\vec{\boldsymbol{\omega}}_s$ has magnitude ω_s and is directed as shown in Figure 22.21. Assume that the gyroscope approximation holds, $|\Omega_z| << \omega_s$. Neglect the mass of the axle. What is the z-component of the precessional angular velocity Ω_z ? Does the gyroscope rotate clockwise or counterclockwise about the vertical axis (as seen from above)?

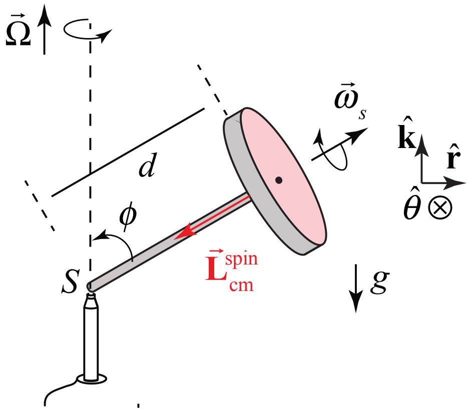


Figure 22.21 Example 22.3

Solution: The gravitational force acts at the center of mass and is directed downward, $\vec{\mathbf{F}}^g = -mg\,\hat{\mathbf{k}}$. Let S denote the contact point between the pylon and the axle. The contact force between the pylon and the axle is acting at S so it does not contribute to the torque about S. Only the gravitational force contributes to the torque. Let's choose cylindrical coordinates. The torque about S is

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,\mathrm{cm}} \times \vec{\mathbf{F}}^g = (d\sin\phi\,\hat{\mathbf{r}} + d\cos\phi\,\hat{\mathbf{k}}) \times mg(-\hat{\mathbf{k}}) = mgd\sin\phi\,\hat{\boldsymbol{\theta}}, \quad (22.4.1)$$

which is into the page in Figure 22.21. Because we are assuming that $|\Omega_z| << \omega_s$, we only consider contribution from the spinning about the flywheel axle to the spin angular momentum,

$$\vec{\boldsymbol{\omega}}_s = -\omega_s \sin\phi\,\hat{\mathbf{r}} - \omega_s \cos\phi\,\hat{\mathbf{k}} \quad (22.4.2)$$

The spin angular momentum has a vertical and radial component,

$$\vec{\mathbf{L}}^{\mathrm{spin}}_{\mathrm{cm}} = -I_{\mathrm{cm}}\omega_s \sin\phi\,\hat{\mathbf{r}} - I_{\mathrm{cm}}\omega_s \cos\phi\,\hat{\mathbf{k}}\,. \quad (22.4.3)$$

We assume that the spin angular velocity ω_s is constant. As the wheel precesses, the time derivative of the spin angular momentum arises from the change in the direction of the radial component of the spin angular momentum,

$$\frac{d}{dt}\vec{\mathbf{L}}^{\mathrm{spin}}_{\mathrm{cm}} = -I_{\mathrm{cm}}\omega_s \sin\phi \frac{d\hat{\mathbf{r}}}{dt} = -I_{\mathrm{cm}}\omega_s \sin\phi \frac{d\theta}{dt}\,\hat{\boldsymbol{\theta}}\,. \quad (22.4.4)$$

where we used the fact that

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\theta}{dt}\,\hat{\boldsymbol{\theta}}\,. \quad (22.4.5)$$

The z-component of the angular velocity of the flywheel about the vertical axis is defined to be

$$\Omega_z \equiv \frac{d\theta}{dt}\,. \quad (22.4.6)$$

Therefore the rate of change of the spin angular momentum is then

$$\frac{d}{dt}\vec{\mathbf{L}}^{\mathrm{spin}}_{\mathrm{cm}} = -I_{\mathrm{cm}}\omega_s \sin\phi\,\Omega_z\,\hat{\boldsymbol{\theta}}\,. \quad (22.4.7)$$

The torque about S induces the spin angular momentum about S to change,

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}}{dt}. \qquad (22.4.8)$$

Now substitute Equation (22.4.1) for the torque about S, and Equation (22.4.7) for the rate of change of the spin angular momentum into Equation (22.4.8), yielding

$$mgd\sin\phi\,\hat{\boldsymbol{\theta}} = -I_{\text{cm}}\omega_s \sin\phi\,\Omega_z\,\hat{\boldsymbol{\theta}}. \qquad (22.4.9)$$

Solving Equation (22.2.18) for the z-component of the precessional angular velocity of the gyroscope yields

$$\Omega_z = -\frac{d\,mg}{I_{\text{cm}}\,\omega_s}. \qquad (22.4.10)$$

The z-component of the precessional angular velocity is independent of the angle ϕ. Because $\Omega_z < 0$, the direction of the precessional angular velocity, $\vec{\Omega} = \Omega_z\hat{\mathbf{k}}$, is in the negative z-direction. That means that the gyroscope precesses in the clockwise direction when seen from above (Figure 21.22).

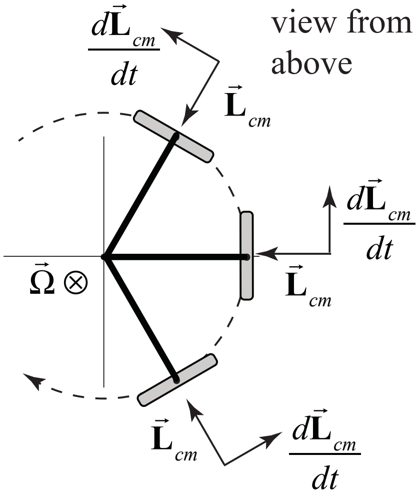


Figure 21.22 Precessional angular velocity of tilted gyroscope as seen from above

Both the torque and the time derivative of the spin angular momentum point in the $\hat{\boldsymbol{\theta}}$-direction indicating that the gyroscope will precess clockwise when seen from above in agreement with the calculation that $\Omega_z < 0$.

Example 22.4 Gyroscope on Rotating Platform

A gyroscope consists of an axle of negligible mass and a disk of mass M and radius R mounted on a platform that rotates with angular speed Ω. The gyroscope is spinning with angular speed ω. Forces F_a and F_b act on the gyroscopic mounts. What are the magnitudes of the forces F_a and F_b (Figure 22.22)? You may assume that the moment of inertia of the gyroscope about an axis passing through the center of mass normal to the plane of the disk is given by I_{cm}.

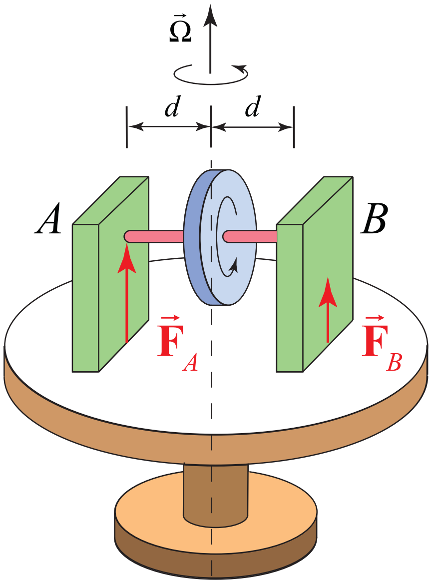


Figure 22.22 Example 22.4

Solution: Figure 22.23 shows a choice of coordinate system and force diagram on the gyroscope.

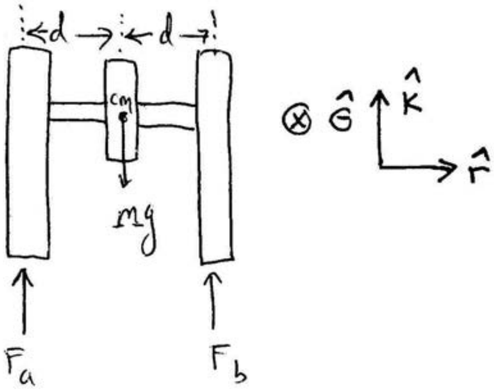


Figure 22.23 Free-body force diagram

The vertical forces sum to zero since there is no vertical motion

$$F_a + F_b - Mg = 0 \tag{22.4.11}$$

Using the coordinate system depicted in the Figure 22.23, torque about the center of mass is

$$\vec{\boldsymbol{\tau}}_{\text{cm}} = d(F_a - F_b)\hat{\boldsymbol{\theta}} \tag{22.4.12}$$

The spin angular momentum is (gyroscopic approximation)

$$\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} \simeq I_{\text{cm}}\omega\,\hat{\mathbf{r}} \tag{22.4.13}$$

Looking down on the gyroscope from above (Figure 2.23), the radial component of the angular momentum about the center of mass is rotating counterclockwise.

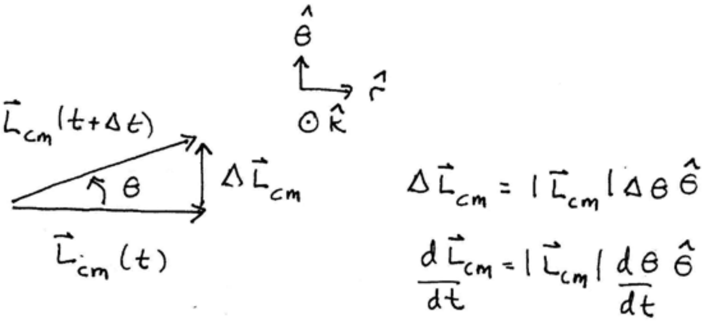


Figure 22.24 Change in angular momentum

During a very short time interval Δt, the change in the spin angular momentum is $\Delta\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = I_{\text{cm}}\omega\Delta\theta\,\hat{\boldsymbol{\theta}}$, (Figure 22.24). Taking limits we have that

$$\frac{d\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}}{dt} = \lim_{\Delta t \to 0}\frac{\Delta\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}}{\Delta t} = \lim_{\Delta t \to 0} I_{\text{cm}}\omega\frac{\Delta\theta}{\Delta t}\hat{\boldsymbol{\theta}} = I_{\text{cm}}\omega\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} \tag{22.4.14}$$

We can now apply the torque law

$$\vec{\boldsymbol{\tau}}_{\text{cm}} = \frac{d\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}}{dt}. \tag{22.4.15}$$

Substitute Eqs. (22.4.12) and (22.4.14) into Eq. (22.4.15) and just taking the component of the resulting vector equation yields

$$d(F_a - F_b) = I_{\text{cm}}\omega\,\Omega_z. \tag{22.4.16}$$

We can divide Eq. (22.4.16) by the quantity d yielding

$$F_a - F_b = \frac{I_{\text{cm}}\omega\,\Omega_z}{d}. \tag{22.4.17}$$

We can now use Eqs. (22.4.17) and (22.4.11) to solve for the forces F_a and F_b,

$$F_a = \frac{1}{2}\left(Mg + \frac{I_{\text{cm}}\omega\,\Omega_z}{d} \right) \tag{22.4.18}$$

$$F_b = \frac{1}{2}\left(Mg - \frac{I_{\text{cm}}\omega\,\Omega_z}{d} \right). \tag{22.4.19}$$

Note that if $\Omega_z = Mgd / I_{\text{cm}}\omega$ then $F_b = 0$ and one could remove the right hand support in the Figure 22.22. The simple pivoted gyroscope that we already analyzed Section 22.2 satisfied this condition. The forces we just found are the forces that the mounts must exert on the gyroscope in order to cause it to move in the desired direction. It is important to understand that the gyroscope is exerting equal and opposite forces on the mounts, i.e. the structure that is holding it. This is a manifestation of Newton's Third Law.

Example 22.5 Grain Mill

In a mill, grain is ground by a massive wheel that rolls without slipping in a circle on a flat horizontal millstone driven by a vertical shaft. The rolling wheel has mass M , radius b and is constrained to roll in a horizontal circle of radius R at angular speed Ω (Figure 22.25). The wheel pushes down on the lower millstone with a force equal to twice its weight (normal force). The mass of the axle of the wheel can be neglected. What is the precessional angular frequency Ω ?

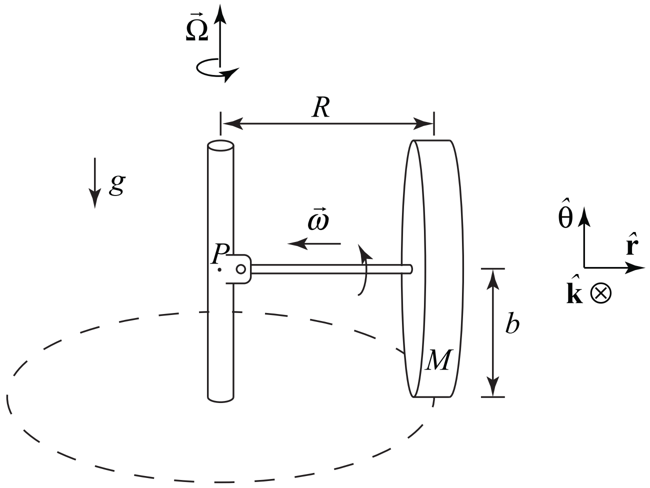


Figure 22.25 Example 22.5

Solution: Figure 22.5 shows the pivot point along with some convenient coordinate axes. For rolling without slipping, the speed of the center of mass of the wheel is related to the angular spin speed by

$$v_{cm} = b\omega\,. \tag{22.4.20}$$

Also the speed of the center of mass is related to the angular speed about the vertical axis associated with the circular motion of the center of mass by

$$v_{cm} = R\Omega\,. \tag{22.4.21}$$

Therefore equating Eqs. (22.4.20) and (22.4.21) we have that

$$\omega = \Omega R / b\,. \tag{22.4.22}$$

Assuming a uniform millwheel, $I_{\text{cm}} = (1/2)Mb^2$, the magnitude of the horizontal component of the spin angular momentum about the center of mass is

$$L_{\text{cm}}^{\text{spin}} = I_{\text{cm}}\omega = \frac{1}{2}Mb^2\omega = \frac{1}{2}\Omega\, MRb\,. \tag{22.4.23}$$

The horizontal component of $\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}$ is directed inward, and in vector form is given by

$$\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = -\frac{\Omega\, MRb}{2}\,\hat{\mathbf{r}}\,. \tag{22.4.24}$$

The axle exerts both a force and torque on the wheel, and this force and torque would be quite complicated. That’s why we consider the forces and torques on the axle/wheel combination. The normal force of the wheel on the ground is equal in magnitude to $N_{\text{W,G}} = 2mg$ so the third-law counterpart; the normal force of the ground on the wheel has the same magnitude $N_{\text{G,W}} = 2mg$. The joint (or hinge) at point P therefore must exert a force $\vec{\mathbf{F}}_{\text{H,A}}$ on the end of the axle that has two components, an inward force $\vec{\mathbf{F}}_2$ to maintain the circular motion and a downward force $\vec{\mathbf{F}}_1$ to reflect that the upward normal force is larger in magnitude than the weight (Figure 22.26).

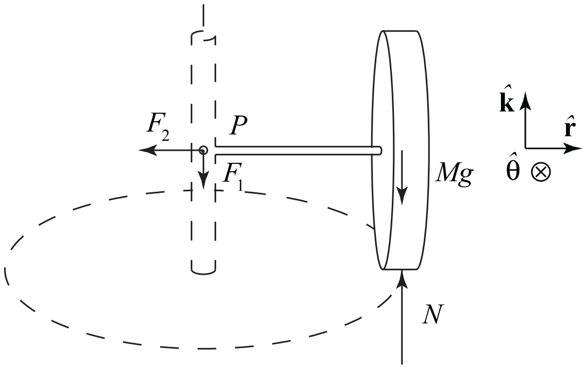


Figure 22.26 Free-body force diagram on wheel

About point P, $\vec{\mathbf{F}}_{\text{H,A}}$ exerts no torque. The normal force exerts a torque of magnitude $N_{\text{G,W}}R = 2mgR$, directed out of the page, or, in vector form, $\vec{\tau}_{P,N} = -2mgR\hat{\boldsymbol{\theta}}$. The weight exerts a toque of magnitude mgR, directed into the page, or, in vector form, $\vec{\tau}_{P,mg} = mgR\hat{\boldsymbol{\theta}}$. The torque about P is then

$$\vec{\tau}_P = \vec{\tau}_{P,N} + \vec{\tau}_{P,mg} = -2mgR\hat{\boldsymbol{\theta}} + mgR\hat{\boldsymbol{\theta}} = -mgR\hat{\boldsymbol{\theta}}. \tag{22.4.25}$$

As the wheel rolls, the horizontal component of the angular momentum about the center of mass will rotate, and the inward-directed vector will change in the negative $\hat{\boldsymbol{\theta}}$-direction. The angular momentum about the point P has orbital and spin decomposition

$$\vec{\mathbf{L}}_P = \vec{\mathbf{L}}_P^{\text{orbital}} + \vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}. \tag{22.4.26}$$

The orbital angular momentum about the point P is

$$\vec{\mathbf{L}}_P^{\text{orbital}} = \vec{\mathbf{r}}_{P,cm} \times m\vec{\mathbf{v}}_{cm} = R\,\hat{\mathbf{r}} \times mb\Omega\,\hat{\boldsymbol{\theta}} = mRb\Omega_z\,\hat{\mathbf{k}}. \tag{22.4.27}$$

The magnitude of the orbital angular momentum about P is nearly constant and the direction does not change. Therefore

$$\frac{d\vec{\mathbf{L}}_P^{\text{orbital}}}{dt} = \vec{\mathbf{0}}. \tag{22.4.28}$$

Therefore the change in angular momentum about the point P is

$$\frac{d\vec{\mathbf{L}}_P}{dt} = \frac{d\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}}{dt} = \frac{d}{dt}\left(\frac{\Omega mRb}{2}(-\hat{\mathbf{r}})\right) = \frac{1}{2}\Omega mRb\Omega(-\hat{\boldsymbol{\theta}}), \tag{22.4.29}$$

where we used Eq. (22.4.24) for the magnitude of the horizontal component of the angular momentum about the center of mass. This is consistent with the torque about P pointing out of the plane of Figure 22.26. We can now apply the rotational equation of motion,

$$\vec{\boldsymbol{\tau}}_P = \frac{d\vec{\mathbf{L}}_P}{dt}. \tag{22.4.30}$$

Substitute Eqs.(22.4.25) and (22.4.29) into Eq. (22.4.30) yielding

$$mgR(-\hat{\boldsymbol{\theta}}) = \frac{1}{2}\Omega^2 mRb(-\hat{\boldsymbol{\theta}}). \tag{22.4.31}$$

We can now solve Eq. (22.4.31) for the angular speed about the vertical axis

$$\Omega = \sqrt{\frac{2g}{b}}\,. \tag{22.4.32}$$

Chapter 23 Simple Harmonic Motion

Chapter 23 Simple Harmonic Motion

> *…Indeed it is not in the nature of a simple pendulum to provide equal and reliable measurements of time, since the wide lateral excursions often made may be observed to be slower than more narrow ones; however, we have been led in a different direction by geometry, from which we have found a means of suspending the pendulum, with which we were previously unacquainted, and by giving close attention to a line with a certain curvature, the time of the swing can be chosen equal to some calculated value and is seen clearly in practice to be in wonderful agreement with that ratio. As we have checked the lapses of time measured by these clocks after making repeated land and sea trials, the effects of motion are seen to have been avoided, so sure and reliable are the measurements; now it can be seen that both astronomical studies and the art of navigation will be greatly helped by them…*[1]
>
> *Christian Huygens*

23.1 Introduction: Periodic Motion

There are two basic ways to measure time: by duration or periodic motion. Early clocks measured duration by calibrating the burning of incense or wax, or the flow of water or sand from a container. Our calendar consists of years determined by the motion of the sun; months determined by the motion of the moon; days by the rotation of the earth; hours by the motion of cyclic motion of gear trains; and seconds by the oscillations of springs or pendulums. In modern times a second is defined by a specific number of vibrations of radiation, corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.

Sundials calibrate the motion of the sun through the sky, including seasonal corrections. A clock escapement is a device that can transform continuous movement into discrete movements of a gear train. The early escapements used oscillatory motion to stop and start the turning of a weight-driven rotating drum. Soon, complicated escapements were regulated by pendulums, the theory of which was first developed by the physicist Christian Huygens in the mid 17th century. The accuracy of clocks was increased and the size reduced by the discovery of the oscillatory properties of springs by Robert Hooke. By the middle of the 18th century, the technology of timekeeping advanced to the point that William Harrison developed timekeeping devices that were accurate to one second in a century.

23.1.1 Simple Harmonic Motion: Quantitative

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[1] Christian Huygens, *The Pendulum Clock or The Motion of Pendulums Adapted to Clocks By Geometrical Demonstrations,* tr. Ian Bruce, p. 1.

One of the most important examples of periodic motion is ***simple harmonic motion*** (SHM), in which some physical quantity varies sinusoidally. Suppose a function of time has the form of a sine wave function,

$$y(t) = A\sin(2\pi t / T) \tag{23.1.1}$$

where $A > 0$ is the ***amplitude*** (maximum value). The function $y(t)$ varies between A and $-A$, because a sine function varies between +1 and -1. A plot of $y(t)$ *vs.* time is shown in Figure 23.1.

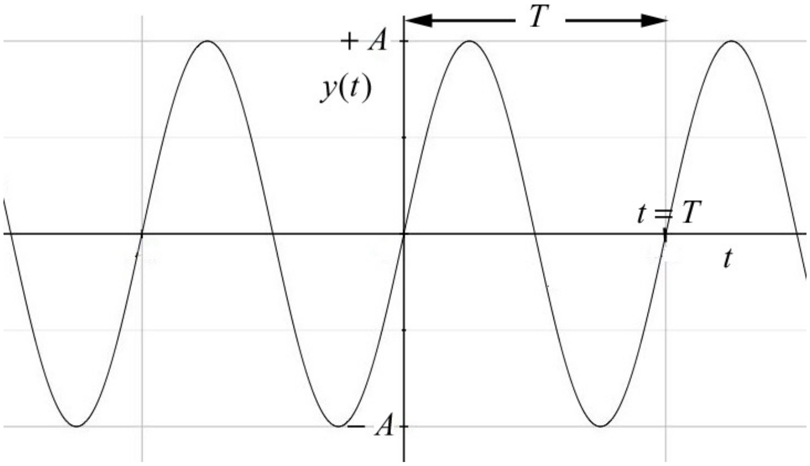


Figure 23.1 Sinusoidal function of time

The sine function is periodic in time. This means that the value of the function at time t will be exactly the same at a later time $t' = t + T$, where T is the ***period***. That the sine function satisfies the periodic condition can be seen from

$$y(t+T) = A\sin\left[\frac{2\pi}{T}(t+T)\right] = A\sin\left[\frac{2\pi}{T}t + 2\pi\right] = A\sin\left[\frac{2\pi}{T}t\right] = y(t)\,. \tag{23.1.2}$$

The ***frequency***, f, is defined to be

$$f \equiv 1/T\,. \tag{23.1.3}$$

The SI unit of frequency is inverse seconds, $\left[\mathrm{s}^{-1}\right]$, or hertz $[\mathrm{Hz}]$. The ***angular frequency*** of oscillation is defined to be

$$\omega_0 \equiv 2\pi / T = 2\pi f\,, \tag{23.1.4}$$

and is measured in radians per second. (The angular frequency of oscillation is denoted by ω_0 to distinguish from the angular speed $\omega = |d\theta / dt|$.) One oscillation per second, $1\,\mathrm{Hz}$, corresponds to an angular frequency of $2\pi\ \mathrm{rad}\cdot\mathrm{s}^{-1}$. (Unfortunately, the same symbol ω is used for angular speed in circular motion. For uniform circular motion the angular speed is equal to the angular frequency but for non-uniform motion the angular speed is not constant. The angular frequency for simple harmonic motion is a constant by definition.) We therefore have several different mathematical representations for sinusoidal motion

$$y(t) = A\sin(2\pi t / T) = A\sin(2\pi f\, t) = A\sin(\omega_0 t) \ . \tag{23.1.5}$$

23.2 Simple Harmonic Motion: Analytic

Our first example of a system that demonstrates simple harmonic motion is a spring-object system on a frictionless surface, shown in Figure 23.2

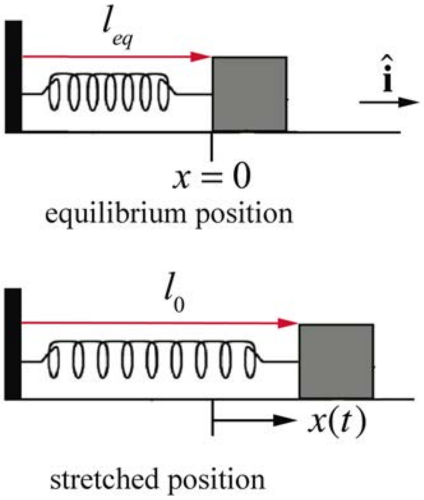


Figure 23.2 Spring-object system

The object is attached to one end of a spring. The other end of the spring is attached to a wall at the left in Figure 23.2. Assume that the object undergoes one-dimensional motion. The spring has a spring constant k and equilibrium length l_{eq}. Choose the origin at the equilibrium position and choose the positive x-direction to the right in the Figure 23.2. In the figure, $x > 0$ corresponds to an extended spring, and $x < 0$ to a compressed spring. Define $x(t)$ to be the position of the object with respect to the equilibrium position. The force acting on the spring is a linear restoring force, $F_x = -k\,x$ (Figure 23.3). The initial conditions are as follows. The spring is initially stretched a distance l_0 and given some initial speed v_0 to the right away from the equilibrium position. The initial position of the stretched spring from the equilibrium position (our choice of origin) is $x_0 = (l_0 - l_{eq}) > 0$ and its initial x-component of the velocity is $v_{x,0} = v_0 > 0$.

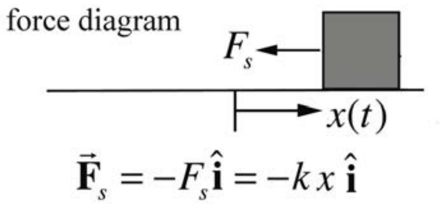


Figure 23.3 Free-body force diagram for spring-object system

Newton's Second law in the x-direction becomes

$$-k\,x = m\frac{d^2x}{dt^2}. \tag{23.2.1}$$

This equation of motion, Eq. (23.2.1), is called the ***simple harmonic oscillator equation*** (SHO). Because the spring force depends on the distance x, the acceleration is not constant. Eq. (23.2.1) is a second order linear differential equation, in which the second derivative of the dependent variable is proportional to the negative of the dependent variable,

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x. \tag{23.2.2}$$

In this case, the constant of proportionality is k/m,

Eq. (23.2.2) can be solved from energy considerations or other advanced techniques but instead we shall first guess the solution and then verify that the guess satisfies the SHO differential equation (see Appendix 22.3.A for a derivation of the solution).

We are looking for a position function $x(t)$ such that the second time derivative position function is proportional to the negative of the position function. Since the sine and cosine functions both satisfy this property, we make a preliminary *ansatz* (educated guess) that our position function is given by

$$x(t) = A\cos((2\pi/T)t) = A\cos(\omega_0\,t), \tag{23.2.3}$$

where ω_0 is the angular frequency (as of yet, undetermined).

We shall now find the condition that the angular frequency ω_0 must satisfy in order to insure that the function in Eq. (23.2.3) solves the simple harmonic oscillator equation, Eq. (23.2.1). The first and second derivatives of the position function are given by

$$\begin{aligned}\frac{dx}{dt} &= -\omega_0 A \sin(\omega_0 t) \\ \frac{d^2x}{dt^2} &= -\omega_0^2 A \cos(\omega_0\, t) = -\omega_0^2 x.\end{aligned} \tag{23.2.4}$$

Substitute the second derivative, the second expression in Eq. (23.2.4), and the position function, Equation (23.2.3), into the SHO Equation (23.2.1), yielding

$$-\omega_0^2 A\cos(\omega_0\, t) = -\frac{k}{m} A\cos(\omega_0\, t)\,. \tag{23.2.5}$$

Eq. (23.2.5) is valid for all times provided that

$$\omega_0 = \sqrt{\frac{k}{m}}\,. \tag{23.2.6}$$

The period of oscillation is then

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}\,. \tag{23.2.7}$$

One possible solution for the position of the block is

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}\,t\right), \tag{23.2.8}$$

and therefore by differentiation, the x -component of the velocity of the block is

$$v_x(t) = -\sqrt{\frac{k}{m}}\,A\sin\left(\sqrt{\frac{k}{m}}\,t\right). \tag{23.2.9}$$

Note that at $t = 0$, the position of the object is $x_0 \equiv x(t = 0) = A$ since $\cos(0) = 1$ and the velocity is $v_{x,0} \equiv v_x(t = 0) = 0$ since $\sin(0) = 0$. The solution in (23.2.8) describes an object that is released from rest at an initial position $A = x_0$ but does not satisfy the initial velocity condition, $v_x(t = 0) = v_{x,0} \neq 0$. We can try a sine function as another possible solution,

$$x(t) = B\sin\left(\sqrt{\frac{k}{m}}\,t\right). \tag{23.2.10}$$

This function also satisfies the simple harmonic oscillator equation because

$$\frac{d^2x}{dt^2} = -\frac{k}{m} B\sin\left(\sqrt{\frac{k}{m}}\,t\right) = -\omega_0{}^2\, x\,, \tag{23.2.11}$$

where $\omega_0 = \sqrt{k/m}$. The x-component of the velocity associated with Eq. (23.2.10) is

$$v_x(t) = \frac{dx}{dt} = \sqrt{\frac{k}{m}} B\cos\left(\sqrt{\frac{k}{m}}\,t\right). \tag{23.2.12}$$

The proposed solution in Eq. (23.2.10) has initial conditions $x_0 \equiv x(t=0) = 0$ and $v_{x,0} \equiv v_x(t=0) = (\sqrt{k/m})B$, thus $B = v_{x,0} / \sqrt{k/m}$. This solution describes an object that is initially at the equilibrium position but has an initial non-zero x-component of the velocity, $v_{x,0} \neq 0$.

23.2.1 General Solution of Simple Harmonic Oscillator Equation

Suppose $x_1(t)$ and $x_2(t)$ are both solutions of the simple harmonic oscillator equation,

$$\begin{aligned} \frac{d^2}{dt^2} x_1(t) &= -\frac{k}{m} x_1(t) \\ \frac{d^2}{dt^2} x_2(t) &= -\frac{k}{m} x_2(t). \end{aligned} \tag{23.2.13}$$

Then the sum $x(t) = x_1(t) + x_2(t)$ of the two solutions is also a solution. To see this, consider

$$\frac{d^2x(t)}{dt^2} = \frac{d^2}{dt^2}(x_1(t) + x_2(t)) = \frac{d^2x_1(t)}{dt^2} + \frac{d^2x_2(t)}{dt^2}. \tag{23.2.14}$$

Using the fact that $x_1(t)$ and $x_2(t)$ both solve the simple harmonic oscillator equation (23.2.13), we see that

$$\begin{aligned} \frac{d^2}{dt^2} x(t) &= -\frac{k}{m} x_1(t) + -\frac{k}{m} x_2(t) = -\frac{k}{m}\big(x_1(t) + x_2(t)\big) \\ &= -\frac{k}{m} x(t). \end{aligned} \tag{23.2.15}$$

Thus the *linear combination* $x(t) = x_1(t) + x_2(t)$ is also a solution of the SHO equation, Eq. (23.2.1). Therefore the sum of the sine and cosine solutions is the *general solution*,

$$x(t) = C\cos(\omega_0\, t) + D\sin(\omega_0\, t)\,, \tag{23.2.16}$$

where the constant coefficients C and D depend on a given set of initial conditions $x_0 \equiv x(t=0)$ and $v_{x,0} \equiv v_x(t=0)$ where x_0 and $v_{x,0}$ are constants. For this general solution, the x-component of the velocity of the object at time t is then obtained by differentiating the position function,

$$v_x(t) = \frac{dx}{dt} = -\omega_0 C \sin(\omega_0 t) + \omega_0 D \cos(\omega_0 t). \qquad (23.2.17)$$

To find the constants C and D, substitute $t = 0$ into the Eqs. (23.2.16) and (23.2.17). Because $\cos(0) = 1$ and $\sin(0) = 0$, the initial position at time $t = 0$ is

$$x_0 \equiv x(t=0) = C. \qquad (23.2.18)$$

The x-component of the velocity at time $t = 0$ is

$$v_{x,0} = v_x(t=0) = -\omega_0 C \sin(0) + \omega_0 D \cos(0) = \omega_0 D. \qquad (23.2.19)$$

Thus

$$C = x_0 \text{ and } D = \frac{v_{x,0}}{\omega_0}. \qquad (23.2.20)$$

The position of the object-spring system is then given by

$$x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + \frac{v_{x,0}}{\sqrt{k/m}} \sin\left(\sqrt{\frac{k}{m}}\, t\right) \qquad (23.2.21)$$

and the x-component of the velocity of the object-spring system is

$$v_x(t) = -\sqrt{\frac{k}{m}} x_0 \sin\left(\sqrt{\frac{k}{m}}\, t\right) + v_{x,0} \cos\left(\sqrt{\frac{k}{m}}\, t\right). \qquad (23.2.22)$$

Although we had previously specified $x_0 > 0$ and $v_{x,0} > 0$, Eq. (23.2.21) is seen to be a valid solution of the SHO equation for any values of x_0 and $v_{x,0}$.

Example 23.1: Phase and Amplitude

Show that $x(t) = C\cos\omega_0 t + D\sin\omega_0 t = A\cos(\omega_0 t + \phi)$, where $A = (C^2 + D^2)^{1/2} > 0$, and $\phi = \tan^{-1}(-D/C)$.

Solution: Use the identity $A\cos(\omega_0 t+\phi) = A\cos(\omega_0 t)\cos(\phi) - A\sin(\omega_0 t)\sin(\phi)$. Thus $C\cos(\omega_0 t) + D\sin(\omega_0 t) = A\cos(\omega_0 t)\cos(\phi) - A\sin(\omega_0 t)\sin(\phi)$. Comparing coefficients we see that $C = A\cos\phi$ and $D = -A\sin\phi$. Therefore

$$(C^2 + D^2)^{1/2} = A^2(\cos^2\phi + \sin^2\phi) = A^2 \ .$$

We choose the positive square root to ensure that $A > 0$, and thus

$$A = (C^2 + D^2)^{1/2} \tag{23.2.23}$$

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{-D / A}{C / A} = -\frac{D}{C},$$

$$\phi = \tan^{-1}(-D / C). \tag{23.2.24}$$

Thus the position as a function of time can be written as

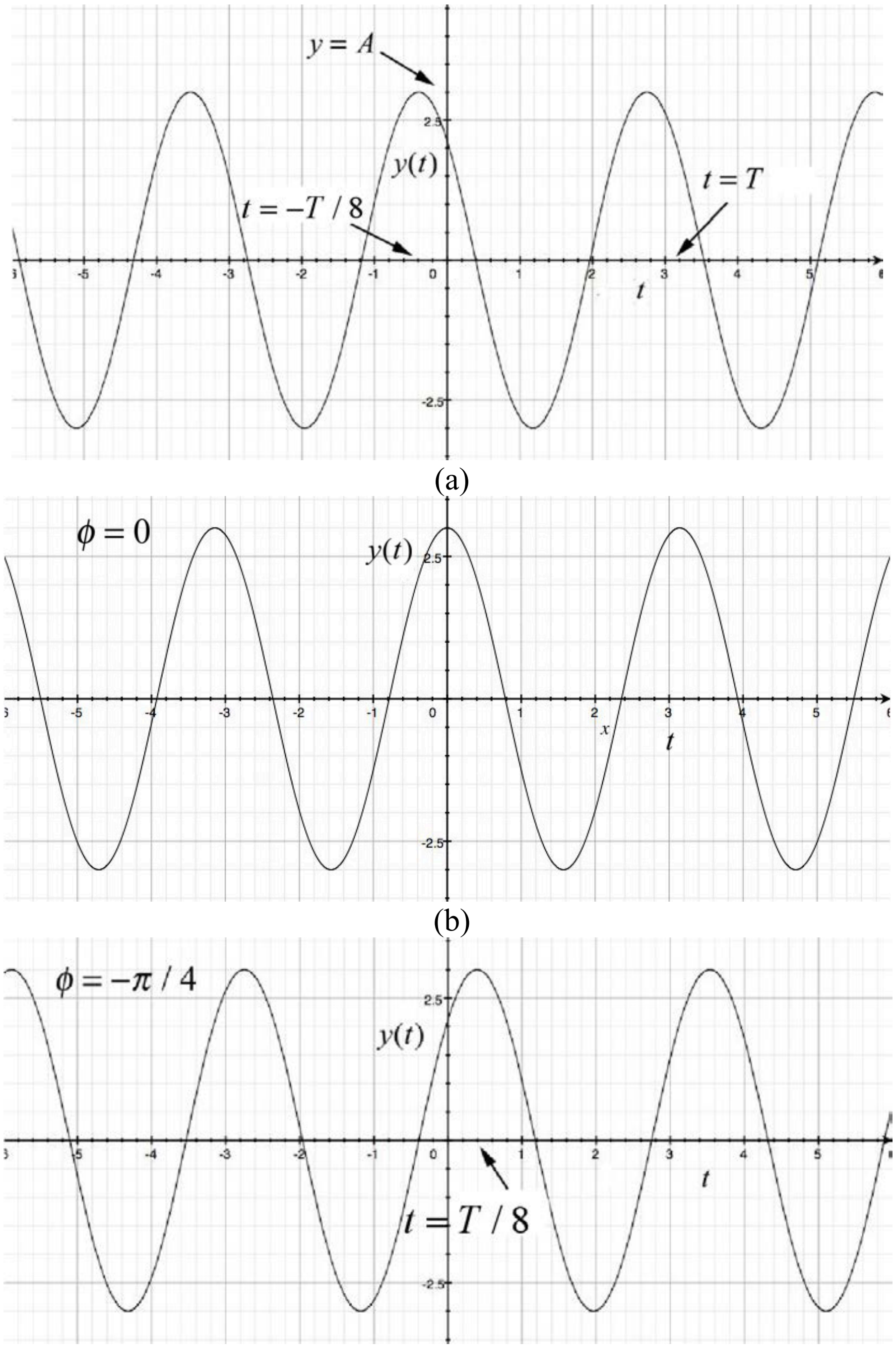
$$x(t) = A\cos(\omega_0 t + \phi). \tag{23.2.25}$$

In Eq. (23.2.25) the quantity $\omega_0 t + \phi$ is called the ***phase***, and ϕ is called the ***phase constant.*** Because $\cos(\omega_0 t + \phi)$ varies between $+1$ and -1, and $A > 0$, A is the amplitude defined earlier. We now substitute Eq. (23.2.20) into Eq. (23.2.23) and find that the amplitude of the motion described in Equation (23.2.21), that is, the maximum value of $x(t)$, and the phase are given by

$$A = \sqrt{x_0^2 + (v_{x,0} / \omega_0)^2} \ . \tag{23.2.26}$$

$$\phi = \tan^{-1}(-v_{x,0} / \omega_0 x_0). \tag{23.2.27}$$

A plot of $x(t)$ vs. t is shown in Figure 23.4a with the values $A = 3$, $T = \pi$, and $\phi = \pi / 4$. Note that $x(t) = A\cos(\omega_0 t + \phi)$ takes on its maximum value when $\cos(\omega_0 t + \phi) = 1$. This occurs when $\omega_0 t + \phi = 2\pi n$ where $n = 0, \pm 1, \pm 2, \cdots$. The maximum value associated with $n = 0$ occurs when $\omega_0 t + \phi = 0$ or $t = -\phi / \omega_0$. For the case shown in Figure 23.4a where $\phi = \pi / 4$, this maximum occurs at the instant $t = -T / 8$. Let's plot $x(t) = A\cos(\omega_0 t + \phi)$ vs. t for $\phi = 0$ (Figure 23.4b). For $\phi > 0$, Figure 23.4a shows the plot $x(t) = A\cos(\omega_0 t + \phi)$ vs. t. Notice that when $\phi > 0$, $x(t)$ is shifted to the left compared with the case $\phi = 0$ (compare Figures 23.4a with 23.4b). The function $x(t) = A\cos(\omega_0 t + \phi)$ with $\phi > 0$ reaches its maximum value at an earlier time than the function $x(t) = A\cos(\omega_0 t)$. The difference in phases for these two cases is $(\omega_0 t + \phi) - \omega_0 t = \phi$ and ϕ is sometimes referred to as the *phase shift*. When $\phi < 0$, the function $x(t) = A\cos(\omega_0 t + \phi)$ reaches its maximum value at a later time $t = T / 8$ than the function $x(t) = A\cos(\omega_0 t)$ as shown in Figure 23.4c.



(a)

(b)

(c)

Figure 23.4 Phase shift of $x(t) = A\cos(\omega_0 t + \phi)$ (a) to the left by $\phi = \pi / 4$, (b) no shift $\phi = 0$, (c) to the right $\phi = -\pi / 4$

Example 23.2: Block-Spring System

A block of mass m is attached to a spring with spring constant k and is free to slide along a horizontal frictionless surface. At $t = 0$, the block-spring system is stretched an amount $x_0 > 0$ from the equilibrium position and is released from rest, $v_{x,0} = 0$. What is the period of oscillation of the block? What is the velocity of the block when it first comes back to the equilibrium position?

Solution: The position of the block can be determined from Eq. (23.2.21) by substituting the initial conditions $x_0 > 0$, and $v_{x,0} = 0$ yielding

$$x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}\,t\right), \tag{23.2.28}$$

and the x-component of its velocity is given by Eq. (23.2.22),

$$v_x(t) = -\sqrt{\frac{k}{m}}\,x_0 \sin\left(\sqrt{\frac{k}{m}}\,t\right). \tag{23.2.29}$$

The angular frequency of oscillation is $\omega_0 = \sqrt{k / m}$ and the period is given by Eq. (23.2.7),

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}\,. \tag{23.2.30}$$

The block first reaches equilibrium when the position function first reaches zero. This occurs at time t_1 satisfying

$$\sqrt{\frac{k}{m}}\,t_1 = \frac{\pi}{2}, \quad t_1 = \frac{\pi}{2}\sqrt{\frac{m}{k}} = \frac{T}{4}\,. \tag{23.2.31}$$

The x-component of the velocity at time t_1 is then

$$v_x(t_1) = -\sqrt{\frac{k}{m}}x_0 \sin\left(\sqrt{\frac{k}{m}}\,t_1\right) = -\sqrt{\frac{k}{m}}x_0 \sin(\pi / 2) = -\sqrt{\frac{k}{m}}x_0 = -\omega_0\, x_0 \tag{23.2.32}$$

Note that the block is moving in the negative x-direction at time t_1; the block has moved from a positive initial position to the equilibrium position (Figure 23.4(b)).

23.3 Energy and the Simple Harmonic Oscillator

Let's consider the block-spring system of Example 23.2 in which the block is initially stretched an amount $x_0 > 0$ from the equilibrium position and is released from rest, $v_{x,0} = 0$. We shall consider three states: state 1, the initial state; state 2, at an arbitrary time in which the position and velocity are non-zero; and state 3, when the object first comes back to the equilibrium position. We shall show that the mechanical energy has the same value for each of these states and is constant throughout the motion. Choose the equilibrium position for the zero point of the potential energy.

State 1: all the energy is stored in the object-spring potential energy, $U_1 = (1/2)k\,x_0^2$. The object is released from rest so the kinetic energy is zero, $K_1 = 0$. The total mechanical energy is then

$$E_1 = U_1 = \frac{1}{2}k\,x_0^2 . \tag{23.3.1}$$

State 2: at some time t, the position and x-component of the velocity of the object are given by

$$\begin{aligned} x(t) &= x_0 \cos\left(\sqrt{\frac{k}{m}}t\right) \\ v_x(t) &= -\sqrt{\frac{k}{m}}x_0 \sin\left(\sqrt{\frac{k}{m}}t\right). \end{aligned} \tag{23.3.2}$$

The kinetic energy is

$$K_2 = \frac{1}{2}m\,v^2 = \frac{1}{2}k\,x_0^2 \sin^2\left(\sqrt{\frac{k}{m}}\,t\right), \tag{23.3.3}$$

and the potential energy is

$$U_2 = \frac{1}{2}k\,x^2 = \frac{1}{2}k\,x_0^2 \cos^2\left(\sqrt{\frac{k}{m}}\,t\right). \tag{23.3.4}$$

The mechanical energy is the sum of the kinetic and potential energies

$$\begin{aligned} E_2 &= K_2 + U_2 = \frac{1}{2}m\,{v_x}^2 + \frac{1}{2}k\,x^2 \\ &= \frac{1}{2}k\,x_0^2\left(\cos^2\left(\sqrt{\frac{k}{m}}\,t\right) + \sin^2\left(\sqrt{\frac{k}{m}}t\right)\right) \\ &= \frac{1}{2}k\,x_0^2, \end{aligned} \tag{23.3.5}$$

where we used the identity that $\cos^2\omega_0 t + \sin^2\omega_0 t = 1$, and that $\omega_0 = \sqrt{k/m}$ (Eq. (23.2.6)).

The mechanical energy in state 2 is equal to the initial potential energy in state 1, so the mechanical energy is constant. This should come as no surprise; we isolated the object-spring system so that there is no external work performed on the system and no internal non-conservative forces doing work.

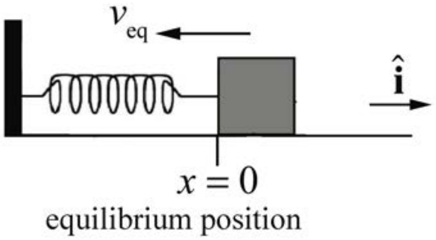


Figure 23.5 State 3 at equilibrium and in motion

State 3: now the object is at the equilibrium position so the potential energy is zero, $U_3 = 0$, and the mechanical energy is in the form of kinetic energy (Figure 23.5).

$$E_3 = K_3 = \frac{1}{2} m v_{\text{eq}}^2 . \tag{23.3.6}$$

Because the system is closed, mechanical energy is constant,

$$E_1 = E_3 . \tag{23.3.7}$$

Therefore the initial stored potential energy is released as kinetic energy,

$$\frac{1}{2} k x_0^2 = \frac{1}{2} m v_{\text{eq}}^2 , \tag{23.3.8}$$

and the x-component of velocity at the equilibrium position is given by

$$v_{\text{x,eq}} = \pm\sqrt{\frac{k}{m}} x_0 . \tag{23.3.9}$$

Note that the plus-minus sign indicates that when the block is at equilibrium, there are two possible motions: in the positive x-direction or the negative x-direction. If we take $x_0 > 0$, then the block starts moving towards the origin, and $v_{\text{x,eq}}$ will be negative the first time the block moves through the equilibrium position.

We can show more generally that the mechanical energy is constant at all times as follows. The mechanical energy at an arbitrary time is given by

$$E = K + U = \frac{1}{2} m v_x^{\,2} + \frac{1}{2} k x^2 \ . \qquad (23.3.10)$$

Differentiate Eq. (23.3.10)

$$\frac{dE}{dt} = m v_x \frac{dv_x}{dt} + k x \frac{dx}{dt} = v_x \left(m \frac{d^2x}{dt^2} + k x \right) \ . \qquad (23.3.11)$$

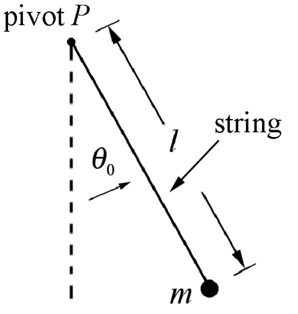
Now substitute the simple harmonic oscillator equation of motion, (Eq. (23.2.1)) into Eq. (23.3.11) yielding

$$\frac{dE}{dt} = 0 \ , \qquad (23.3.12)$$

demonstrating that the mechanical energy is a constant of the motion.

23.3.1 Simple Pendulum: Force Approach

A pendulum consists of an object hanging from the end of a string or rigid rod pivoted about the point P. The object is pulled to one side and allowed to oscillate. If the object has negligible size and the string or rod is massless, then the pendulum is called a ***simple pendulum***. Consider a simple pendulum consisting of a massless string of length l and a point-like object of mass m attached to one end, called the *bob*. Suppose the string is fixed at the other end and is initially pulled out at an angle θ_0 from the vertical and released from rest (Figure 23.6). Neglect any dissipation due to air resistance or frictional forces acting at the pivot.



object released from rest

Figure 23.6 Simple pendulum

Let's choose polar coordinates for the pendulum as shown in Figure 23.7a along with the free-body force diagram for the suspended object (Figure 23.7b). The angle θ is defined with respect to the equilibrium position. When $\theta > 0$, the bob is has moved to the right, and when $\theta < 0$, the bob has moved to the left. The object will move in a circular arc centered at the pivot point. The forces on the object are the tension in the string

$\vec{\mathbf{T}} = -T\,\hat{\mathbf{r}}$ and gravity $m\vec{\mathbf{g}}$. The gravitation force on the object has $\hat{\mathbf{r}}$ - and $\hat{\boldsymbol{\theta}}$ -components given by

$$m\vec{\mathbf{g}} = mg(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})\,. \tag{23.3.13}$$

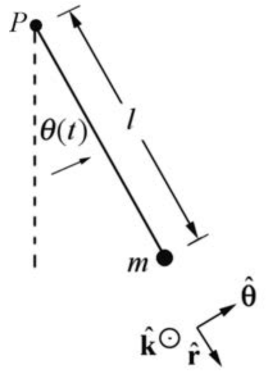


Figure 23.7 (a) Coordinate system

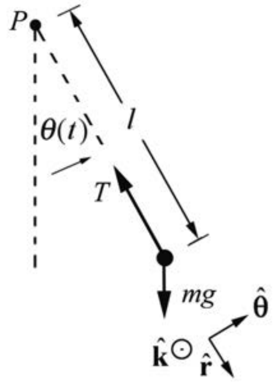


Figure 23.7 (b) free-body force diagram

Our concern is with the tangential component of the gravitational force,

$$F_\theta = -mg\sin\theta\,. \tag{23.3.14}$$

The sign in Eq. (23.3.14) is crucial; the tangential force tends to restore the pendulum to the equilibrium value $\theta = 0$. If $\theta > 0$, $F_\theta < 0$ and if $\theta < 0$, $F_\theta > 0$, where we are that because the string is flexible, the angle θ is restricted to the range $-\pi/2 < \theta < \pi/2$. (For angles $|\theta| > \pi/2$, the string would go slack.) In both instances the tangential component of the force is directed towards the equilibrium position. The tangential component of acceleration is

$$a_\theta = l\alpha = l\frac{d^2\theta}{dt^2}\,. \tag{23.3.15}$$

Newton's Second Law, $F_\theta = m\,a_\theta$, yields

$$-mgl\sin\theta = ml^2\frac{d^2\theta}{dt^2}\,. \tag{23.3.16}$$

We can rewrite this equation is the form

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta\,. \tag{23.3.17}$$

This is not the simple harmonic oscillator equation although it still describes periodic motion. In the limit of small oscillations, $\sin\theta \cong \theta$, Eq. (23.3.17) becomes

$$\frac{d^2\theta}{dt^2} \cong -\frac{g}{l}\theta. \tag{23.3.18}$$

This equation is similar to the object-spring simple harmonic oscillator differential equation

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x. \tag{23.3.19}$$

By comparison with Eq. (23.2.6) the angular frequency of oscillation for the pendulum is approximately

$$\omega_0 \simeq \sqrt{\frac{g}{l}}, \tag{23.3.20}$$

with period

$$T = \frac{2\pi}{\omega_0} \simeq 2\pi\sqrt{\frac{l}{g}}. \tag{23.3.21}$$

The solutions to Eq. (23.3.18) can be modeled after Eq. (23.2.21). With the initial conditions that the pendulum is released from rest, $\frac{d\theta}{dt}(t=0)=0$, at a small angle $\theta(t=0)=\theta_0$, the angle the string makes with the vertical as a function of time is given by

$$\theta(t) = \theta_0\cos(\omega_0 t) = \theta_0\cos\left(\frac{2\pi}{T}t\right) = \theta_0\cos\left(\sqrt{\frac{g}{l}}\,t\right). \tag{23.3.22}$$

The z-component of the angular velocity of the bob is

$$\omega_z(t) = \frac{d\theta}{dt}(t) = -\sqrt{\frac{g}{l}}\theta_0\sin\left(\sqrt{\frac{g}{l}}\,t\right). \tag{23.3.23}$$

Keep in mind that the component of the angular velocity $\omega_z = d\theta / dt$ changes with time in an oscillatory manner (sinusoidally in the limit of small oscillations). The angular frequency ω_0 is a parameter that describes the system. The z-component of the angular velocity $\omega_z(t)$, besides being time-dependent, depends on the amplitude of oscillation θ_0. In the limit of small oscillations, ω_0 does not depend on the amplitude of oscillation.

The fact that the period is independent of the mass of the object follows algebraically from the fact that the mass appears on both sides of Newton's Second Law and hence cancels. Consider also the argument that is attributed to Galileo: if a pendulum, consisting of two identical masses joined together, were set to oscillate, the two halves would not exert forces on each other. So, if the pendulum were split into two pieces, the pieces would oscillate the same as if they were one piece. This argument can be extended to simple pendula of arbitrary masses.

23.3.2 Simple Pendulum: Energy Approach

We can use energy methods to find the differential equation describing the time evolution of the angle θ. When the string is at an angle θ with respect to the vertical, the gravitational potential energy (relative to a choice of zero potential energy at the bottom of the swing where $\theta = 0$ as shown in Figure 23.8) is given by

$$U = mgl(1-\cos\theta) \tag{23.3.24}$$

The θ-component of the velocity of the object is given by $v_\theta = l(d\theta / dt)$ so the kinetic energy is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2 . \tag{23.3.25}$$

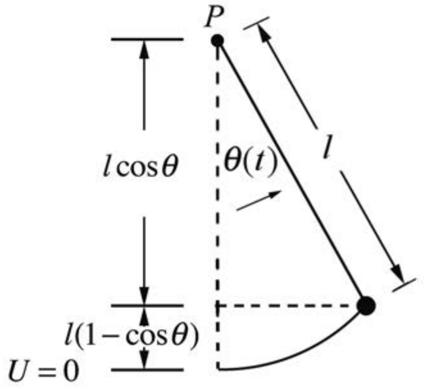


Figure 23.8 Energy diagram for simple pendulum

The mechanical energy of the system is then

$$E = K + U = \frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2 + mgl\left(1-\cos\theta\right). \tag{23.3.26}$$

Because we assumed that there is no non-conservative work (i.e. no air resistance or frictional forces acting at the pivot), the energy is constant, hence

$$\begin{aligned} 0 &= \frac{dE}{dt} = \frac{1}{2}m2l^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + mgl\sin\theta\frac{d\theta}{dt} \\ &= ml^2\frac{d\theta}{dt}\left(\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta\right). \end{aligned} \tag{23.3.27}$$

There are two solutions to this equation; the first one $d\theta / dt = 0$ is the equilibrium solution. That the z-component of the angular velocity is zero means the suspended object is not moving. The second solution is the one we are interested in

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0\,, \tag{23.3.28}$$

which is the same differential equation (Eq. (23.3.16)) that we found using the force method.

We can find the time t_1 that the object first reaches the bottom of the circular arc by setting $\theta(t_1) = 0$ in Eq. (23.3.22)

$$0 = \theta_0 \cos\left(\sqrt{\frac{g}{l}}\, t_1\right). \tag{23.3.29}$$

This zero occurs when the argument of the cosine satisfies

$$\sqrt{\frac{g}{l}}\, t_1 = \frac{\pi}{2}. \tag{23.3.30}$$

The z-component of the angular velocity at time t_1 is therefore

$$\frac{d\theta}{dt}(t_1) = -\sqrt{\frac{g}{l}}\,\theta_0 \sin\left(\sqrt{\frac{g}{l}}\, t_1\right) = -\sqrt{\frac{g}{l}}\,\theta_0 \sin\left(\frac{\pi}{2}\right) = -\sqrt{\frac{g}{l}}\,\theta_0\,. \tag{23.3.31}$$

Note that the negative sign means that the bob is moving in the negative $\hat{\boldsymbol{\theta}}$-direction when it first reaches the bottom of the arc. The θ-component of the velocity at time t_1 is therefore

$$v_\theta(t_1) \equiv v_1 = l\frac{d\theta}{dt}(t_1) = -l\sqrt{\frac{g}{l}}\,\theta_0 \sin\left(\sqrt{\frac{g}{l}}\, t_1\right) = -\sqrt{lg}\,\theta_0 \sin\left(\frac{\pi}{2}\right) = -\sqrt{lg}\,\theta_0\,. \tag{23.3.32}$$

We can also find the components of both the velocity and angular velocity using energy methods. When we release the bob from rest, the energy is only potential energy

$$E = U_0 = mgl\left(1 - \cos\theta_0\right) \cong mgl\frac{\theta_0^2}{2}\,, \tag{23.3.33}$$

where we used the approximation that $\cos\theta_0 \cong 1 - \theta_0^2 / 2$. When the bob is at the bottom of the arc, the only contribution to the mechanical energy is the kinetic energy given by

$$K_1 = \frac{1}{2} m v_1^{\,2} . \qquad (23.3.34)$$

Because the energy is constant, we have that $U_0 = K_1$ or

$$mgl \frac{\theta_0^2}{2} = \frac{1}{2} m v_1^{\,2} . \qquad (23.3.35)$$

We can solve for the θ-component of the velocity at the bottom of the arc

$$v_{\theta,1} = \pm \sqrt{gl}\, \theta_0 . \qquad (23.3.36)$$

The two possible solutions correspond to the different directions that the motion of the bob can have when at the bottom. The z-component of the angular velocity is then

$$\frac{d\theta}{dt}(t_1) = \frac{v_1}{l} = \pm \sqrt{\frac{g}{l}}\, \theta_0 , \qquad (23.3.37)$$

in agreement with our previous calculation.

If we do not make the small angle approximation, we can still use energy techniques to find the θ-component of the velocity at the bottom of the arc by equating the energies at the two positions

$$mgl\left(1 - \cos\theta_0\right) = \frac{1}{2} m v_1^{\,2} , \qquad (23.3.38)$$

$$v_{\theta,1} = \pm \sqrt{2gl\left(1 - \cos\theta_0\right)} . \qquad (23.3.39)$$

23.4 Worked Examples

Example 23.3: Rolling Without Slipping Oscillating Cylinder

Attach a solid cylinder of mass M and radius R to a horizontal massless spring with spring constant k so that it can roll without slipping along a horizontal surface. At time t, the center of mass of the cylinder is moving with speed V_{cm} and the spring is compressed a distance x from its equilibrium length. What is the period of simple harmonic motion for the center of mass of the cylinder?

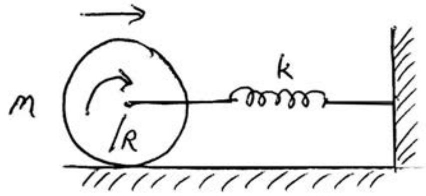


Figure 23.9 Example 23.3

Solution: At time t, the energy of the rolling cylinder and spring system is

$$E = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}kx^2 \ . \tag{23.4.1}$$

where x is the amount the spring has compressed, $I_{cm} = (1/2)MR^2$, and because it is rolling without slipping

$$\frac{d\theta}{dt} = \frac{V_{cm}}{R} \ . \tag{23.4.2}$$

Therefore the energy is

$$E = \frac{1}{2}MV_{cm}^2 + \frac{1}{4}MR^2\left(\frac{V_{cm}}{R}\right)^2 + \frac{1}{2}kx^2 = \frac{3}{4}MV_{cm}^2 + \frac{1}{2}kx^2 \ . \tag{23.4.3}$$

The energy is constant (no non-conservative force is doing work on the system) so

$$0 = \frac{dE}{dt} = \frac{3}{4}2MV_{cm}\frac{dV_{cm}}{dt} + \frac{1}{2}k2x\frac{dx}{dt} = V_{cm}(\frac{3}{2}M\frac{d^2x}{dt^2} + kx) \tag{23.4.4}$$

Because V_{cm} is non-zero most of the time, the displacement of the spring satisfies a simple harmonic oscillator equation

$$\frac{d^2x}{dt^2} + \frac{2k}{3M}x = 0 \ . \tag{23.4.5}$$

Hence the period is

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{3M}{2k}} \ . \tag{23.4.6}$$

Example 23.4: U-Tube

A U-tube open at both ends is filled with an incompressible fluid of density ρ. The cross-sectional area A of the tube is uniform and the total length of the fluid in the tube is L. A piston is used to depress the height of the liquid column on one side by a distance x_0, (raising the other side by the same distance) and then is quickly removed (Figure 23.10). What is the angular frequency of the ensuing simple harmonic motion? Neglect any resistive forces and at the walls of the U-tube.

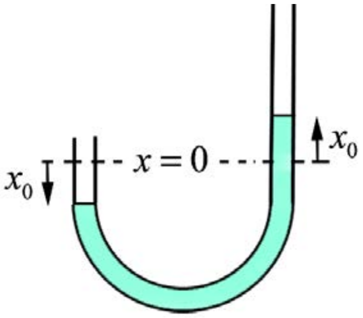


Figure 23.10 Example 23.4

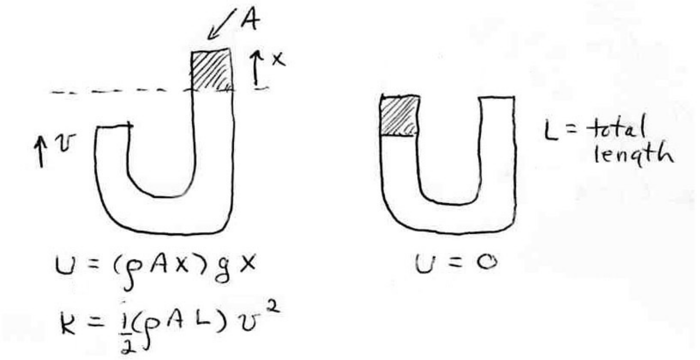


Figure 23.11 Energy diagram for water

Solution: We shall use conservation of energy. First choose as a zero for gravitational potential energy in the configuration where the water levels are equal on both sides of the tube. When the piston on one side depresses the fluid, it rises on the other. At a given instant in time when a portion of the fluid of mass $\Delta m = \rho A x$ is a height x above the equilibrium height (Figure 23.11), the potential energy of the fluid is given by

$$U = \Delta m g x = (\rho A x) g x = \rho A g x^2 \ . \tag{23.4.7}$$

At that same instant the entire fluid of length L and mass $m = \rho A L$ is moving with speed v, so the kinetic energy is

$$K = \frac{1}{2} m v^2 = \frac{1}{2} \rho A L v^2 \ . \tag{23.4.8}$$

Thus the total energy is

$$E = K + U = \frac{1}{2} \rho A L v^2 + \rho A g x^2 \ . \tag{23.4.9}$$

By neglecting resistive force, the mechanical energy of the fluid is constant. Therefore

$$0 = \frac{dE}{dt} = \rho A L v \frac{dv}{dt} + 2 \rho A g x \frac{dx}{dt} . \tag{23.4.10}$$

If we just consider the top of the fluid above the equilibrium position on the right arm in Figure 23.13, we rewrite Eq. (23.4.10) as

$$0 = \frac{dE}{dt} = \rho A L v_x \frac{dv_x}{dt} + 2 \rho A g x \frac{dx}{dt} \ , \tag{23.4.11}$$

where $v_x = dx / dt$. We now rewrite the energy condition using $dv_x / dt = d^2 x / dt^2$ as

$$0 = v_x \rho A \left(L\frac{d^2x}{dt^2} + 2gx \right) . \tag{23.4.12}$$

This condition is satisfied when $v_x = 0$, i.e. the equilibrium condition or when

$$0 = L\frac{d^2x}{dt^2} + 2gx . \tag{23.4.13}$$

This last condition can be written as

$$\frac{d^2x}{dt^2} = -\frac{2g}{L}x . \tag{23.4.14}$$

This last equation is the simple harmonic oscillator equation. Using the same mathematical techniques as we used for the spring-block system, the solution for the height of the fluid above the equilibrium position is given by

$$x(t) = B\cos(\omega_0 t) + C\sin(\omega_0 t) , \tag{23.4.15}$$

where

$$\omega_0 = \sqrt{\frac{2g}{L}} \tag{23.4.16}$$

is the angular frequency of oscillation. The x-component of the velocity of the fluid on the right-hand side of the U-tube is given by

$$v_x(t) = \frac{dx(t)}{dt} = -\omega_0 B\sin(\omega_0 t) + \omega_0 C\cos(\omega_0 t) . \tag{23.4.17}$$

The coefficients B and C are determined by the initial conditions. At $t = 0$, the height of the fluid is $x(t = 0) = B = x_0$. At $t = 0$, the speed is zero so $v_x(t = 0) = \omega_0 C = 0$, hence $C = 0$. The height of the fluid above the equilibrium position on the right hand-side of the U-tube as a function of time is thus

$$x(t) = x_0 \cos\left(\sqrt{\frac{2g}{L}}t \right) . \tag{23.4.18}$$

23.5 Damped Oscillatory Motion

Let’s now consider our spring-block system moving on a horizontal frictionless surface but now the block is attached to a damper that resists the motion of the block due to viscous friction. This damper, commonly called a *dashpot,* is shown in Figure 23.13. The viscous force arises when objects move through fluids at speeds slow enough so that there is no turbulence. When the viscous force opposes the motion and is proportional to the velocity, so that

$$\vec{\mathbf{f}}_{\text{vis}} = -b\vec{\mathbf{v}}\,, \tag{23.5.1}$$

the dashpot is referred to as a *linear dashpot*. The constant of proportionality b depends on the properties of the dashpot.

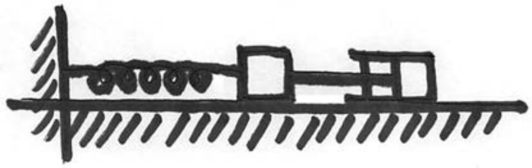


Figure 23.12 Spring-block system connected to a linear dashpot

Choose the origin at the equilibrium position and choose the positive x-direction to the right in the Figure 23.13. Define $x(t)$ to be the position of the object with respect to the equilibrium position. The x-component of the total force acting on the spring is the sum of the linear restoring spring force, and the viscous friction force (Figure 23.13),

$$F_x = -k\,x - b\frac{dx}{dt} \tag{23.5.2}$$

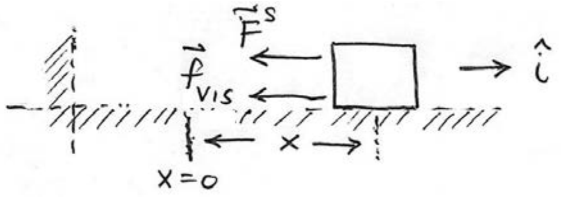


Figure 23.13 Free-body force diagram for spring-object system with linear dashpot

Newton’s Second law in the x-direction becomes

$$-k\,x - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}\,. \tag{23.5.3}$$

We can rewrite Eq. (23.5.3) as

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0\,. \tag{23.5.4}$$

When $(b/m)^2 < 4k/m$, the oscillator is called *underdamped*, and the solution to Eq. (23.5.4) is given by

$$x(t) = x_{\text{m}}e^{-\alpha t}\cos(\gamma t + \phi) \tag{23.5.5}$$

where $\gamma = (k/m - (b/2m)^2)^{1/2}$ is the angular frequency of oscillation, $\alpha = b/2m$ is a parameter that measured the exponential decay of the oscillations, x_{m} is a constant and ϕ is the phase constant. Recall the undamped oscillator has angular frequency $\omega_0 = (k/m)^{1/2}$, so the angular frequency of the underdamped oscillator can be expressed as

$$\gamma = (\omega_0{}^2 - \alpha^2)^{1/2} . \tag{23.5.6}$$

In Appendix 23B: Complex Numbers, we introduce complex numbers and use them to solve Eq.(23.5.4) in Appendix 23C: Solution to the Underdamped Simple Harmonic Oscillator Equation.

The x-component of the velocity of the object is given by

$$v_x(t) = dx/dt = \left(-\gamma x_{\mathrm{m}} \sin(\gamma t + \phi) - \alpha x_{\mathrm{m}} \cos(\gamma t + \phi)\right) e^{-\alpha t} . \tag{23.5.7}$$

The position and the x-component of the velocity of the object oscillate but the amplitudes of the oscillations decay exponentially. In Figure 23.14, the position is plotted as a function of time for the underdamped system for the special case $\phi = 0$. For that case

$$x(t) = x_{\mathrm{m}} e^{-\alpha t} \cos(\gamma t) . \tag{23.5.8}$$

and

$$v_x(t) = dx/dt = \left(-\gamma x_{\mathrm{m}} \sin(\gamma t) - \alpha x_{\mathrm{m}} \cos(\gamma t)\right) e^{-\alpha t} . \tag{23.5.9}$$

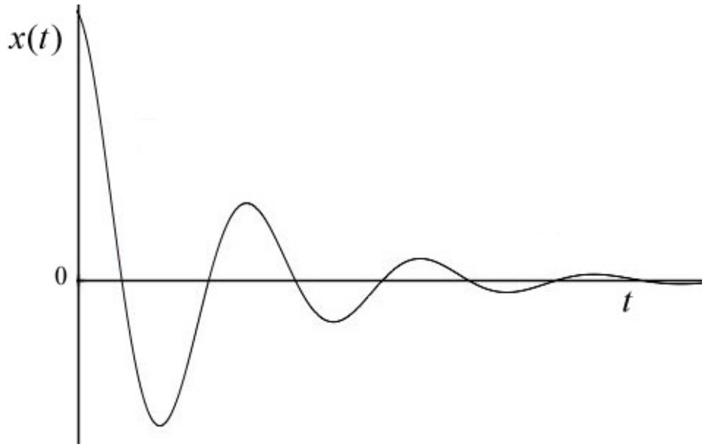


Figure 23.14 Plot of position $x(t)$ of object for underdamped oscillator with $\phi = 0$

Because the coefficient of exponential decay $\alpha = b/2m$ is proportional to the b, we see that the position will decay more rapidly if the viscous force increases. We can introduce a *time constant*

$$\tau = 1/\alpha = 2m/b . \tag{23.5.10}$$

When $t = \tau$, the position is

$$x(t = \tau) = x_{\mathrm{m}} \cos(\gamma\tau) e^{-1} . \qquad (23.5.11)$$

The envelope of exponential decay has now decreases by a factor of e^{-1}, i.e. the amplitude can be at most $x_{\mathrm{m}} e^{-1}$. During this time interval $[0, \tau]$, the position has undergone a number of oscillations. The total number of radians associated with those oscillations is given by

$$\gamma\tau = (k / m - (b / 2m)^2)^{1/2} (2m / b) . \qquad (23.5.12)$$

The closest integral number of cycles is then

$$n = \left[\gamma\tau / 2\pi \right] = \left[(k / m - (b / 2m)^2)^{1/2} (m / \pi b) \right] . \qquad (23.5.13)$$

If the system is very weakly damped, such that $(b / m)^2 << 4k / m$, then we can approximate the number of cycles by

$$n = \left[\gamma\tau / 2\pi \right] \simeq \left[(k / m)^{1/2} (m / \pi b) \right] = \left[\omega_0 (m / \pi b) \right] , \qquad (23.5.14)$$

where $\omega_0 = (k / m)^{1/2}$ is the angular frequency of the undamped oscillator.

We define the *quality*, Q, of this oscillating system to be proportional to the number of integral cycles it takes for the exponential envelope of the position function to fall off by a factor of e^{-1}. The constant of proportionality is chosen to be π. Thus

$$Q = n\pi . \qquad (23.5.15)$$

For the weakly damped case, we have that

$$Q \simeq \omega_0 (m / b) . \qquad (23.5.16)$$

23.5.1 Energy in the Underdamped Oscillator

For the underdamped oscillator, $(b / m)^2 < 4k / m$, $\gamma = (k / m - (b / 2m)^2)^{1/2}$, and $\alpha = b/2m$. Let's choose $t = 0$ such that the phase shift is zero $\phi = 0$. The stored energy in the system will decay due to the energy loss due to dissipation. The mechanical energy stored in the potential and kinetic energies is then given by

$$E = \frac{1}{2} k x^2 + \frac{1}{2} m v^2 . \qquad (23.5.17)$$

where the position and the x-component of the velocity are given by Eqs. (23.5.8) and (23.5.9). The mechanical energy is then

$$E = \frac{1}{2} k x_{\mathrm{m}}{}^{2} \cos^{2}(\gamma t) e^{-2\alpha t} + \frac{1}{2} m \left(-\gamma x_{\mathrm{m}} \sin(\gamma t) - \alpha x_{\mathrm{m}} \cos(\gamma t)\right)^{2} e^{-2\alpha t}. \quad (23.5.18)$$

Expanding this expression yields

$$E = \frac{1}{2}(k + m\alpha^{2}) x_{\mathrm{m}}{}^{2} \cos^{2}(\gamma t) e^{-2\alpha t} + m\gamma\alpha x_{\mathrm{m}}{}^{2} \sin(\gamma t)\cos(\gamma t) e^{-2\alpha t} + \frac{1}{2} m\gamma^{2} x_{\mathrm{m}}{}^{2} \sin^{2}(\gamma t) e^{-2\alpha t} \quad (23.5.19)$$

The kinetic energy, potential energy, and mechanical energy are shown in Figure 23.15.

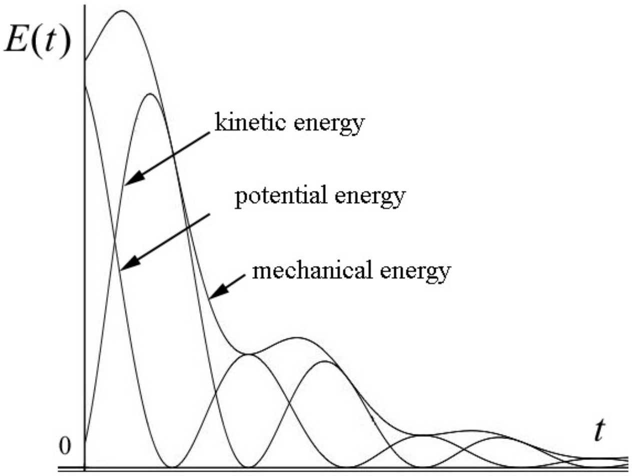


Figure 23.15 Kinetic, potential and mechanical energy for the underdamped oscillator

The stored energy at time $t = 0$ is

$$E(t = 0) = \frac{1}{2}(k + m\alpha^{2}) x_{\mathrm{m}}{}^{2} \quad (23.5.20)$$

The mechanical energy at the conclusion of one cycle, with $\gamma T = 2\pi$, is

$$E(t = T) = \frac{1}{2}(k + m\alpha^{2}) x_{\mathrm{m}}{}^{2} e^{-2\alpha T} \quad (23.5.21)$$

The change in the mechanical energy for one cycle is then

$$E(t = T) - E(t = 0) = -\frac{1}{2}(k + m\alpha^{2}) x_{\mathrm{m}}{}^{2} (1 - e^{-2\alpha T}). \quad (23.5.22)$$

Recall that $\alpha^2 = b^2/4m^2$. Therefore

$$E(t=T) - E(t=0) = -\frac{1}{2}(k + b^2/4m)x_{\mathrm{m}}{}^2(1 - e^{-2\alpha T}) . \tag{23.5.23}$$

We can show (although the calculation is lengthy) that the energy dissipated by the viscous force over one cycle is given by the integral

$$E_{\mathrm{dis}} = \int_0^T \vec{\mathbf{F}}_{\mathrm{vis}} \cdot \vec{\mathbf{v}}\, dt = -\left(k + \frac{b^2}{4m} \right) \frac{x_{\mathrm{m}}{}^2}{2} (1 - e^{-2\alpha t}) . \tag{23.5.24}$$

By comparison with Eq. (23.5.23), the change in the mechanical energy in the underdamped oscillator during one cycle is equal to the energy dissipated due to the viscous force during one cycle.

23.6 Forced Damped Oscillator

Let's drive our damped spring-object system by a sinusoidal force. Suppose that the x-component of the driving force is given by

$$F_x(t) = F_0 \cos(\omega t) \ , \tag{23.6.1}$$

where F_0 is called the *amplitude* (maximum value) and ω is the *driving angular frequency.* The force varies between F_0 and $-F_0$ because the cosine function varies between $+1$ and -1. Define $x(t)$ to be the position of the object with respect to the equilibrium position. The x-component of the force acting on the object is now the sum

$$F_x = F_0 \cos(\omega t) - kx - b\frac{dx}{dt} \ . \tag{23.6.2}$$

Newton's Second law in the x-direction becomes

$$F_0 \cos(\omega t) - kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2} \ . \tag{23.6.3}$$

We can rewrite Eq. (23.6.3) as

$$F_0 \cos(\omega t) = m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx \ . \tag{23.6.4}$$

We derive the solution to Eq. (23.6.4) in Appendix 23E: Solution to the forced Damped Oscillator Equation. The solution to is given by the function

$$x(t) = x_0 \cos(\omega t + \phi) \ , \tag{23.6.5}$$

where the amplitude x_0 is a function of the driving angular frequency ω and is given by

$$x_0(\omega) = \frac{F_0 / m}{\left((b/m)^2 \omega^2 + (\omega_0^{\ 2} - \omega^2)^2\right)^{1/2}} \ . \tag{23.6.6}$$

The phase constant ϕ is also a function of the driving angular frequency ω and is given by

$$\phi(\omega) = \tan^{-1}\left(\frac{(b/m)\omega}{\omega^2 - \omega_0^{\ 2}} \right) . \tag{23.6.7}$$

In Eqs. (23.6.6) and (23.6.7)

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{23.6.8}$$

is the natural angular frequency associated with the undriven undamped oscillator. The x -component of the velocity can be found by differentiating Eq. (23.6.5),

$$v_x(t) = \frac{dx}{dt}(t) = -\omega x_0 \sin(\omega t + \phi) \ , \tag{23.6.9}$$

where the amplitude $x_0(\omega)$ is given by Eq. (23.6.6) and the phase constant $\phi(\omega)$ is given by Eq. (23.6.7).

23.6.1 Resonance

When $b / m << 2\omega_0$ we say that the oscillator is *lightly damped*. For a lightly-damped driven oscillator, after a transitory period, the position of the object will oscillate with the same angular frequency as the driving force. The plot of amplitude $x_0(\omega)$ vs. driving angular frequency ω for a lightly damped forced oscillator is shown in Figure 23.16. If the angular frequency is increased from zero, the amplitude of the $x_0(\omega)$ will increase until it reaches a maximum when the angular frequency of the driving force is the same as the natural angular frequency, ω_0, associated with the undamped oscillator. This is called **resonance.** When the driving angular frequency is increased above the natural angular frequency the amplitude of the position oscillations diminishes.

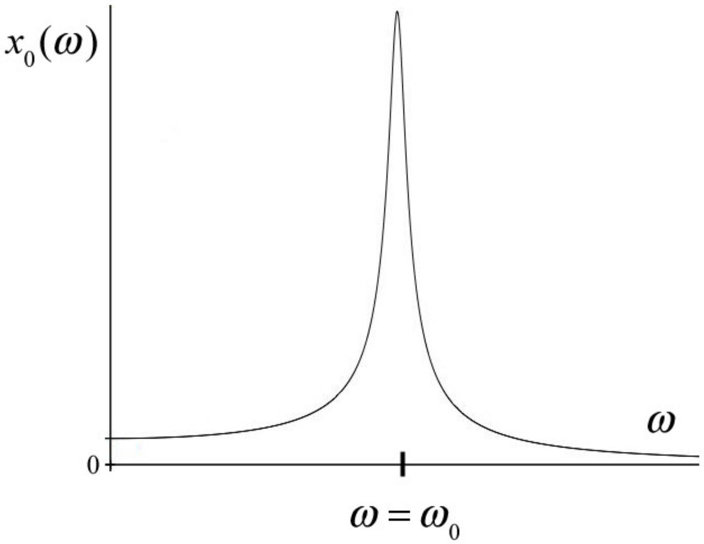


Figure 23.16 Plot of amplitude $x_0(\omega)$ vs. driving angular frequency ω for a lightly damped oscillator with $b/m << 2\omega_0$

We can find the angular frequency such that the amplitude $x_0(\omega)$ is at a maximum by setting the derivative of Eq. (23.6.6) equal to zero,

$$0 = \frac{d}{dt}x_0(\omega) = -\frac{F_0(2\omega)}{2m}\frac{((b/m)^2 - 2(\omega_0{}^2 - \omega^2))}{\left((b/m)^2\omega^2 + (\omega_0{}^2 - \omega^2)^2\right)^{3/2}} \ . \tag{23.6.10}$$

This vanishes when

$$\omega = (\omega_0{}^2 - (b/m)^2/2)^{1/2} \ . \tag{23.6.11}$$

For the lightly-damped oscillator, $\omega_0 >> (1/2)b/m$, and so the maximum value of the amplitude occurs when

$$\omega \simeq \omega_0 = (k/m)^{1/2} \ . \tag{23.6.12}$$

The amplitude at resonance is then

$$x_0(\omega = \omega_0) = \frac{F_0}{b\omega_0} \ \text{(lightly damped)} \ . \tag{23.6.13}$$

The plot of phase constant $\phi(\omega)$ vs. driving angular frequency ω for a lightly damped forced oscillator is shown in Figure 23.17.

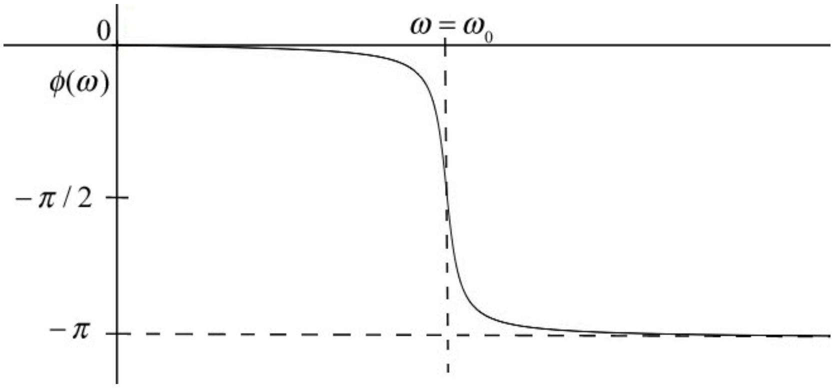


Figure 23.17 Plot of phase constant $\phi(\omega)$ vs. driving angular frequency ω for a lightly damped oscillator with $b/m << 2\omega_0$

The phase constant at resonance is zero,

$$\phi(\omega = \omega_0) = 0 \ . \tag{23.6.14}$$

At resonance, the x-component of the velocity is given by

$$v_x(t) = \frac{dx}{dt}(t) = -\frac{F_0}{b}\sin(\omega_0 t) \quad \text{(lightly damped)} \ . \tag{23.6.15}$$

When the oscillator is not lightly damped ($b/m \simeq \omega_0$), the resonance peak is shifted to the left of $\omega = \omega_0$ as shown in the plot of amplitude vs. angular frequency in Figure 23.18. The corresponding plot of phase constant vs. angular frequency for the non-lightly damped oscillator is shown in Figure 23.19.

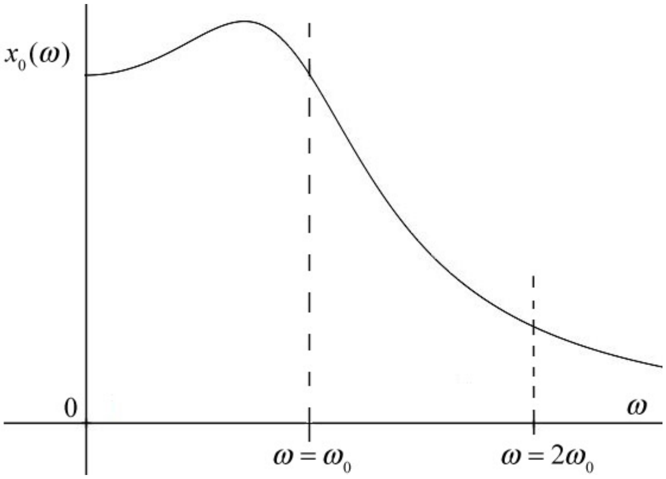


Figure 23.18 Plot of amplitude vs. angular frequency for lightly-damped driven oscillator where $b/m \simeq \omega_0$

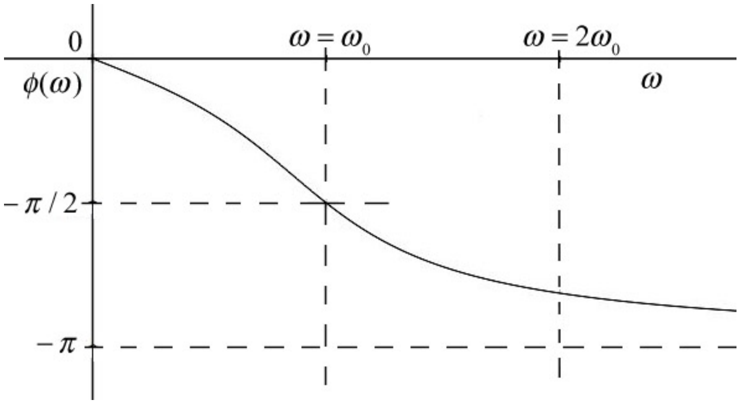


Figure 23.19 Plot of phase constant vs. angular frequency for lightly-damped driven oscillator where $b/m \simeq \omega_0$

23.6.2 Mechanical Energy

The kinetic energy for the driven damped oscillator is given by

$$K(t) = \frac{1}{2}mv^2(t) = \frac{1}{2}m\omega^2 x_0{}^2 \sin^2(\omega t + \phi) \ . \tag{23.6.16}$$

The potential energy is given by

$$U(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kx_0{}^2 \cos^2(\omega t + \phi) \ . \tag{23.6.17}$$

The mechanical energy is then

$$E(t) = \frac{1}{2}mv^2(t) + \frac{1}{2}kx^2(t) = \frac{1}{2}m\omega^2 x_0{}^2 \sin^2(\omega t + \phi) + \frac{1}{2}kx_0{}^2 \cos^2(\omega t + \phi) \ . \tag{23.6.18}$$

Example 23.5: Time-Averaged Mechanical Energy

The period of one cycle is given by $T = 2\pi/\omega$. Show that

(i)
$$\frac{1}{T}\int_0^T \sin^2(\omega t + \phi)\, dt = \frac{1}{2} \ , \tag{23.6.19}$$

(ii)
$$\frac{1}{T}\int_0^T \cos^2(\omega t + \phi)\, dt = \frac{1}{2} \ , \tag{23.6.20}$$

(iii)
$$\frac{1}{T}\int_0^T \sin(\omega t)\cos(\omega t)\, dt = 0 \ . \tag{23.6.21}$$

Solution: (i) We use the trigonometric identity

$$\sin^2(\omega t+\phi)) = \frac{1}{2}(1-\cos(2(\omega t+\phi)) \qquad (23.6.22)$$

to rewrite the integral in Eq. (23.6.19) as

$$\frac{1}{T}\int_0^T \sin^2(\omega t+\phi))\,dt = \frac{1}{2T}\int_0^T (1-\cos(2(\omega t+\phi))\,dt \qquad (23.6.23)$$

Integration yields

$$\begin{aligned}\frac{1}{2T}\int_0^T (1-\cos(2(\omega t+\phi))\,dt &= \frac{1}{2}-\left(\frac{\sin(2(\omega t+\phi))}{2\omega}\right)\Bigg|_{T=0}^{T=2\pi/\omega} \\ &= \frac{1}{2}-\left(\frac{\sin(4\pi+2\phi)}{2\omega}-\frac{\sin(2\phi)}{2\omega}\right)=\frac{1}{2},\end{aligned} \qquad (23.6.24)$$

where we used the trigonometric identity that

$$\sin(4\pi+2\phi) = \sin(4\pi)\cos(2\phi)+\sin(2\phi)\cos(4\pi) = \sin(2\phi)\ , \qquad (23.6.25)$$

proving Eq. (23.6.19).

(ii) We use a similar argument starting with the trigonometric identity that

$$\cos^2(\omega t+\phi)) = \frac{1}{2}(1+\cos(2(\omega t+\phi))\ . \qquad (23.6.26)$$

Then

$$\frac{1}{T}\int_0^T \cos^2(\omega t+\phi))\,dt = \frac{1}{2T}\int_0^T (1+\cos(2(\omega t+\phi))\,dt\ . \qquad (23.6.27)$$

Integration yields

$$\begin{aligned}\frac{1}{2T}\int_0^T (1+\cos(2(\omega t+\phi))\,dt &= \frac{1}{2}+\left(\frac{\sin(2(\omega t+\phi))}{2\omega}\right)\Bigg|_{T=0}^{T=2\pi/\omega} \\ &= \frac{1}{2}+\left(\frac{\sin(4\pi+2\phi)}{2\omega}-\frac{\sin(2\phi)}{2\omega}\right)=\frac{1}{2}.\end{aligned} \qquad (23.6.28)$$

(iii) We first use the trigonometric identity that

$$\sin(\omega t)\cos(\omega t) = \frac{1}{2}\sin(\omega t) \ . \tag{23.6.29}$$

Then

$$\begin{aligned}\frac{1}{T}\int_0^T \sin(\omega t)\cos(\omega t)\,dt &= \frac{1}{T}\int_0^T \sin(\omega t)\,dt \\ = -\frac{1}{T}\frac{\cos(\omega t)}{2\omega}\bigg|_0^T &= -\frac{1}{2\omega T}(1-1) = 0.\end{aligned} \tag{23.6.30}$$

The values of the integrals in Example 23.5 are called the **time-averaged values**. We denote the time-average value of a function $f(t)$ over one period by

$$\langle f \rangle \equiv \frac{1}{T}\int_0^T f(t)\,dt \ . \tag{23.6.31}$$

In particular, the time-average kinetic energy as a function of the angular frequency is given by

$$\langle K(\omega)\rangle = \frac{1}{4}m\omega^2 x_0^{\ 2} \ . \tag{23.6.32}$$

The time-averaged potential energy as a function of the angular frequency is given by

$$\langle U(\omega)\rangle = \frac{1}{4}kx_0^{\ 2} \ . \tag{23.6.33}$$

The time-averaged value of the mechanical energy as a function of the angular frequency is given by

$$\langle E(\omega)\rangle = \frac{1}{4}m\omega^2 x_0^{\ 2} + \frac{1}{4}kx_0^{\ 2} = \frac{1}{4}(m\omega^2 + k)x_0^{\ 2} \ . \tag{23.6.34}$$

We now substitute Eq. (23.6.6) for the amplitude into Eq. (23.6.34) yielding

$$\langle E(\omega)\rangle = \frac{F_0^{\ 2}}{4m}\frac{(\omega_0^{\ 2} + \omega^2)}{\left((b/m)^2\omega^2 + \left(\omega_0^{\ 2} - \omega^2\right)^2\right)} \ . \tag{23.6.35}$$

A plot of the time-averaged energy versus angular frequency for the lightly-damped case ($b/m << 2\omega_0$) is shown in Figure 23.20.

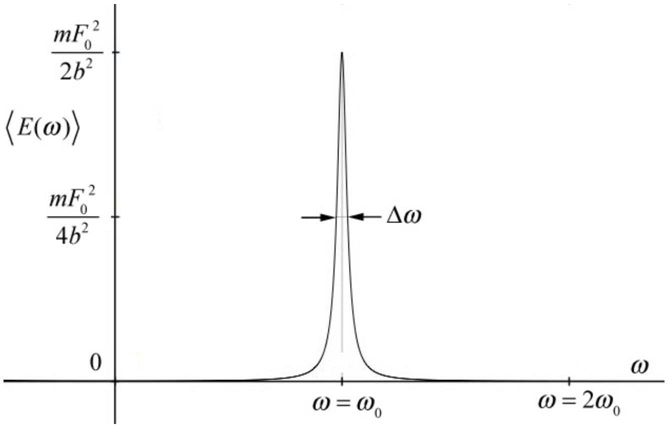


Figure 23.20 Plot of the time-averaged energy versus angular frequency for the lightly-damped case ($b/m << 2\omega_0$)

We can simplify the expression for the time-averaged energy for the lightly-damped case by observing that the time-averaged energy is nearly zero everywhere except where $\omega=\omega_0$, (see Figure 23.20). We first substitute $\omega=\omega_0$ everywhere in Eq. (23.6.35) except the term $\omega_0{}^2-\omega^2$ that appears in the denominator, yielding

$$\langle E(\omega)\rangle=\frac{F_0{}^2}{2m}\frac{(\omega_0{}^2)}{\left((b/m)^2\omega_0{}^2+\left(\omega_0{}^2-\omega^2\right)^2\right)}. \tag{23.6.36}$$

We can approximate the term

$$\omega_0{}^2-\omega^2=(\omega_0-\omega)(\omega_0+\omega)\simeq 2\omega_0(\omega_0-\omega) \tag{23.6.37}$$

Then Eq. (23.6.36) becomes

$$\langle E(\omega)\rangle=\frac{F_0{}^2}{2m}\frac{1}{\left((b/m)^2+4(\omega_0-\omega)^2\right)} \quad \text{(lightly damped)}. \tag{23.6.38}$$

The right-hand expression of Eq. (23.6.38) takes on its maximum value when the denominator has its minimum value. By inspection, this occurs when $\omega=\omega_0$. Alternatively, to find the maximum value, we set the derivative of Eq. (23.6.35) equal to zero and solve for ω ,

$$
\begin{aligned}
0 &= \frac{d}{d\omega}\left\langle E(\omega)\right\rangle = \frac{d}{d\omega}\frac{F_0^{\,2}}{2m}\frac{1}{\left((b/m)^2+4(\omega_0-\omega)^2\right)} \\
&= \frac{4F_0^{\,2}}{m}\frac{(\omega_0-\omega)}{\left((b/m)^2+4(\omega_0-\omega)^2\right)^2}
\end{aligned}
. \qquad (23.6.39)
$$

The maximum occurs when occurs at $\omega = \omega_0$ and has the value

$$
\left\langle E(\omega_0)\right\rangle = \frac{mF_0^{\,2}}{2b^2} \quad \text{(underdamped)}. \qquad (23.6.40)
$$

23.6.3 The Time-averaged Power

The time-averaged power delivered by the driving force is given by the expression

$$
\left\langle P(\omega)\right\rangle = \frac{1}{T}\int_0^T F_x v_x \, dt = -\frac{1}{T}\int_0^T \frac{F_0^{\,2}\omega\cos(\omega t)\sin(\omega t+\phi)}{m\left((b/m)^2\omega^2+(\omega_0^{\,2}-\omega^2)^2\right)^{1/2}}dt \ , \qquad (23.6.41)
$$

where we used Eq. (23.6.1) for the driving force, and Eq. (23.6.9) for the x-component of the velocity of the object. We use the trigonometric identity

$$
\sin(\omega t+\phi) = \sin(\omega t)\cos(\phi) + \cos(\omega t)\sin(\phi) \qquad (23.6.42)
$$

to rewrite the integral in Eq. (23.6.41) as two integrals

$$
\begin{aligned}
\left\langle P(\omega)\right\rangle &= -\frac{1}{T}\int_0^T \frac{F_0^{\,2}\omega\cos(\omega t)\sin(\omega t)\cos(\phi)}{m\left((b/m)^2\omega^2+(\omega_0^{\,2}-\omega^2)^2\right)^{1/2}}dt \\
&-\frac{1}{T}\int_0^T \frac{F_0^{\,2}\omega\cos^2(\omega t)\sin(\phi)}{m\left((b/m)^2\omega^2+(\omega_0^{\,2}-\omega^2)^2\right)^{1/2}}dt.
\end{aligned}
\qquad (23.6.43)
$$

Using the time-averaged results from Example 23.5, we see that the first term in Eq. (23.6.43) is zero and the second term becomes

$$
\left\langle P(\omega)\right\rangle = \frac{F_0^{\,2}\omega\sin(\phi)}{2m\left((b/m)^2\omega^2+(\omega_0^{\,2}-\omega^2)^2\right)^{1/2}} \qquad (23.6.44)
$$

For the underdamped driven oscillator, we make the same approximations in Eq. (23.6.44) that we made for the time-averaged energy. In the term in the numerator and the term on the left in the denominator, we set $\omega \simeq \omega_0$, and we use Eq. (23.6.37) in the term on the right in the denominator yielding

$$\left\langle P(\omega) \right\rangle = \frac{F_0^{\;2}\sin(\phi)}{2m\left((b/m)^2 + 2(\omega_0 - \omega)\right)^{1/2}} \qquad \text{(underdamped)} \; . \qquad (23.6.45)$$

The time-averaged power dissipated by the resistive force is given by

$$\begin{aligned}\left\langle P_{\text{dis}}(\omega) \right\rangle &= \frac{1}{T}\int_0^T (F_x)_{\text{dis}} v_x \, dt = -\frac{1}{T}\int_0^T b v_x^{\;2}\, dt = \frac{1}{T}\int_0^T \frac{F_0^{\;2}\omega^2 \sin^2(\omega t + \phi)dt}{m^2\left((b/m)^2\omega^2 + (\omega_0^{\;2} - \omega^2)^2\right)} \\ &= \frac{F_0^{\;2}\omega^2 dt}{2m^2\left((b/m)^2\omega^2 + (\omega_0^{\;2} - \omega^2)^2\right)} .\end{aligned} \quad ,(23.6.46)$$

where we used Eq. (23.5.1) for the dissipative force, Eq. (23.6.9) for the x-component of the velocity of the object, and Eq. (23.6.19) for the time-averaging.

23.6.4 Quality Factor

The plot of the time-averaged energy vs. the driving angular frequency for the underdamped oscullator has a width, $\Delta\omega$ (Figure 23.20). One way to characterize this width is to define $\Delta\omega = \omega_+ - \omega_-$, where $\omega_\pm$ are the values of the angular frequency such that time-averaged energy is equal to one half its maximum value

$$\left\langle E(\omega_\pm) \right\rangle = \frac{1}{2}\left\langle E(\omega_0) \right\rangle = \frac{mF_0^{\;2}}{4b^2} . \qquad (23.6.47)$$

The quantity $\Delta\omega$ is called the *line width at half energy maximum* also known as the *resonance width.* We can now solve for $\omega_\pm$ by setting

$$\left\langle E(\omega_\pm) \right\rangle = \frac{F_0^{\;2}}{2m}\frac{1}{\left((b/m)^2 + 4(\omega_0 - \omega_\pm)^2\right)} = \frac{mF_0^{\;2}}{4b^2} , \qquad (23.6.48)$$

yielding the condition that

$$(b/m)^2 = 4(\omega_0 - \omega_\pm)^2 . \qquad (23.6.49)$$

Taking square roots of Eq. (23.6.49) yields

$$\mp(b/2m) = \omega_0 - \omega_\pm . \qquad (23.6.50)$$

Therefore

$$\omega_{\pm} = \omega_0 \pm (b/2m) . \qquad (23.6.51)$$

The half-width is then

$$\Delta\omega = \omega_{+} - \omega_{-} = (\omega_0 + (b/2m)) - (\omega_0 - (b/2m)) = b/m \ . \qquad (23.6.52)$$

We define the quality Q of the resonance as the ratio of the resonant angular frequency to the line width,

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{b/m} \ . \qquad (23.6.53)$$

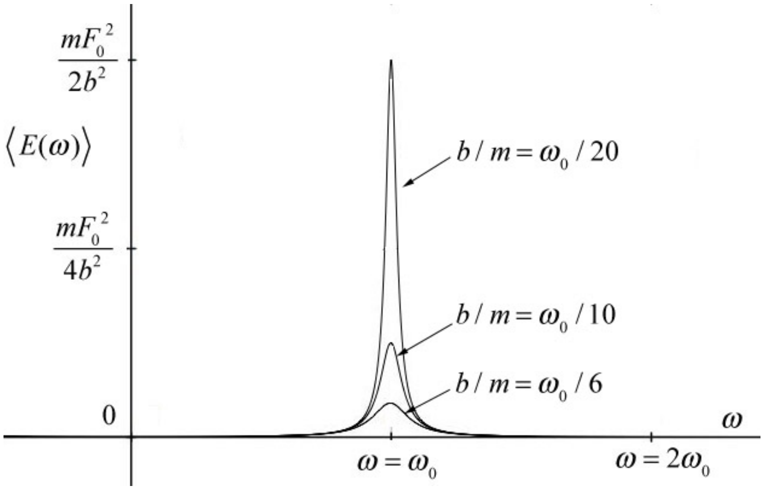


Figure 23.21 Plot of time-averaged energy vs. angular frequency for different values of b/m

In Figure 23.21 we plot the time-averaged energy vs. angular frequency for several different values of the quality factor $Q = 10, 5,$ and 3. Recall that this was the same result that we had for the quality of the free oscillations of the damped oscillator, Eq. (23.5.16) (because we chose the factor π in Eq. (23.5.16)).

23.7 Small Oscillations

Any object moving subject to a force associated with a potential energy function that is quadratic will undergo simple harmonic motion,

$$U(x) = U_0 + \frac{1}{2}k(x - x_{eq})^2 . \qquad (23.7.1)$$

where k is a "spring constant", x_{eq} is the equilibrium position, and the constant U_0 just depends on the choice of reference point x_{ref} for zero potential energy, $U(x_{ref}) = 0$,

$$0 = U(x_{ref}) = U_0 + \frac{1}{2}k(x_{ref} - x_{eq})^2 . \qquad (23.7.2)$$

Therefore the constant is

$$U_0 = -\frac{1}{2}k(x_{ref} - x_{eq})^2 . \qquad (23.7.3)$$

The minimum of the potential x_0 corresponds to the point where the x-component of the force is zero,

$$\left.\frac{dU}{dx}\right|_{x=x_0} = 2k(x_0 - x_{eq}) = 0 \Rightarrow x_0 = x_{eq} , \qquad (23.7.4)$$

corresponding to the equilibrium position. Therefore the constant is $U(x_0) = U_0$ and we rewrite our potential function as

$$U(x) = U(x_0) + \frac{1}{2}k(x - x_0)^2 . \qquad (23.7.5)$$

Now suppose that a potential energy function is not quadratic but still has a minimum at x_0. For example, consider the potential energy function

$$U(x) = -U_1\left(\left(\frac{x}{x_1}\right)^3 - \left(\frac{x}{x_1}\right)^2\right), \qquad (23.7.6)$$

(Figure 23.22), which has a stable minimum at x_0.

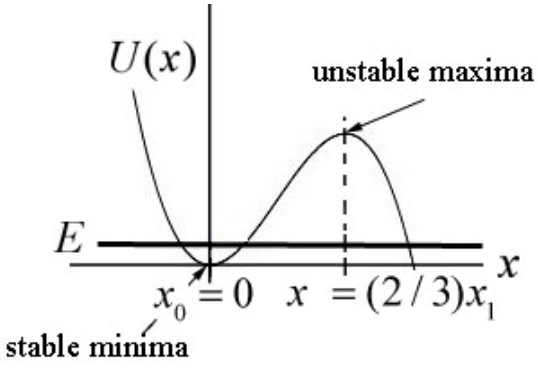


Figure 23.22 Potential energy function with stable minima and unstable maxima

When the energy of the system is very close to the value of the potential energy at the minimum $U(x_0)$, we shall show that the system will undergo small oscillations about the minimum value x_0. We shall use the Taylor formula to approximate the potential function as a polynomial. We shall show that near the minimum x_0, we can approximate the potential function by a quadratic function similar to Eq. (23.7.5) and show that the system undergoes simple harmonic motion for small oscillations about the minimum x_0.

We begin by expanding the potential energy function about the minimum point using the Taylor formula

$$U(x) = U(x_0) + \left.\frac{dU}{dx}\right|_{x=x_0}(x - x_0) + \frac{1}{2!}\left.\frac{d^2U}{dx^2}\right|_{x=x_0}(x - x_0)^2 + \frac{1}{3!}\left.\frac{d^3U}{dx^3}\right|_{x=x_0}(x - x_0)^3 + \cdots \quad (23.7.7)$$

where $\frac{1}{3!}\left.\frac{d^3U}{dx^3}\right|_{x=x_0}(x - x_0)^3$ is a third order term in that it is proportional to $(x - x_0)^3$, and $\left.\frac{d^3U}{dx^3}\right|_{x=x_0}$, $\left.\frac{d^2U}{dx^2}\right|_{x=x_0}$, and $\left.\frac{dU}{dx}\right|_{x=x_0}$ are constants. If x_0 is the minimum of the potential energy, then the linear term is zero, because

$$\left.\frac{dU}{dx}\right|_{x=x_0} = 0 \quad (23.7.8)$$

and so Eq. ((23.7.7)) becomes

$$U(x) \simeq U(x_0) + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x=x_0}(x - x_0)^2 + \frac{1}{3!}\left.\frac{d^3U}{dx^3}\right|_{x=x_0}(x - x_0)^3 + \cdots \quad (23.7.9)$$

For small displacements from the equilibrium point such that $|x - x_0|$ is sufficiently small, the third order term and higher order terms are very small and can be ignored. Then the potential energy function is approximately a quadratic function,

$$U(x) \simeq U(x_0) + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x=x_0}(x - x_0)^2 = U(x_0) + \frac{1}{2}k_{eff}(x - x_0)^2 \quad (23.7.10)$$

where we define k_{eff}, the ***effective spring constant***, by

$$k_{eff} \equiv \left.\frac{d^2U}{dx^2}\right|_{x=x_0}. \quad (23.7.11)$$

Because the potential energy function is now approximated by a quadratic function, the system will undergo simple harmonic motion for small displacements from the minimum with a force given by

$$F_x = -\frac{dU}{dx} = -k_{eff}(x - x_0)\,. \tag{23.7.12}$$

At $x = x_0$, the force is zero

$$F_x(x_0) = \frac{dU}{dx}(x_0) = 0\,. \tag{23.7.13}$$

We can determine the period of oscillation by substituting Eq. (23.7.12) into Newton's Second Law

$$-k_{eff}(x - x_0) = m_{eff}\frac{d^2x}{dt^2} \tag{23.7.14}$$

where m_{eff} is the ***effective mass.*** For a two-particle system, the effective mass is the reduced mass of the system.

$$m_{eff} = \frac{m_1 m_2}{m_1 + m_2} \equiv \mu_{red}\,, \tag{23.7.15}$$

Eq. (23.7.14) has the same form as the spring-object ideal oscillator. Therefore the angular frequency of small oscillations is given by

$$\omega_0 = \sqrt{\frac{k_{eff}}{m_{eff}}} = \sqrt{\frac{\left.\dfrac{d^2U}{dx^2}\right|_{x=x_0}}{m_{eff}}}\;. \tag{23.7.16}$$

Example 23.6: Quartic Potential

A system with effective mass m has a potential energy given by

$$U(x) = U_0\left(-2\left(\frac{x}{x_0}\right)^2 + \left(\frac{x}{x_0}\right)^4\right), \tag{23.7.17}$$

where U_0 and x_0 are positive constants and $U(0) = 0$. (a) Find the points where the force on the particle is zero. Classify these points as stable or unstable. Calculate the value of $U(x)/U_0$ at these equilibrium points. (b) If the particle is given a small displacement from an equilibrium point, find the angular frequency of small oscillation.

Solution: (a) A plot of $U(x)/U_0$ as a function of x/x_0 is shown in Figure 23.23.

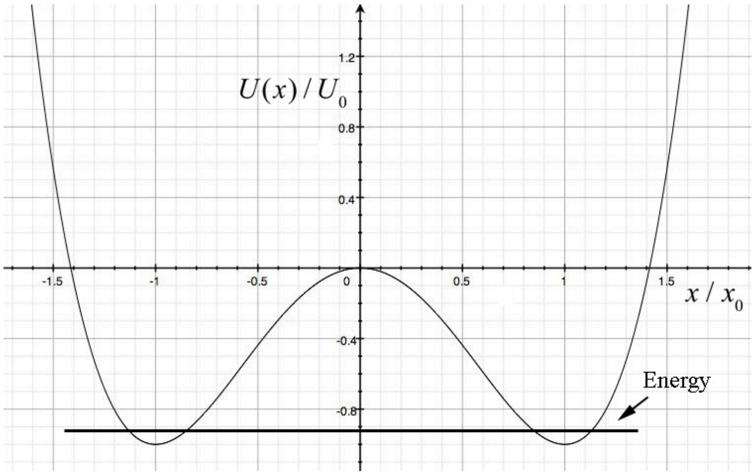


Figure 22.23 Plot of $U(x)/U_0$ as a function of x/x_0

The force on the particle is zero at the minimum of the potential energy,

$$\begin{aligned} 0 = \frac{dU}{dx} &= U_0\left(-4\left(\frac{1}{x_0}\right)^2 x + 4\left(\frac{1}{x_0}\right)^4 x^3\right) \\ &= -4U_0 x\left(\frac{1}{x_0}\right)^2\left(1-\left(\frac{x}{x_0}\right)^2\right) \Rightarrow x^2 = {x_0}^2 \text{ and } x = 0. \end{aligned} \tag{23.7.18}$$

The equilibrium points are at $x = \pm x_0$ which are stable and $x = 0$ which is unstable. The second derivative of the potential energy is given by

$$\frac{d^2U}{dx^2} = U_0\left(-4\left(\frac{1}{x_0}\right)^2 + 12\left(\frac{1}{x_0}\right)^4 x^2\right) . \tag{23.7.19}$$

If the particle is given a small displacement from $x = x_0$ then

$$\left.\frac{d^2U}{dx^2}\right|_{x=x_0} = U_0\left(-4\left(\frac{1}{x_0}\right)^2 + 12\left(\frac{1}{x_0}\right)^4 x_0^2\right) = U_0\frac{8}{x_0^2} . \tag{23.7.20}$$

(b) The angular frequency of small oscillations is given by

$$\omega_0 = \sqrt{\left.\frac{d^2U}{dx^2}\right|_{x=x_0} / m} = \sqrt{\frac{8U_0}{mx_0^2}} . \tag{23.7.21}$$

Example 23.7: Lennard-Jones 6-12 Potential

A commonly used potential energy function to describe the interaction between two atoms is the Lennard-Jones 6-12 potential

$$U(r) = U_0\left[(r_0/r)^{12} - 2(r_0/r)^6\right];\ r > 0\ , \tag{23.7.22}$$

where r is the distance between the atoms. Find the angular frequency of small oscillations about the stable equilibrium position for two identical atoms bound to each other by the Lennard-Jones interaction. Let m denote the effective mass of the system of two atoms.

Solution: The equilibrium points are found by setting the first derivative of the potential energy equal to zero,

$$0 = \frac{dU}{dr} = U_0\left[-12r_0^{12}r^{-13} + 12r_0^{6}r^{-7}\right] = U_0 12r_0^{6}r^{-7}\left[-\left(\frac{r_0}{r}\right)^6 + 1\right]. \tag{23.7.23}$$

The equilibrium point occurs when $r = r_0$. The second derivative of the potential energy function is

$$\frac{d^2U}{dr^2} = U_0\left[+(12)(13)r_0^{12}r^{-14} - (12)(7)r_0^{6}r^{-8}\right]. \tag{23.7.24}$$

Evaluating this at $r = r_0$ yields

$$\left.\frac{d^2U}{dr^2}\right|_{r=r_0} = 72U_0 r_0^{-2}\ . \tag{23.7.25}$$

The angular frequency of small oscillation is therefore

$$\omega_0 = \sqrt{\left.\frac{d^2U}{dr^2}\right|_{r=r_0} / m} = \sqrt{72U_0 / mr_0^{2}}\ . \tag{23.7.26}$$

Appendix 23A: Solution to Simple Harmonic Oscillator Equation

In our analysis of the solution of the simple harmonic oscillator equation of motion, Equation (23.2.1),

$$-k\,x = m\frac{d^2x}{dt^2}, \tag{23.A.1}$$

we assumed that the solution was a linear combination of sinusoidal functions,

$$x(t) = A\cos(\omega_0\, t) + B\sin(\omega_0\, t), \tag{23.A.2}$$

where $\omega_0 = \sqrt{k/m}$. We shall now derive Eq. (23.A.2).

Assume that the mechanical energy of the spring-object system is given by the constant E. Choose the reference point for potential energy to be the unstretched position of the spring. Let x denote the amount the spring has been compressed ($x < 0$) or stretched ($x > 0$) from equilibrium at time t and denote the amount the spring has been compressed or stretched from equilibrium at time $t = 0$ by $x(t=0) \equiv x_0$. Let $v_x = dx/dt$ denote the x-component of the velocity at time t and denote the x-component of the velocity at time $t = 0$ by $v_x(t=0) \equiv v_{x,0}$. The constancy of the mechanical energy is then expressed as

$$E = K + U = \frac{1}{2}k\,x^2 + \frac{1}{2}m\,v^2. \tag{23.A.3}$$

We can solve Eq. (23.A.3) for the square of the x-component of the velocity,

$$v_x^{\,2} = \frac{2E}{m} - \frac{k}{m}x^2 = \frac{2E}{m}\left(1 - \frac{k}{2E}x^2\right). \tag{23.A.4}$$

Taking square roots, we have

$$\frac{dx}{dt} = \sqrt{\frac{2E}{m}}\sqrt{1 - \frac{k}{2E}x^2}\,. \tag{23.A.5}$$

(why we take the positive square root will be explained below).

Let $a_1 \equiv \sqrt{2E/m}$ and $a_2 \equiv k/2E$. It's worth noting that a_1 has dimensions of velocity and w has dimensions of [length]$^{-2}$. Eq. (23.A.5) is separable,

$$\begin{aligned}\frac{dx}{dt} &= a_1\sqrt{1-a_2\,x^2}\\ \frac{dx}{\sqrt{1-a_2\,x^2}} &= a_1\,dt.\end{aligned} \tag{23.A.6}$$

We now integrate Eq. (23.A.6),

$$\int \frac{dx}{\sqrt{1-a_1\,x^2}} = \int a_1\,dt \ . \tag{23.A.7}$$

The integral on the left in Eq. (23.A.7) is well known, and a derivation is presented here. We make a change of variables $\cos\theta = \sqrt{a_2}\ x$ with the differentials $d\theta$ and dx related by $-\sin\theta\ d\theta = \sqrt{a_2}\ dx$. The integration variable is

$$\theta = \cos^{-1}\left(\sqrt{a_2}\ x\right). \tag{23.A.8}$$

Eq. (23.A.7) then becomes

$$\int \frac{-\sin\theta\, d\theta}{\sqrt{1-\cos^2\theta}} = \int \sqrt{a_2}\ a_1\ dt\ . \tag{23.A.9}$$

This is a good point at which to check the dimensions. The term on the left in Eq. (23.A.9) is dimensionless, and the product $\sqrt{a_2}\ a_1$ on the right has dimensions of inverse time, $[\text{length}]^{-1}[\text{length}\cdot\text{time}^{-1}] = [\text{time}^{-1}]$, so $\sqrt{a_2}\ a_1\ dt$ is dimensionless. Using the trigonometric identity $\sqrt{1-\cos^2\theta} = \sin\theta$, Eq. (23.A.9) reduces to

$$\int d\theta = -\int \sqrt{a_2}\ a_1\ \ dt\ . \tag{23.A.10}$$

Although at this point in the derivation we don’t know that $\sqrt{a_2}\ a_1$, which has dimensions of frequency, is the angular frequency of oscillation, we’ll use some foresight and make the identification

$$\omega_0 \equiv \sqrt{a_2}\ a_1 = \sqrt{\frac{k}{2E}}\sqrt{\frac{2E}{m}} = \sqrt{\frac{k}{m}}\,, \tag{23.A.11}$$

and Eq. (23.A.10) becomes

$$\int_{\theta=\theta_0}^{\theta} d\theta = -\int_{t=0}^{t} \omega_0\, dt\,. \tag{23.A.12}$$

After integration we have

$$\theta - \theta_0 = -\omega_0\, t\,, \tag{23.A.13}$$

where $\theta_0 \equiv -\phi$ is the constant of integration. Because $\theta = \cos^{-1}\left(\sqrt{a_2}\; x(t)\right)$, Eq. (23.A.13) becomes

$$\cos^{-1}\left(\sqrt{a_2}\; x(t)\right) = -(\omega_0\, t + \phi)\,. \tag{23.A.14}$$

Take the cosine of each side of Eq. (23.A.14), yielding

$$x(t) = \frac{1}{\sqrt{a_2}}\cos(-(\omega_0\, t + \phi)) = \sqrt{\frac{2E}{k}}\cos(\omega_0\, t + \phi)\,. \tag{23.A.15}$$

At $t = 0$,

$$x_0 \equiv x(t = 0) = \sqrt{\frac{2E}{k}}\cos\phi\,. \tag{23.A.16}$$

The x-component of the velocity as a function of time is then

$$v_x(t) = \frac{dx(t)}{dt} = -\omega_0\sqrt{\frac{2E}{k}}\sin(\omega_0\, t + \phi)\,. \tag{23.A.17}$$

At $t = 0$,

$$v_{x,0} \equiv v_x(t = 0) = -\omega_0\sqrt{\frac{2E}{k}}\sin\phi\,. \tag{23.A.18}$$

We can determine the constant ϕ by dividing the expressions in Eqs. (23.A.18) and (23.A.16),

$$-\frac{v_{x,0}}{\omega_0 x_0} = \tan\phi. \tag{23.A.19}$$

Thus the constant ϕ can be determined by the initial conditions and the angular frequency of oscillation,

$$\phi = \tan^{-1}\left(-\frac{v_{x,0}}{\omega_0 x_0}\right). \tag{23.A.20}$$

Use the identity

$$\cos(\omega_0 t + \phi) = \cos(\omega_0 t)\cos(\phi) - \sin(\omega_0 t)\sin(\phi) \tag{23.A.21}$$

to expand Eq. (23.A.15) yielding

$$x(t) = \sqrt{\frac{2E}{k}}\cos(\omega_0 t)\cos(\phi) - \sqrt{\frac{2E}{k}}\sin(\omega_0 t)\sin(\phi), \tag{23.A.22}$$

and substituting Eqs. (23.A.16) and (23.A.18) into Eq. (23.A.22) yields

$$x(t) = x_0 \cos\omega_0 t + \frac{v_{x,0}}{\omega_0}\sin\omega_0 t, \tag{23.A.23}$$

agreeing with Eq. (23.2.21).

So, what about the missing ± that should have been in Eq. (23.A.5)? Strictly speaking, we would need to redo the derivation for the block moving in different directions. Mathematically, this would mean replacing ϕ by $\pi - \phi$ (or $\phi - \pi$) when the block's velocity changes direction. Changing from the positive square root to the negative *and* changing ϕ to $\pi - \phi$ have the collective action of reproducing Eq. (23.A.23).

Appendix 23B: Complex Numbers

A complex number z can be written as a sum of a real number x and a purely imaginary number iy where $i = \sqrt{-1}$,

$$z = x + iy. \tag{23.B.1}$$

The complex number can be represented as a point in the x-y plane as show in Figure 23B.1.

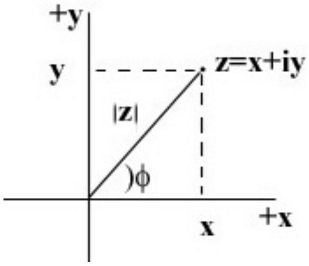


Figure 23B.1 Complex numbers

The complex conjugate $\overline{z}$ of a complex number z is defined to be

$$\overline{z} = x - iy. \tag{23.B.2}$$

The modulus of a complex number is

$$\left|z\right| = (z\overline{z})^{1/2} = ((x+iy)(x-iy))^{1/2} = (x^2 + y^2)^{1/2}. \tag{23.B.3}$$

where we used the fact that $i^2 = -1$. The modulus $|z|$ represents the length of the ray from the origin to the complex number z in Figure 23B.1. Let ϕ denote the angle that the ray with the positive x-axis in Figure 23B.1. Then

$$x = |z|\cos\phi, \tag{23.B.4}$$

$$y = |z|\sin\phi. \tag{23.B.5}$$

Hence the angle ϕ is given by

$$\phi = \tan^{-1}(y/x). \tag{23.B.6}$$

The inverse of a complex number is then

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x - iy}{(x^2 + y^2)}. \tag{23.B.7}$$

The modulus of the inverse is the inverse of the modulus;

$$\left|\frac{1}{z}\right| = \frac{1}{(x^2 + y^2)^{1/2}} = \frac{1}{|z|}. \tag{23.B.8}$$

The sum of two complex numbers, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, is the complex number

$$z_3 = z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) = x_3 + iy_3, \tag{23.B.9}$$

where $x_3 = x_1 + x_2$, $y_3 = y_1 + y_2$. We can represent this by the vector sum in Figure 23B.2,

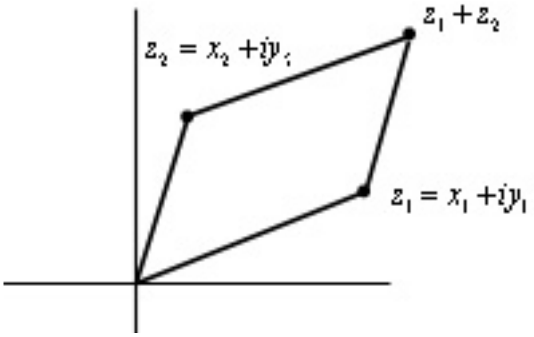


Figure 23B.2 Sum of two complex numbers

The product of two complex numbers is given by

$$z_3 = z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) = x_3 + iy_3, \quad (23.B.10)$$

where $x_3 = x_1 x_2 - y_1 y_2$, and $y_3 = x_1 y_2 + x_2 y_1$.

One of the most important identities in mathematics is the Euler formula,

$$e^{i\phi} = \cos\phi + i\sin\phi. \quad (23.B.11)$$

This identity follows from the power series representations for the exponential, sine, and cosine functions,

$$e^{i\phi} = \sum_{n=0}^{n=\infty} \frac{1}{n!}(i\phi)^n = 1 + i\phi - \frac{\phi^2}{2} - i\frac{\phi^3}{3!} + \frac{\phi^4}{4!} + i\frac{\phi^5}{5!}..., \quad (23.B.12)$$

$$\cos\phi = 1 - \frac{\phi^2}{2} + \frac{\phi^4}{4!} - ..., \quad (23.B.13)$$

$$\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - .... \quad (23.B.14)$$

We define two projection operators. The first one takes the complex number $e^{i\phi}$ and gives its real part,

$$\operatorname{Re} e^{i\phi} = \cos\phi. \quad (23.B.15)$$

The second operator takes the complex number $e^{i\phi}$ and gives its imaginary part, which is the real number

$$\operatorname{Im} e^{i\phi} = \sin\phi \; . \quad (23.B.16)$$

A complex number $z = x + iy$ can also be represented as the product of a modulus $|z|$ and a phase factor $e^{i\phi}$,

$$z = |z| e^{i\phi}. \quad (23.B.17)$$

The inverse of a complex number is then

$$\frac{1}{z} = \frac{1}{|z| e^{i\phi}} = \frac{1}{|z|} e^{-i\phi}, \quad (23.B.18)$$

where we used the fact that

$$\frac{1}{e^{i\phi}} = e^{-i\phi}. \quad (23.B.19)$$

In terms of modulus and phase, the sum of two complex numbers, $z_1 = |z_1| e^{i\phi_1}$ and $z_2 = |z_2| e^{i\phi_2}$, is

$$z_1 + z_2 = |z_1| e^{i\phi_1} + |z_2| e^{i\phi_2} . \tag{23.B.20}$$

A special case of this result is when the phase angles are equal, $\phi_1 = \phi_2$, then the sum $z_1 + z_2$ has the same phase factor $e^{i\phi_1}$ as z_1 and z_2,

$$z_1 + z_2 = |z_1| e^{i\phi_1} + |z_2| e^{i\phi_1} = \left(|z_1| + |z_2|\right) e^{i\phi_1} . \tag{23.B.21}$$

The product of two complex numbers, $z_1 = |z_1| e^{i\phi_1}$, and $z_2 = |z_2| e^{i\phi_2}$ is

$$z_1 z_2 = |z_1| e^{i\phi_1} |z_2| e^{i\phi_2} = |z_1||z_2| e^{i\phi_1 + \phi_2} . \tag{23.B.22}$$

When the phases are equal, the product does not have the same factor as z_1 and z_2,

$$z_1 z_2 = |z_1| e^{i\phi_1} |z_2| e^{i\phi_1} = |z_1||z_2| e^{i2\phi_1} . \tag{23.B.23}$$

Appendix 23C: Solution to the Underdamped Simple Harmonic Oscillator

Consider the underdamped simple harmonic oscillator equation (Eq. (23.5.4)),

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 . \tag{23.C.1}$$

When $(b/m)^2 < 4k/m$, we show that the equation has a solution of the form

$$x(t) = x_{\mathrm{m}} e^{-\alpha t} \cos(\gamma t + \phi) . \tag{23.C.2}$$

Solution: Let's suppose the function $x(t)$ has the form

$$x(t) = A\,\mathrm{Re}(e^{zt}) \tag{23.C.3}$$

where z is a number (possibly complex) and A is a real number. Then

$$\frac{dx}{dt} = zAe^{zt} \tag{23.C.4}$$

$$\frac{d^2x}{dt^2} = z^2 Ae^{zt} \tag{23.C.5}$$

We now substitute Eqs. (23.C.3), (23.C.4), and (23.C.5), into Eq. (23.C.1) resulting in

$$z^2 Ae^{zt} + \frac{b}{m} zAe^{zt} + \frac{k}{m} Ae^{zt} = 0 . \tag{23.C.6}$$

Collecting terms in Eq. (23.C.6) yields

$$\left(z^2 + \frac{b}{m} z + \frac{k}{m} \right) Ae^{zt} = 0 \tag{23.C.7}$$

The condition for the solution is that

$$z^2 + \frac{b}{m} z + \frac{k}{m} = 0 \ . \tag{23.C.8}$$

This quadratic equation has solutions

$$z = \frac{-(b/m) \pm ((b/m)^2 - 4k/m)^{1/2}}{2} \ . \tag{23.C.9}$$

When $(b/m)^2 < 4k/m$, the oscillator is called *underdamped*, and we have two solutions for z, however the solutions are complex numbers. Let

$$\gamma = (k/m - (b/2m)^2)^{1/2} ; \tag{23.C.10}$$

and

$$\alpha = b/2m . \tag{23.C.11}$$

.

Recall that the imaginary number $i = \sqrt{-1}$. The two solutions are then $z_1 = -\alpha + i\gamma t$ and $z_2 = -\alpha - i\gamma t$. Because our system is linear, our general solution is a linear combination of these two solutions,

$$x(t) = A_1 e^{-\alpha + i\gamma t} + A_2 e^{-\alpha - i\gamma t} = (A_1 e^{i\gamma t} + A_2 e^{-i\gamma t}) e^{-\alpha t} , \tag{23.C.12}$$

where A_1 and A_2 are constants. We shall transform this expression into a more familiar equation involving sine and cosine functions with help from the Euler formula,

$$e^{\pm i\gamma t} = \cos(\gamma t) \pm i \sin(\gamma t) . \tag{23.C.13}$$

Therefore we can rewrite our solution as

$$x(t) = \left(A_1 (\cos(\gamma t) + i \sin(\gamma t)) + A_2 (\cos(\gamma t) - i \sin(\gamma t)) \right) e^{-\alpha t} . \tag{23.C.14}$$

A little rearrangement yields

$$x(t) = \big((A_1 + A_2)\cos(\gamma t) + i(A_1 - A_2)\sin(\gamma t)\big)e^{-\alpha t}. \tag{23.C.15}$$

Define two new constants $C = A_1 + A_2$ and $D = i(A_1 - A_2)$. Then our solution looks like

$$x(t) = (C\cos(\gamma t) + D\sin(\gamma t))e^{-\alpha t}. \tag{23.C.16}$$

Recall from Example 23.5 that we can rewrite

$$C\cos(\gamma t) + D\sin(\gamma t) = x_{\mathrm{m}}\cos(\gamma t + \phi) \tag{23.C.17}$$

,

where

$$x_{\mathrm{m}} = (C^2 + D^2)^{1/2}\text{, and } \phi = \tan^{-1}(D / C).$$

Then our general solution for the underdamped case (Eq. (23.C.16)) can be written as

$$x(t) = x_{\mathrm{m}}e^{-\alpha t}\cos(\gamma t + \phi). \tag{23.C.18}$$

There are two other possible cases which we shall not analyze: when $(b / m)^2 > 4k / m$, a case referred to as *overdamped*, and when $(b / m)^2 = 4k / m$, a case referred to as *critically damped*.

Appendix 23D: Solution to the Forced Damped Oscillator Equation

We shall now use complex numbers to solve the differential equation

$$F_0\cos(\omega t) = m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx \ . \tag{23.D.1}$$

We begin by assuming a solution of the form

$$x(t) = x_0\cos(\omega t + \phi) \ . \tag{23.D.2}$$

where the amplitude x_0 and the phase constant ϕ need to be determined. We begin by defining the complex function

$$z(t) = x_0 e^{i(\omega t + \phi)} \ . \tag{23.D.3}$$

Our desired solution can be found by taking the real projection

$$x(t) = \mathrm{Re}(z(t)) = x_0 \cos(\omega t + \phi) \ . \tag{23.D.4}$$

Our differential equation can now be written as

$$F_0 e^{i\omega t} = m\frac{d^2 z}{dt^2} + b\frac{dz}{dt} + kz \ . \tag{23.D.5}$$

We take the first and second derivatives of Eq. (23.D.3),

$$\frac{dz}{dt}(t) = i\omega x_0 e^{i(\omega t+\phi)} = i\omega z \ . \tag{23.D.6}$$

$$\frac{d^2 z}{dt^2}(t) = -\omega^2 x_0 e^{i(\omega t+\phi)} = -\omega^2 z \ . \tag{23.D.7}$$

We substitute Eqs. (23.D.3), (23.D.6), and (23.D.7) into Eq. (23.D.5) yielding

$$F_0 e^{i\omega t} = (-\omega^2 m + bi\omega + k)z = (-\omega^2 m + bi\omega + k)x_0 e^{i(\omega t+\phi)} \ . \tag{23.D.8}$$

We divide Eq. (23.D.8) through by $e^{i\omega t}$ and collect terms using yielding

$$x_0 e^{i\phi} = \frac{F_0 / m}{((\omega_0{}^2 - \omega^2) + i(b/m)\omega)} \ . \tag{23.D.9}$$

where we have used $\omega_0{}^2 = k/m$. Introduce the complex number

$$z_1 = (\omega_0{}^2 - \omega^2) + i(b/m)\omega \ . \tag{23.D.10}$$

Then Eq. (23.D.9) can be written as

$$x_0 e^{i\phi} = \frac{F_0}{my} \ . \tag{23.D.11}$$

Multiply the numerator and denominator of Eq. (23.D.11) by the complex conjugate $\overline{z}_1 = (\omega_0{}^2 - \omega^2) - i(b/m)\omega$ yielding

$$x_0 e^{i\phi} = \frac{F_0 \overline{z}_1}{m z_1 \overline{z}_1} = \frac{F_0}{m}\frac{((\omega_0{}^2 - \omega^2) - i(b/m)\omega)}{((\omega_0{}^2 - \omega^2)^2 + (b/m)^2\omega^2)} \equiv u + iv \ . \tag{23.D.12}$$

where

$$u = \frac{F_0}{m}\frac{(\omega_0{}^2 - \omega^2)}{((\omega_0{}^2 - \omega^2)^2 + (b/m)^2\omega^2)} \ , \tag{23.D.13}$$

$$v = -\frac{F_0}{m}\frac{(b/m)\omega}{((\omega_0{}^2-\omega^2)^2+(b/m)^2\omega^2)} \quad . \tag{23.D.14}$$

Therefore the modulus x_0 is given by

$$x_0 = (u^2+v^2)^{1/2} = \frac{F_0/m}{((\omega_0{}^2-\omega^2)^2+(b/m)^2\omega^2)} \quad , \tag{23.D.15}$$

and the phase is given by

$$\phi = \tan^{-1}(v/u) = \frac{-(b/m)\omega}{(\omega_0{}^2-\omega^2)} \quad . \tag{23.D.16}$$

Chapter 24 Physical Pendulum

Chapter 24 Physical Pendulum

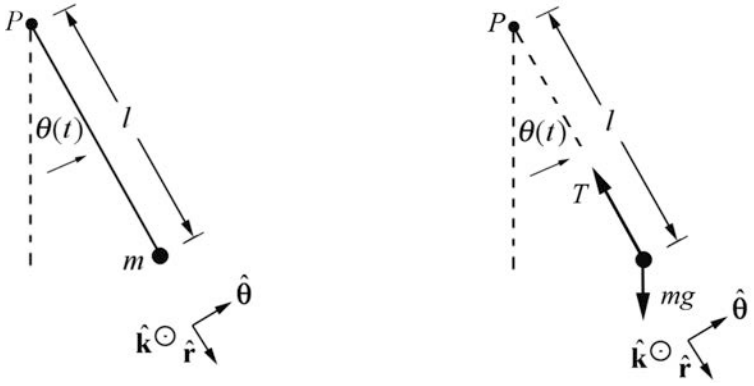
> *…. I had along with me….the Descriptions, with some Drawings of the principal Parts of the Pendulum-Clock which I had made, and as also of them of my then intended Timekeeper for the Longitude at Sea.*[1]
>
> *John Harrison*

24.1 Introduction

We have already used Newton’s Second Law or Conservation of Energy to analyze systems like the spring-object system that oscillate. We shall now use torque and the rotational equation of motion to study oscillating systems like pendulums and torsional springs.

24.1.1 Simple Pendulum: Torque Approach

Recall the simple pendulum from Chapter 23.3.1.The coordinate system and force diagram for the simple pendulum is shown in Figure 24.1.



(a) (b)

Figure 24.1 (a) Coordinate system and (b) torque diagram for simple pendulum

The torque about the pivot point P is given by

$$\vec{\boldsymbol{\tau}}_P = \vec{\mathbf{r}}_{P,m} \times m\vec{\mathbf{g}} = l\,\hat{\mathbf{r}} \times m\,g(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}) = -l\,m\,g\sin\theta\,\hat{\mathbf{k}} \qquad (24.1.1)$$

The z -component of the torque about point P

$$(\tau_P)_z = -mgl\sin\theta\,. \qquad (24.1.2)$$

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[1] J. Harrison, *A Description Concerning Such Mechanisms as will Afford a Nice, or True Mensuration of Time;…*(London, 1775), p.19.

When $\theta > 0$, $(\tau_P)_z < 0$ and the torque about P is directed in the negative $\hat{\mathbf{k}}$-direction (into the plane of Figure 24.1b) when $\theta < 0$, $(\tau_P)_z > 0$ and the torque about P is directed in the positive $\hat{\mathbf{k}}$-direction (out of the plane of Figure 24.1b). The moment of inertia of a point mass about the pivot point P is $I_P = ml^2$. The rotational equation of motion is then

$$\begin{aligned}(\tau_P)_z &= I_P\alpha_z \equiv I_P\frac{d^2\theta}{dt^2}\\ -mgl\sin\theta &= ml^2\frac{d^2\theta}{dt^2}.\end{aligned} \tag{24.1.3}$$

Thus we have

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta\,, \tag{24.1.4}$$

agreeing with Eq. 23. 3.14. When the angle of oscillation is small, we may use the small angle approximation

$$\sin\theta \cong \theta\,, \tag{24.1.5}$$

and Eq. (24.1.4) reduces to the simple harmonic oscillator equation

$$\frac{d^2\theta}{dt^2} \cong -\frac{g}{l}\theta\,. \tag{24.1.6}$$

We have already studied the solutions to this equation in Chapter 23.3. A procedure for determining the period when the small angle approximation does not hold is given in Appendix 24A.

24.2 Physical Pendulum

A ***physical pendulum*** consists of a rigid body that undergoes fixed axis rotation about a fixed point S (Figure 24.2).

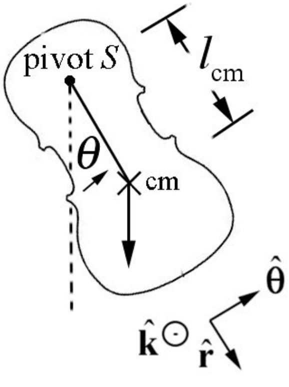


Figure 24.2 Physical pendulum

The gravitational force acts at the center of mass of the physical pendulum. Denote the distance of the center of mass to the pivot point S by l_{cm}. The torque analysis is nearly identical to the simple pendulum. The torque about the pivot point S is given by

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,\text{cm}} \times m\vec{\mathbf{g}} = l_{\text{cm}}\hat{\mathbf{r}} \times mg(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) = -l_{\text{cm}}mg\sin\theta\,\hat{\mathbf{k}}\,. \qquad (24.2.1)$$

Following the same steps that led from Equation (24.1.1) to Equation (24.1.4), the rotational equation for the physical pendulum is

$$-mgl_{\text{cm}}\sin\theta = I_S\frac{d^2\theta}{dt^2}\,, \qquad (24.2.2)$$

where I_S the moment of inertia about the pivot point S. As with the simple pendulum, for small angles $\sin\theta \approx \theta$, Equation (24.2.2) reduces to the simple harmonic oscillator equation

$$\frac{d^2\theta}{dt^2} \simeq -\frac{mgl_{\text{cm}}}{I_S}\theta\,. \qquad (24.2.3)$$

The equation for the angle $\theta(t)$ is given by

$$\theta(t) = A\cos(\omega_0\,t) + B\sin(\omega_0\,t)\,, \qquad (24.2.4)$$

where the angular frequency is given by

$$\omega_0 \simeq \sqrt{\frac{mg\,l_{\text{cm}}}{I_S}} \qquad (\text{physical pendulum})\,, \qquad (24.2.5)$$

and the period is

$$T = \frac{2\pi}{\omega_0} \simeq 2\pi\sqrt{\frac{I_S}{mg\,l_{\text{cm}}}} \qquad (\text{physical pendulum})\,. \qquad (24.2.6)$$

Substitute the parallel axis theorem, $I_S = ml_{\text{cm}}^2 + I_{\text{cm}}$, into Eq. (24.2.6) with the result that

$$T \simeq 2\pi\sqrt{\frac{l_{\text{cm}}}{g} + \frac{I_{\text{cm}}}{mg\,l_{\text{cm}}}} \qquad (\text{physical pendulum})\,. \qquad (24.2.7)$$

Thus, if the object is “small” in the sense that $I_{\text{cm}} << ml_{\text{cm}}^2$, the expressions for the physical pendulum reduce to those for the simple pendulum. The z-component of the angular velocity is given by

$$\omega_z(t) = \frac{d\theta}{dt}(t) = -\omega_0 A\sin(\omega_0\, t) + \omega_0 B\cos(\omega_0\, t)\,. \qquad (24.2.8)$$

The coefficients A and B can be determined form the initial conditions by setting $t = 0$ in Eqs. (24.2.4) and (24.2.8) resulting in the conditions that

$$\begin{aligned} A &= \theta(t=0) \equiv \theta_0 \\ B &= \frac{\omega_z(t=0)}{\omega_0} \equiv \frac{\omega_{z,0}}{\omega_0}. \end{aligned} \qquad (24.2.9)$$

Therefore the equations for the angle $\theta(t)$ and $\omega_z(t) = \frac{d\theta}{dt}(t)$ are given by

$$\theta(t) = \theta_0 \cos(\omega_0\, t) + \frac{\omega_{z,0}}{\omega_0}\sin(\omega_0\, t)\,, \qquad (24.2.10)$$

$$\omega_z(t) = \frac{d\theta}{dt}(t) = -\omega_0\theta_0 \sin(\omega_0\, t) + \omega_{z,0}\cos(\omega_0\, t)\,. \qquad (24.2.11)$$

24.3 Worked Examples

Example 24.1 Oscillating Rod

A physical pendulum consists of a uniform rod of length d and mass m pivoted at one end. The pendulum is initially displaced to one side by a small angle θ_0 and released from rest with $\theta_0 << 1$. Find the period of the pendulum. Determine the period of the pendulum using (a) the torque method and (b) the energy method.

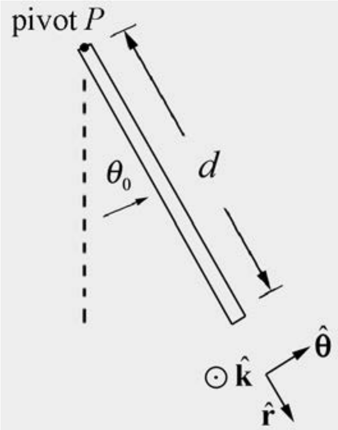


Figure 24.3 Oscillating rod

(a) Torque Method: with our choice of rotational coordinate system the angular acceleration is given by

$$\vec{\boldsymbol{\alpha}} = \frac{d^2\theta}{dt^2}\hat{\mathbf{k}}\,. \qquad (24.3.1)$$

The force diagram on the pendulum is shown in Figure 24.4. In particular, there is an unknown pivot force and the gravitational force acts at the center of mass of the rod.

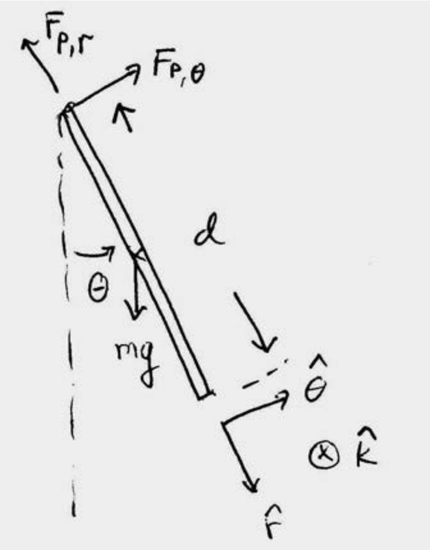


Figure 24.4 Free-body force diagram on rod

The torque about the pivot point P is given by

$$\vec{\boldsymbol{\tau}}_P = \vec{\mathbf{r}}_{P,\text{cm}} \times m\vec{\mathbf{g}}\;. \qquad (24.3.2)$$

The rod is uniform, therefore the center of mass is a distance $d/2$ from the pivot point. The gravitational force acts at the center of mass, so the torque about the pivot point P is given by

$$\vec{\boldsymbol{\tau}}_P = (d/2)\hat{\mathbf{r}} \times mg(-\sin\theta\,\hat{\boldsymbol{\theta}} + \cos\hat{\mathbf{r}}) = -(d/2)mg\sin\theta\,\hat{\mathbf{k}}\,. \qquad (24.3.3)$$

The rotational equation of motion about P is then

$$\vec{\boldsymbol{\tau}}_P = I_P\vec{\boldsymbol{\alpha}}\,. \qquad (24.3.4)$$

Substituting Eqs. (24.3.3) and (24.3.1) into Eq. (24.3.4) yields

$$-(d/2)mg\sin\theta\,\hat{\mathbf{k}} = I_P\frac{d^2\theta}{dt^2}\hat{\mathbf{k}}\,. \qquad (24.3.5)$$

When the angle of oscillation is small, we may use the small angle approximation $\sin\theta \cong \theta$, then Eq. (24.3.5) becomes

$$\frac{d^2\theta}{dt^2} + \frac{(d/2)mg}{I_P}\theta \simeq 0\,, \tag{24.3.6}$$

which is a simple harmonic oscillator equation. The angular frequency of small oscillations for the pendulum is

$$\omega_0 \simeq \sqrt{\frac{(d/2)mg}{I_P}}\,. \tag{24.3.7}$$

The moment of inertia of a rod about the end point P is $I_P = (1/3)md^2$ therefore the angular frequency is

$$\omega_0 \simeq \sqrt{\frac{(d/2)mg}{(1/3)md^2}} = \sqrt{\frac{(3/2)g}{d}} \tag{24.3.8}$$

with period

$$T = \frac{2\pi}{\omega_0} \simeq 2\pi\sqrt{\frac{2}{3}\frac{d}{g}}\,. \tag{24.3.9}$$

(b) Energy Method: Take the zero point of gravitational potential energy to be the point where the center of mass of the pendulum is at its lowest point (Figure 24.5), that is, $\theta = 0$.

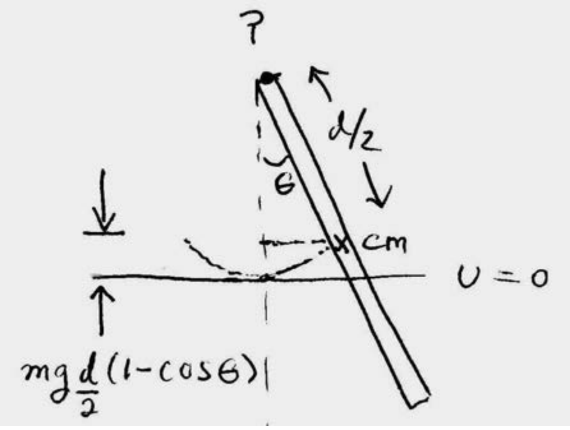


Figure 24.5 Energy diagram for rod

When the pendulum is at an angle θ the potential energy is

$$U = mg\frac{d}{2}(1-\cos\theta)\,. \tag{24.3.10}$$

The kinetic energy of rotation about the pivot point is

$$K^{\text{rot}} = \frac{1}{2} I_p \omega_z^{\ 2} . \qquad (24.3.11)$$

The mechanical energy is then

$$E = U + K^{\text{rot}} = m g \frac{d}{2}\left(1 - \cos\theta\right) + \frac{1}{2} I_p \omega_z^{\ 2} , \qquad (24.3.12)$$

with $I_P = (1/3)md^2$. There are no non-conservative forces acting (by assumption), so the mechanical energy is constant, and therefore the time derivative of energy is zero,

$$0 = \frac{dE}{dt} = m g \frac{d}{2} \sin\theta \frac{d\theta}{dt} + I_p \omega_z \frac{d\omega_z}{dt} . \qquad (24.3.13)$$

Recall that $\omega_z = d\theta / dt$ and $\alpha_z = d\omega_z / dt = d^2\theta / dt^2$, so Eq. (24.3.13) becomes

$$0 = \omega_z \left(m g \frac{d}{2} \sin\theta + I_p \frac{d^2\theta}{dt^2} \right) . \qquad (24.3.14)$$

There are two solutions, $\omega_z = 0$, in which case the rod remains at the bottom of the swing,

$$0 = m g \frac{d}{2} \sin\theta + I_p \frac{d^2\theta}{dt^2} . \qquad (24.3.15)$$

Using the small angle approximation, we obtain the simple harmonic oscillator equation (Eq. (24.3.6))

$$\frac{d^2\theta}{dt^2} + \frac{m g (d/2)}{I_p} \theta \simeq 0 . \qquad (24.3.16)$$

Example 24.3 Torsional Oscillator

A disk with moment of inertia about the center of mass I_{cm} rotates in a horizontal plane. It is suspended by a thin, massless rod. If the disk is rotated away from its equilibrium position by an angle θ, the rod exerts a restoring torque about the center of the disk with magnitude given by $\tau_{\text{cm}} = b\theta$ (Figure 24.6), where b is a positive constant. At $t = 0$, the disk is released from rest at an angular displacement of θ_0. Find the subsequent time dependence of the angular displacement $\theta(t)$.

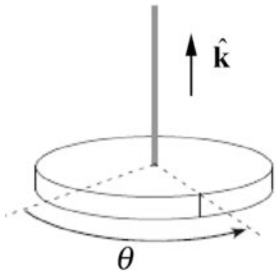


Figure 24.6 Example 24.3 with exaggerated angle θ

Solution: Choose a coordinate system such that $\hat{\mathbf{k}}$ is pointing upwards (Figure 24.6), then the angular acceleration is given by

$$\vec{\boldsymbol{\alpha}} = \frac{d^2\theta}{dt^2}\hat{\mathbf{k}}. \tag{24.3.17}$$

The torque about the center of mass is given in the statement of the problem as a restoring torque, therefore

$$\vec{\boldsymbol{\tau}}_{\text{cm}} = -b\,\theta\,\hat{\mathbf{k}}. \tag{24.3.18}$$

The z-component of the rotational equation of motion is

$$-b\,\theta = I_{\text{cm}}\frac{d^2\theta}{dt^2}. \tag{24.3.19}$$

This is a simple harmonic oscillator equation with solution

$$\theta(t) = A\cos(\omega_0\,t) + B\sin(\omega_0\,t) \tag{24.3.20}$$

where the angular frequency of oscillation is given by

$$\omega_0 = \sqrt{b/I_{\text{cm}}}\ . \tag{24.3.21}$$

The z-component of the angular velocity is given by

$$\omega_z(t) = \frac{d\theta}{dt}(t) = -\omega_0\,A\sin(\omega_0\,t) + \omega_0 B\cos(\omega_0\,t). \tag{24.3.22}$$

The initial conditions at $t = 0$, are $\theta(t=0) = A = \theta_0$, and $(d\theta/dt)(t=0) = \omega_0 B = 0$. Therefore

$$\theta(t) = \theta_0\cos(\sqrt{b/I_{\text{cm}}}\ t). \tag{24.3.23}$$

Example 24.4 Compound Physical Pendulum

A compound physical pendulum consists of a disk of radius R and mass m_d fixed at the end of a rod of mass m_r and length l (Figure 24.7a). (a) Find the period of the pendulum. (b) How does the period change if the disk is mounted to the rod by a frictionless bearing so that it is perfectly free to spin?

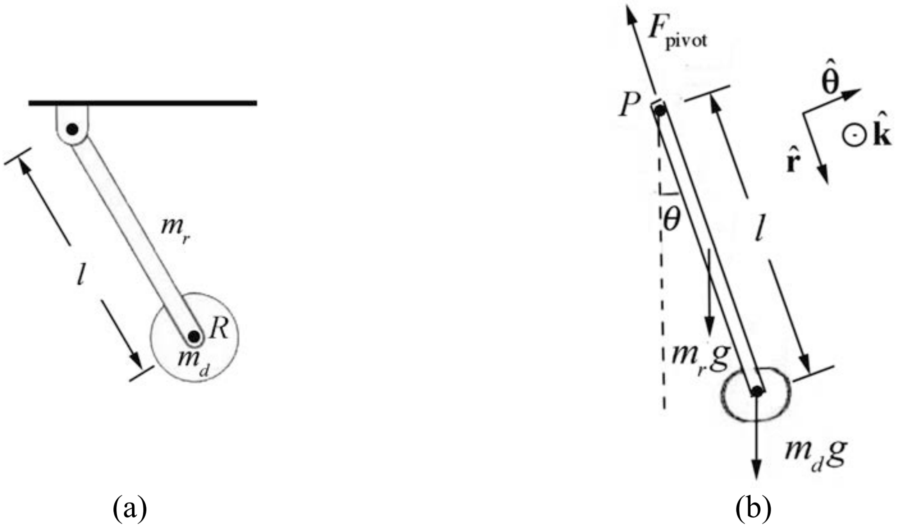


Figure 24.7 (a) Example 24.4 (b) Free-body force diagram

Solution: We begin by choosing coordinates. Let $\hat{\mathbf{k}}$ be normal to the plane of the motion of the pendulum pointing out of the plane of the Figure 24.7b. Choose an angle variable θ such that counterclockwise rotation corresponds to a positive z-component of the angular velocity. Thus a torque that points into the page has a negative z-component and a torque that points out of the page has a positive z-component. The free-body force diagram on the pendulum is also shown in Figure 24.7b. In particular, there is an unknown pivot force, the gravitational force acting at the center of mass of the rod, and the gravitational force acting at the center of mass of the disk. The torque about the pivot point is given by

$$\vec{\boldsymbol{\tau}}_P = \vec{\mathbf{r}}_{P,\text{cm}} \times m_r \vec{\mathbf{g}} + \vec{\mathbf{r}}_{P,\text{disk}} \times m_d \vec{\mathbf{g}} . \tag{24.3.24}$$

Recall that the vector $\vec{\mathbf{r}}_{P,\text{cm}}$ points from the pivot point to the center of mass of the rod a distance $l/2$ from the pivot. The vector $\vec{\mathbf{r}}_{P,\text{disk}}$ points from the pivot point to the center of mass of the disk a distance l from the pivot. Torque diagrams for the gravitational force on the rod and the disk are shown in Figure 24.8. Both torques about the pivot are in the negative $\hat{\mathbf{k}}$-direction (into the plane of Figure 24.8) and hence have negative z-components,

$$\vec{\boldsymbol{\tau}}_P = -(m_r(l/2) + m_d l) g \sin\theta \, \hat{\mathbf{k}} . \tag{24.3.25}$$

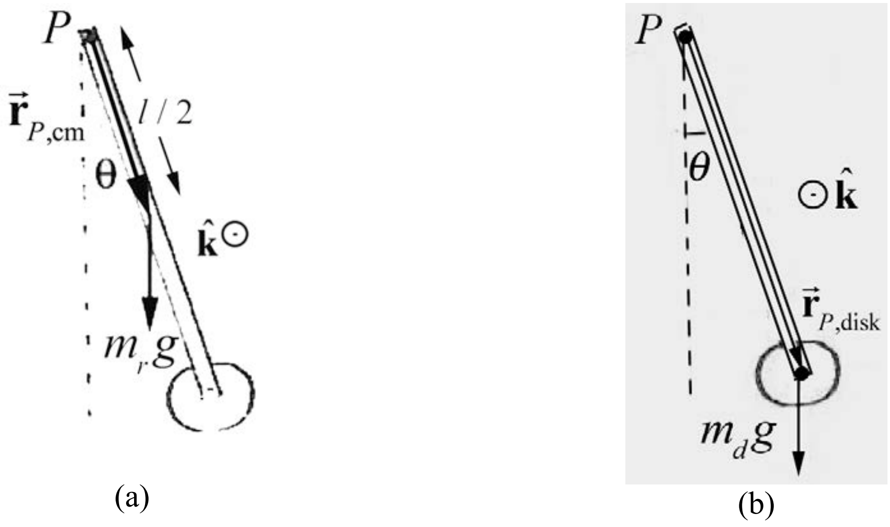


Figure 24.8 Torque diagram for (a) center of mass, (b) disk

In order to determine the moment of inertia of the rigid compound pendulum we will treat each piece separately, the uniform rod of length d and the disk attached at the end of the rod. The moment of inertia about the pivot point P is the sum of the moments of inertia of the two pieces,

$$I_P = I_{P,\mathrm{rod}} + I_{P,\mathrm{disc}} . \tag{24.3.26}$$

We calculated the moment of inertia of a rod about the end point P (Chapter 16.3.3), with the result that

$$I_{P,\mathrm{rod}} = \frac{1}{3} m_r l^2 . \tag{24.3.27}$$

We can use the parallel axis theorem to calculate the moment of inertia of the disk about the pivot point P,

$$I_{P,\mathrm{disc}} = I_{\mathrm{cm,disc}} + m_d l^2 . \tag{24.3.28}$$

We calculated the moment of inertia of a disk about the center of mass (Example 16.3) and determined that

$$I_{\mathrm{cm,disc}} = \frac{1}{2} m_d R^2 . \tag{24.3.29}$$

The moment of inertia of the compound system is then

$$I_P = \frac{1}{3} m_r l^2 + m_d l^2 + \frac{1}{2} m_d R^2 . \tag{24.3.30}$$

Therefore the rotational equation of motion becomes

$$-((1/2)m_r + m_d)gl\sin\theta\,\hat{\mathbf{k}} = \left(((1/3)m_r + m_d)l^2 + (1/2)m_d R^2\right)\frac{d^2\theta}{dt^2}\hat{\mathbf{k}}\,. \quad (24.3.31)$$

When the angle of oscillation is small, we can use the small angle approximation $\sin\theta \simeq \theta$. Then Eq. (24.3.31) becomes a simple harmonic oscillator equation,

$$\frac{d^2\theta}{dt^2} \simeq -\frac{((1/2)m_r + m_d)gl}{((1/3)m_r + m_d)l^2 + (1/2)m_d R^2}\theta\,. \quad (24.3.32)$$

Eq. (24.3.32) describes simple harmonic motion with an angular frequency of oscillation when the disk is fixed in place given by

$$\omega_{\text{fixed}} = \sqrt{\frac{((1/2)m_r + m_d)gl}{((1/3)m_r + m_d)l^2 + (1/2)m_d R^2}}\,. \quad (24.3.33)$$

The period is

$$T_{\text{fixed}} = \frac{2\pi}{\omega_{\text{fixed}}} \simeq 2\pi\sqrt{\frac{((1/3)m_r + m_d)l^2 + (1/2)m_d R^2}{((1/2)m_r + m_d)gl}}\,. \quad (24.3.34)$$

(b) If the disk is not fixed to the rod, then it will not rotate about its center of mass as the pendulum oscillates. Therefore the moment of inertia of the disk about its center of mass does not contribute to the moment of inertia of the physical pendulum about the pivot point. Notice that the pendulum is no longer a rigid body. The total moment of inertia is only due to the rod and the disk treated as a point like object,

$$I_P = \frac{1}{3}m_r l^2 + m_d l^2\,. \quad (24.3.35)$$

Therefore the period of oscillation is given by

$$T_{\text{free}} = \frac{2\pi}{\omega_{\text{free}}} \simeq 2\pi\sqrt{\frac{((1/3)m_r + m_d)l^2}{((1/2)m_r + m_d)gl}}\,. \quad (24.3.36)$$

Comparing Eq. (24.3.36) to Eq. (24.3.34), we see that the period is smaller when the disk is free and not fixed. From an energy perspective we can argue that when the disk is free, it is not rotating about the center of mass. Therefore more of the gravitational potential energy goes into the center of mass translational kinetic energy than when the disk is free. Hence the center of mass is moving faster when the disk is free so it completes one period is a shorter time.

Appendix 24A Higher-Order Corrections to the Period for Larger Amplitudes of a Simple Pendulum

In Section 24.1.1, we found the period for a simple pendulum that undergoes small oscillations is given by

$$T = \frac{2\pi}{\omega_0} \cong 2\pi\sqrt{\frac{l}{g}} \qquad \text{(simple pendulum)}.$$

How good is this approximation? If the pendulum is pulled out to an initial angle θ_0 that is not small (such that our first approximation $\sin\theta \cong \theta$ no longer holds) then our expression for the period is no longer valid. We shall calculate the first-order (or higher-order) correction to the period of the pendulum.

Let's first consider the mechanical energy, a conserved quantity in this system. Choose an initial state when the pendulum is released from rest at an angle θ_i; this need not be at time $t = 0$, and in fact later in this derivation we'll see that it's inconvenient to choose this position to be at $t = 0$. Choose for the final state the position and velocity of the bob at an arbitrary time t. Choose the zero point for the potential energy to be at the bottom of the bob's swing (Figure 24A.1).

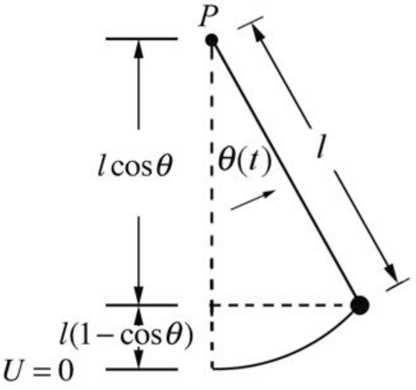


Figure 24A.1 Energy diagram for simple pendulum

The mechanical energy when the bob is released from rest at an angle θ_i is

$$E_i = K_i + U_i = m g l (1 - \cos\theta_i). \tag{24.C.37}$$

The tangential component of the velocity of the bob at an arbitrary time t is given by

$$v_\theta = l\frac{d\theta}{dt}, \tag{24.C.38}$$

and the kinetic energy at that time is

$$K_f = \frac{1}{2} m v_\theta{}^2 = \frac{1}{2} m \left(l \frac{d\theta}{dt} \right)^2 . \tag{24.C.39}$$

The mechanical energy at time t is then

$$E_f = K_f + U_f = \frac{1}{2} m \left(l \frac{d\theta}{dt} \right)^2 + m g l (1 - \cos\theta) . \tag{24.C.40}$$

Because the tension in the string is always perpendicular to the displacement of the bob, the tension does no work, we neglect any frictional forces, and hence mechanical energy is constant, $E_f = E_i$. Thus

$$\begin{aligned} &\frac{1}{2} m \left(l \frac{d\theta}{dt} \right)^2 + m g l (1 - \cos\theta) = m g l (1 - \cos\theta_i) \\ &\left(l \frac{d\theta}{dt} \right)^2 = 2 \frac{g}{l} (\cos\theta - \cos\theta_i). \end{aligned} \tag{24.C.41}$$

We can solve Equation (24.C.41) for the angular velocity as a function of θ,

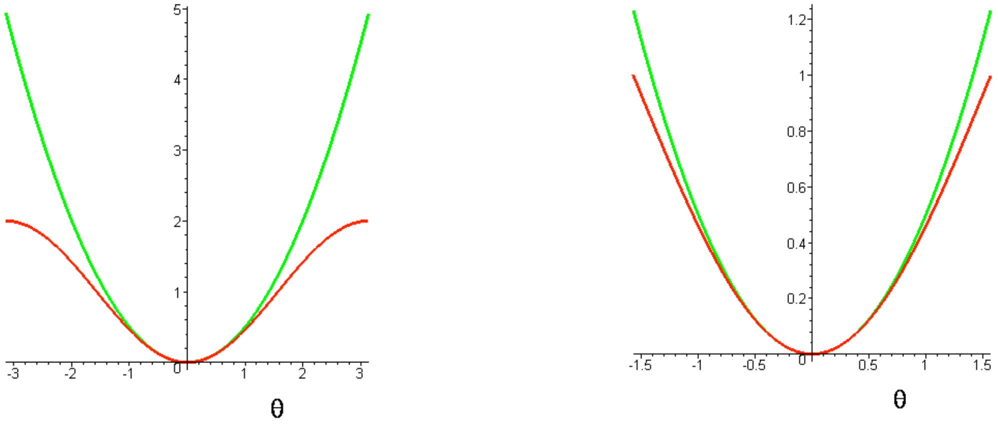
$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}} \sqrt{\cos\theta - \cos\theta_i} . \tag{24.C.42}$$

Note that we have taken the positive square root, implying that $d\theta / dt \geq 0$. This clearly cannot always be the case, and we should change the sign of the square root every time the pendulum's direction of motion changes. For our purposes, this is not an issue. If we wished to find an explicit form for either $\theta(t)$ or $t(\theta)$, we would have to consider the signs in Equation (24.C.42) more carefully.

Before proceeding, it's worth considering the difference between Equation (24.C.42) and the equation for the simple pendulum in the simple harmonic oscillator limit, where $\cos\theta \simeq 1 - (1/2)\theta^2$. Then Eq. (24.C.42) reduces to

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}} \sqrt{\frac{\theta_i^2}{2} - \frac{\theta^2}{2}} . \tag{24.C.43}$$

In both Equations (24.C.42) and (24.C.43) the last term in the square root is proportional to the difference between the initial potential energy and the final potential energy. The final potential energy for the two cases is plotted in Figures 24A.2 for $-\pi < \theta < \pi$ on the left and $-\pi / 2 < \theta < \pi / 2$ on the right (the vertical scale is in units of mgl).



Figures 24A.2 Potential energies as a function of displacement angle

It would seem to be to our advantage to express the potential energy for an arbitrary displacement of the pendulum as the difference between two squares. We do this by first recalling the trigonometric identity

$$1-\cos\theta = 2\sin^2(\theta/2) \tag{24.C.44}$$

with the result that Equation (24.C.42) may be re-expressed as

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}}\sqrt{2(\sin^2(\theta_i/2)-\sin^2(\theta/2))}\,. \tag{24.C.45}$$

Equation (24.C.45) is separable,

$$\frac{d\theta}{\sqrt{\sin^2(\theta_i/2)-\sin^2(\theta/2)}} = 2\sqrt{\frac{g}{l}}\,dt \tag{24.C.46}$$

Rewrite Equation (24.C.46) as

$$\frac{d\theta}{\sin(\theta_i/2)\sqrt{1-\dfrac{\sin^2(\theta/2)}{\sin^2(\theta_i/2)}}} = 2\sqrt{\frac{g}{l}}\,dt\,. \tag{24.C.47}$$

The ratio $\sin(\theta/2)/\sin(\theta_i/2)$ in the square root in the denominator will oscillate (but *not* with simple harmonic motion) between -1 and $+1$, and so we will make the identification

$$\sin\phi = \frac{\sin(\theta/2)}{\sin(\theta_i/2)}\,. \tag{24.C.48}$$

Let $b = \sin(\theta_i / 2)$, so that

$$\begin{aligned} &\sin\frac{\theta}{2} = b\sin\phi \\ &\cos\frac{\theta}{2} = \left(1 - \sin^2\frac{\theta}{2}\right)^{1/2} = (1 - b^2\sin^2\phi)^{1/2}. \end{aligned} \tag{24.C.49}$$

Eq. (24.C.47) can then be rewritten in integral form as

$$\int \frac{d\theta}{b\sqrt{1-\sin^2\phi}} = 2\int\sqrt{\frac{g}{l}}\,dt\,. \tag{24.C.50}$$

From differentiating the first expression in Equation (24.C.49), we have that

$$\begin{aligned} &\frac{1}{2}\cos\frac{\theta}{2}\ d\theta = b\cos\phi\ d\phi \\ &d\theta = 2b\frac{\cos\phi}{\cos(\theta/2)}d\phi = 2b\frac{\sqrt{1-\sin^2\phi}}{\sqrt{1-\sin^2(\theta/2)}}d\phi \\ &= 2b\frac{\sqrt{1-\sin^2\phi}}{\sqrt{1-b^2\sin^2\phi}}d\phi. \end{aligned} \tag{24.C.51}$$

Substituting the last equation in (24.C.51) into the left-hand side of the integral in (24.C.50) yields

$$\int \frac{2b}{b\sqrt{1-\sin^2\phi}}\frac{\sqrt{1-\sin^2\phi}}{\sqrt{1-b^2\sin^2\phi}}d\phi = 2\int\frac{d\phi}{\sqrt{1-b^2\sin^2\phi}}\,. \tag{24.C.52}$$

Thus the integral in Equation (24.C.50) becomes

$$\int \frac{d\phi}{\sqrt{1-b^2\sin^2\phi}} = \int\sqrt{\frac{g}{l}}\,dt\,. \tag{24.C.53}$$

This integral is one of a class of integrals known as ***elliptic integrals***. We find a power series solution to this integral by expanding the function

$$(1-b^2\sin^2\phi)^{-1/2} = 1 + \frac{1}{2}b^2\sin^2\phi + \frac{3}{8}b^4\sin^4\phi + \cdots. \tag{24.C.54}$$

The integral in Equation (24.C.53) then becomes

$$\int\left(1+\frac{1}{2}b^2\sin^2\phi+\frac{3}{8}b^4\sin^4\phi+\cdots\right)d\phi=\int\sqrt{\frac{g}{l}}\,dt\,. \tag{24.C.55}$$

Now let's integrate over one period. Set $t = 0$ when $\theta = 0$, the lowest point of the swing, so that $\sin\phi = 0$ and $\phi = 0$. One period T has elapsed the second time the bob returns to the lowest point, or when $\phi = 2\pi$. Putting in the limits of the ϕ-integral, we can integrate term by term, noting that

$$\begin{aligned}\int_0^{2\pi}\frac{1}{2}b^2\sin^2\phi\,d\phi&=\int_0^{2\pi}\frac{1}{2}b^2\frac{1}{2}(1-\cos(2\phi))\,d\phi\\&=\frac{1}{2}b^2\frac{1}{2}\left(\phi-\frac{\sin(2\phi)}{2}\right)\Bigg|_0^{2\pi}\\&=\frac{1}{2}\pi b^2=\frac{1}{2}\pi\sin^2\frac{\theta_i}{2}.\end{aligned} \tag{24.C.56}$$

Thus, from Equation (24.C.55) we have that

$$\begin{aligned}\int_0^{2\pi}\left(1+\frac{1}{2}b^2\sin^2\phi+\frac{3}{8}b^4\sin^4\phi+\cdots\right)d\phi&=\int_0^{T}\sqrt{\frac{g}{l}}\,dt\\2\pi+\frac{1}{2}\pi\sin^2\frac{\theta_i}{2}+\cdots&=\sqrt{\frac{g}{l}}\,T\end{aligned}\ , \tag{24.C.57}$$

We can now solve for the period,

$$T=2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{4}\sin^2\frac{\theta_i}{2}+\cdots\right). \tag{24.C.58}$$

If the initial angle $\theta_i << 1$ (measured in radians), then $\sin^2(\theta_i/2)\simeq\theta_i^2/4$ and the period is approximately

$$T\cong2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{16}\theta_i^2\right)=T_0\left(1+\frac{1}{16}\theta_i^2\right), \tag{24.C.59}$$

where

$$T_0=2\pi\sqrt{\frac{l}{g}} \tag{24.C.60}$$

is the period of the simple pendulum with the standard small angle approximation. The first order correction to the period of the pendulum is then

$$\Delta T_1 = \frac{1}{16}\theta_i^2 T_0. \tag{24.C.61}$$

Figure 24A.3 below shows the three functions given in Equation (24.C.60) (the horizontal, or red plot if seen in color), Equation (24.C.59) (the middle, parabolic or green plot) and the numerically-integrated function obtained by integrating the expression in Equation (24.C.53) (the upper, or blue plot) between $\phi = 0$ and $\phi = 2\pi$. The plots demonstrate that Equation (24.C.60) is a valid approximation for small values of θ_i, and that Equation (24.C.59) is a very good approximation for all but the largest amplitudes of oscillation. The vertical axis is in units of $\sqrt{l/g}$. Note the displacement of the horizontal axis.

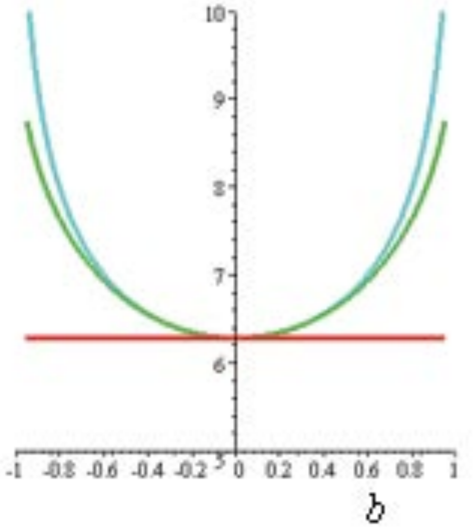


Figure 24A.3 Pendulum Period Approximations as Functions of Amplitude

Chapter 25 Celestial Mechanics

Chapter 25 Celestial Mechanics

> *...and if you want the exact moment in time, it was conceived mentally on 8th March in this year one thousand six hundred and eighteen, but submitted to calculation in an unlucky way, and therefore rejected as false, and finally returning on the 15th of May and adopting a new line of attack, stormed the darkness of my mind. So strong was the support from the combination of my labour of seventeen years on the observations of Brahe and the present study, which conspired together, that at first I believed I was dreaming, and assuming my conclusion among my basic premises. But it is absolutely certain and exact that "the proportion between the periodic times of any two planets is precisely the sesquialterate proportion of their mean distances ..."* [1]
>
> *Johannes Kepler*

25.1 Introduction: The Kepler Problem

Johannes Kepler first formulated the laws that describe planetary motion,

I. Each planet moves in an ellipse with the sun at one focus.

II. The radius vector from the sun to a planet sweeps out equal areas in equal time.

III. The period of revolution T of a planet about the sun is related to the semi-major axis a of the ellipse by $T^2 = k\,a^3$ where k is the same for all planets.[2]

The third law was published in 1619, and efforts to discover and solve the equation of motion of the planets generated two hundred years of mathematical and scientific discovery. In his honor, this problem has been named ***the Kepler Problem*.**

When there are more than two bodies, the problem becomes impossible to solve exactly. The most important "three-body problem" in the 17th and 18th centuries involved finding the motion of the moon, due to gravitational interaction with both the sun and the earth. Newton realized that if the exact position of the moon were known, the longitude of any observer on the earth could be determined by measuring the moon's position with respect to the stars.

In the eighteenth century, Leonhard Euler and other mathematicians spent many years trying to solve the three-body problem, and they raised a deeper question. Do the small contributions from the gravitational interactions of all the planets make the planetary system unstable over long periods of time? At the end of 18th century, Pierre

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[1] Kepler, Johannes, *Harmonice mundi* Book 5, Chapter 3, trans. Aiton, Duncan and Field, p. 411

[2] As stated in *An Introduction to Mechanics*, Daniel Kleppner and Robert Kolenkow, McGraw-Hill, 1973, p 401.

Simon Laplace and others found a series solution to this stability question, but it was unknown whether or not the series solution converged after a long period of time. Henri Poincaré proved that the series actually diverged. Poincaré went on to invent new mathematical methods that produced the modern fields of differential geometry and topology in order to answer the stability question using geometric arguments, rather than analytic methods. Poincaré and others did manage to show that the three-body problem was indeed stable, due to the existence of periodic solutions. Just as in the time of Newton and Leibniz and the invention of calculus, unsolved problems in celestial mechanics became the experimental laboratory for the discovery of new mathematics.

25.2 Planetary Orbits

We now commence a study of the Kepler Problem. We shall determine the equation of motion for the motions of two bodies interacting via a gravitational force (two-body problem) using both force methods and conservation laws.

25.2.1 Reducing the Two-Body Problem into a One-Body Problem

We shall begin by showing how the motion of two bodies interacting via a gravitational force (two-body problem) is mathematically equivalent to the motion of a single body acted on by an external central gravitational force, where the mass of the single body is the *reduced mass* μ,

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \Rightarrow \mu = \frac{m_1 m_2}{m_1 + m_2}. \qquad (25.2.1)$$

Once we solve for the motion of the reduced body in this ***equivalent one-body problem***, we can then return to the real two-body problem and solve for the actual motion of the two original bodies. The reduced mass was introduced in Chapter 13 Appendix A of these notes. That appendix used similar but slightly different notation from that used in this chapter.

Consider the gravitational interaction between two bodies with masses m_1 and m_2 as shown in Figure 25.1.

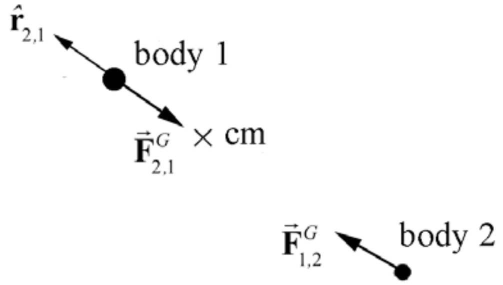


Figure 25.1 Gravitational force between two bodies

Choose a coordinate system with a choice of origin such that body 1 has position $\vec{\mathbf{r}}_1$ and body 2 has position $\vec{\mathbf{r}}_2$ (Figure 25.2). The *relative position vector* $\vec{\mathbf{r}}$ pointing from body 2 to body 1 is $\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$. We denote the magnitude of $\vec{\mathbf{r}}$ by $\left|\vec{\mathbf{r}}\right| = r$, where r is the distance between the bodies, and $\hat{\mathbf{r}}$ is the unit vector pointing from body 2 to body 1, so that $\vec{\mathbf{r}} = r\,\hat{\mathbf{r}}$.

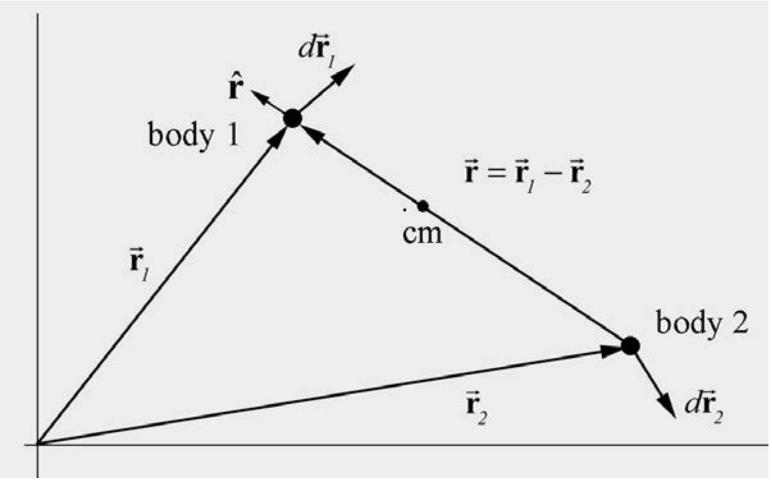


Figure 25.2 Coordinate system for the two-body problem

The force on body 1 (due to the interaction of the two bodies) can be described by Newton's Universal Law of Gravitation

$$\vec{\mathbf{F}}_{2,1} = -F_{2,1}\,\hat{\mathbf{r}} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}\,. \tag{25.2.2}$$

Recall that Newton's Third Law requires that the force on body 2 is equal in magnitude and opposite in direction to the force on body 1,

$$\vec{\mathbf{F}}_{1,2} = -\vec{\mathbf{F}}_{2,1}\,. \tag{25.2.3}$$

Newton's Second Law can be applied individually to the two bodies:

$$\vec{\mathbf{F}}_{2,1} = m_1\frac{d^2\vec{\mathbf{r}}_1}{dt^2}\,, \tag{25.2.4}$$

$$\vec{\mathbf{F}}_{1,2} = m_2\frac{d^2\vec{\mathbf{r}}_2}{dt^2}\,. \tag{25.2.5}$$

Dividing through by the mass in each of Equations (25.2.4) and (25.2.5) yields

$$\frac{\vec{\mathbf{F}}_{2,1}}{m_1} = \frac{d^2\vec{\mathbf{r}}_1}{dt^2}\,, \tag{25.2.6}$$

$$\frac{\vec{\mathbf{F}}_{1,2}}{m_2} = \frac{d^2\vec{\mathbf{r}}_2}{dt^2} . \qquad (25.2.7)$$

Subtracting the expression in Equation (25.2.7) from that in Equation (25.2.6) yields

$$\frac{\vec{\mathbf{F}}_{2,1}}{m_1} - \frac{\vec{\mathbf{F}}_{1,2}}{m_2} = \frac{d^2\vec{\mathbf{r}}_1}{dt^2} - \frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \frac{d^2\vec{\mathbf{r}}}{dt^2} . \qquad (25.2.8)$$

Using Newton's Third Law, Equation (25.2.3), Equation (25.2.8) becomes

$$\vec{\mathbf{F}}_{2,1}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{d^2\vec{\mathbf{r}}}{dt^2} . \qquad (25.2.9)$$

Using the *reduced mass* μ, as defined in Equation (25.2.1), Equation (25.2.9) becomes

$$\begin{gathered} \frac{\vec{\mathbf{F}}_{2,1}}{\mu} = \frac{d^2\vec{\mathbf{r}}}{dt^2} \\ \vec{\mathbf{F}}_{2,1} = \mu\frac{d^2\vec{\mathbf{r}}}{dt^2}, \end{gathered} \qquad (25.2.10)$$

where $\vec{\mathbf{F}}_{2,1}$ is given by Equation (25.2.2).

Our result has a special interpretation using Newton's Second Law. Let μ be the mass of a ***single body*** with position vector $\vec{\mathbf{r}} = r\,\hat{\mathbf{r}}$ with respect to an origin O, where $\hat{\mathbf{r}}$ is the unit vector pointing from the origin O to the single body. Then the equation of motion, Equation (25.2.10), implies that the single body of mass μ is under the influence of an attractive gravitational force pointing toward the origin. So, the original two-body gravitational problem has now been reduced to an equivalent one-body problem, involving a single body with mass μ under the influence of a central force $\vec{\mathbf{F}}^G = -\mathrm{F}_{2,1}\,\hat{\mathbf{r}}$. Note that in this reformulation, there is no body located at the central point (the origin O). The parameter r in the two-body problem is the relative distance between the original two bodies, while the same parameter r in the one-body problem is the distance between the single body and the central point. This reduction generalizes to all central forces.

25.3 Energy and Angular Momentum, Constants of the Motion

The equivalent one-body problem has two constants of the motion, energy E and the angular momentum L about the origin O. Energy is a constant because in our original two-body problem, the gravitational interaction was an internal conservative force. Angular momentum is constant about the origin because the only force is directed towards the origin, and hence the torque about the origin due to that force is zero (the vector from the origin to the single body is anti-parallel to the force vector and $\sin\pi = 0$). Because angular momentum is constant, the orbit of the single body lies in a plane with the angular momentum vector pointing perpendicular to this plane.

In the plane of the orbit, choose polar coordinates (r,θ) for the single body (see Figure 25.3), where r is the distance of the single body from the central point that is now taken as the origin O, and θ is the angle that the single body makes with respect to a chosen direction, and which increases positively in the counterclockwise direction.

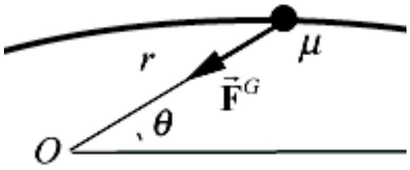


Figure 25.3 Coordinate system for the orbit of the single body

There are two approaches to describing the motion of the single body. We can try to find both the distance from the origin, $r(t)$ and the angle, $\theta(t)$, as functions of the parameter time, but in most cases explicit functions can't be found analytically. We can also find the distance from the origin, $r(\theta)$, as a function of the angle θ. This second approach offers a spatial description of the motion of the single body (see Appendix 25A).

25.3.1 The Orbit Equation for the One-Body Problem

Consider the single body with mass μ given by Equation (25.2.1), orbiting about a central point under the influence of a radially attractive force given by Equation (25.2.2). Since the force is conservative, the potential energy (from the two-body problem) with choice of zero reference point $U(\infty) = 0$ is given by

$$U(r) = -\frac{G m_1 m_2}{r}. \tag{25.3.1}$$

The total energy E is constant, and the sum of the kinetic energy and the potential energy is

$$E = \frac{1}{2}\mu v^2 - \frac{G m_1 m_2}{r}. \tag{25.3.2}$$

The kinetic energy term $\mu v^2 / 2$ is written in terms of the mass μ and the relative speed v of the two bodies. Choose polar coordinates such that

$$\vec{\mathbf{v}} = v_r\,\hat{\mathbf{r}} + v_\theta \hat{\boldsymbol{\theta}},\qquad v = \left|\vec{\mathbf{v}}\right| = \left|\frac{d\vec{\mathbf{r}}}{dt}\right|, \tag{25.3.3}$$

where $v_r = dr / dt$ and $v_\theta = r(d\theta / dt)$. Equation (25.3.2) then becomes

$$E = \frac{1}{2}\mu\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right] - \frac{G m_1 m_2}{r}. \tag{25.3.4}$$

The angular momentum with respect to the origin O is given by

$$\vec{\mathbf{L}}_O = \vec{\mathbf{r}}_O \times \mu\vec{\mathbf{v}} = r\hat{\mathbf{r}} \times \mu(v_r\,\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}) = \mu r v_\theta \hat{\mathbf{k}} = \mu r^2 \frac{d\theta}{dt}\hat{\mathbf{k}} \equiv L\hat{\mathbf{k}} \tag{25.3.5}$$

with magnitude

$$L = \mu r v_\theta = \mu r^2 \frac{d\theta}{dt}. \tag{25.3.6}$$

We shall explicitly eliminate the θ dependence from Equation (25.3.4) by using our expression in Equation (25.3.6),

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2}. \tag{25.3.7}$$

The mechanical energy as expressed in Equation (25.3.4) then becomes

$$E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{L^2}{\mu r^2} - \frac{G m_1 m_2}{r}. \tag{25.3.8}$$

Equation (25.3.8) is a separable differential equation involving the variable r as a function of time t and can be solved for the first derivative dr / dt,

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}}\left(E - \frac{1}{2}\frac{L^2}{\mu r^2} + \frac{G m_1 m_2}{r}\right)^{\frac{1}{2}}. \tag{25.3.9}$$

Equation (25.3.9) can in principle be integrated directly for $r(t)$. In fact, doing the integrals is complicated and beyond the scope of this book. The function $r(t)$ can then, in principle, be substituted into Equation (25.3.7) and can then be integrated to find $\theta(t)$.

Instead of solving for the position of the single body as a function of time, we shall find a geometric description of the orbit by finding $r(\theta)$. We first divide Equation (25.3.7) by Equation (25.3.9) to obtain

$$\frac{d\theta}{dr}=\frac{\frac{d\theta}{dt}}{\frac{dr}{dt}}=\frac{L}{\sqrt{2\mu}}\frac{\left(1/r^2\right)}{\left(E-\frac{L^2}{2\mu r^2}+\frac{G m_1 m_2}{r}\right)^{1/2}}\,. \tag{25.3.10}$$

The variables r and θ are separable;

$$d\theta=\frac{L}{\sqrt{2\mu}}\frac{\left(1/r^2\right)}{\left(E-\frac{L^2}{2\mu r^2}+\frac{G m_1 m_2}{r}\right)^{1/2}}dr\,. \tag{25.3.11}$$

Equation (25.3.11) can be integrated to find the radius as a function of the angle θ; see Appendix 25A for the exact integral solution. The result is called the ***orbit equation*** for the reduced body and is given by

$$r=\frac{r_0}{1-\varepsilon\cos\theta} \tag{25.3.12}$$

where

$$r_0=\frac{L^2}{\mu G m_1 m_2} \tag{25.3.13}$$

is a constant (known as the ***semilatus rectum***) and

$$\varepsilon=\left(1+\frac{2EL^2}{\mu(G m_1 m_2)^2}\right)^{\frac{1}{2}} \tag{25.3.14}$$

is the ***eccentricity*** of the orbit. The two constants of the motion, angular momentum L and mechanical energy E, in terms of r_0 and ε, are

$$L=(\mu G m_1 m_2 r_0)^{1/2} \tag{25.3.15}$$

$$E=\frac{G m_1 m_2(\varepsilon^2-1)}{2r_0}. \tag{25.3.16}$$

The orbit equation as given in Equation (25.3.12) is a general ***conic section*** and is perhaps somewhat more familiar in Cartesian coordinates. Let $x=r\cos\theta$ and $y=r\sin\theta$, with $r^2=x^2+y^2$. The orbit equation can be rewritten as

$$r=r_0+\varepsilon r\cos\theta\,. \tag{25.3.17}$$

Using the Cartesian substitutions for x and y, rewrite Equation (25.3.17) as

$$(x^2 + y^2)^{1/2} = r_0 + \varepsilon x . \tag{25.3.18}$$

Squaring both sides of Equation (25.3.18),

$$x^2 + y^2 = r_0{}^2 + 2\varepsilon x r_0 + \varepsilon^2 x^2 . \tag{25.3.19}$$

After rearranging terms, Equation (25.3.19) is the general expression of a conic section with axis on the x-axis,

$$x^2(1-\varepsilon^2) - 2\varepsilon x r_0 + y^2 = r_0{}^2 . \tag{25.3.20}$$

(We now see that the horizontal axis in Figure 25.3 can be taken to be the x-axis).

For a given $r_0 > 0$, corresponding to a given nonzero angular momentum as in Equation (25.3.12), there are four cases determined by the value of the eccentricity.

Case 1: when $\varepsilon = 0$, $E = E_{\text{min}} < 0$ and $r = r_0$, Equation (25.3.20) is the equation for a *circle*,

$$x^2 + y^2 = r_0{}^2 . \tag{25.3.21}$$

Case 2: when $0 < \varepsilon < 1$, $E_{\text{min}} < E < 0$, Equation (25.3.20) describes an *ellipse*,

$$y^2 + Ax^2 - Bx = k . \tag{25.3.22}$$

where $A > 0$ and k is a positive constant. (Appendix 25C shows how this expression may be expressed in the more traditional form involving the coordinates of the center of the ellipse and the semi-major and semi-minor axes.)

Case 3: when $\varepsilon = 1$, $E = 0$, Equation (25.3.20) describes a *parabola*,

$$x = \frac{y^2}{2r_0} - \frac{r_0}{2} . \tag{25.3.23}$$

Case 4: when $\varepsilon > 1$, $E > 0$, Equation (25.3.20) describes a *hyperbola*,

$$y^2 - Ax^2 - Bx = k , \tag{25.3.24}$$

where $A > 0$ and k is a positive constant.

25.4 Energy Diagram, Effective Potential Energy, and Orbits

The energy (Equation (25.3.8)) of the single body moving in two dimensions can be reinterpreted as the energy of a single body moving in one dimension, the radial direction r, in an ***effective potential energy*** given by two terms,

$$U_{\text{eff}} = \frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r}. \tag{25.4.1}$$

The energy is still the same, but our interpretation has changed,

$$E = K_{\text{eff}} + U_{\text{eff}} = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r}, \tag{25.4.2}$$

where the ***effective kinetic energy*** K_{eff} associated with the one-dimensional motion is

$$K_{\text{eff}} = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2. \tag{25.4.3}$$

The graph of U_{eff} as a function of $u = r / r_0$, where r_0 as given in Equation (25.3.13), is shown in Figure 25.4. The upper red curve is proportional to $L^2 / (2\mu r^2) \sim 1/2r^2$. The lower blue curve is proportional to $-Gm_1 m_2 / r \sim -1/r$. The sum U_{eff} is represented by the middle green curve. The minimum value of U_{eff} is at $r = r_0$, as will be shown analytically below. The vertical scale is in units of $-U_{\text{eff}}(r_0)$. Whenever the one-dimensional kinetic energy is zero, $K_{\text{eff}} = 0$, the energy is equal to the effective potential energy,

$$E = U_{\text{eff}} = \frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r}. \tag{25.4.4}$$

Recall that the potential energy is defined to be the negative integral of the work done by the force. For our reduction to a one-body problem, using the effective potential, we will introduce an ***effective force*** such that

$$U_{\text{eff},B} - U_{\text{eff},A} = -\int_A^B \vec{\mathbf{F}}^{\text{eff}} \cdot d\vec{\mathbf{r}} = -\int_A^B F_r^{\text{eff}}\, dr \tag{25.4.5}$$

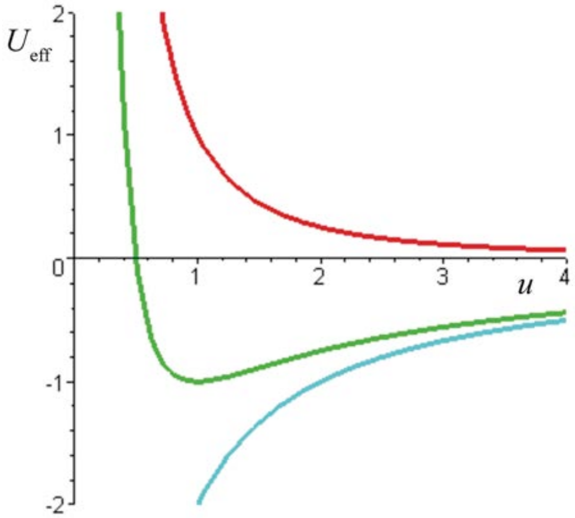


Figure 25.4 Graph of effective potential energy

The fundamental theorem of calculus (for one variable) then states that the integral of the derivative of the effective potential energy function between two points is the effective potential energy difference between those two points,

$$U_{\text{eff},B} - U_{\text{eff},A} = \int_A^B \frac{dU_{\text{eff}}}{dr} dr \tag{25.4.6}$$

Comparing Equation (25.4.6) to Equation (25.4.5) shows that the radial component of the effective force is the negative of the derivative of the effective potential energy,

$$F_r^{\text{eff}} = -\frac{dU_{\text{eff}}}{dr} \tag{25.4.7}$$

The effective potential energy describes the potential energy for a reduced body moving in one dimension. (Note that the effective potential energy is only a function of the variable r and is independent of the variable θ). There are two contributions to the effective potential energy, and the radial component of the force is then

$$F_r^{\text{eff}} = -\frac{d}{dr} U_{\text{eff}} = -\frac{d}{dr}\left(\frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r} \right) \tag{25.4.8}$$

Thus there are two "forces" acting on the reduced body,

$$F_r^{\text{eff}} = F_{r,\,\text{centifugal}} + F_{r,\,\text{gravity}}\,, \tag{25.4.9}$$

with an ***effective centrifugal force*** given by

$$F_{r,\text{centrifugal}} = -\frac{d}{dr}\left(\frac{L^2}{2\mu r^2}\right) = \frac{L^2}{\mu r^3} \qquad (25.4.10)$$

and the centripetal gravitational force given by

$$F_{r,\text{ gravity}} = -\frac{G m_1 m_2}{r^2}. \qquad (25.4.11)$$

With this nomenclature, let's review the four cases presented in Section 25.3.

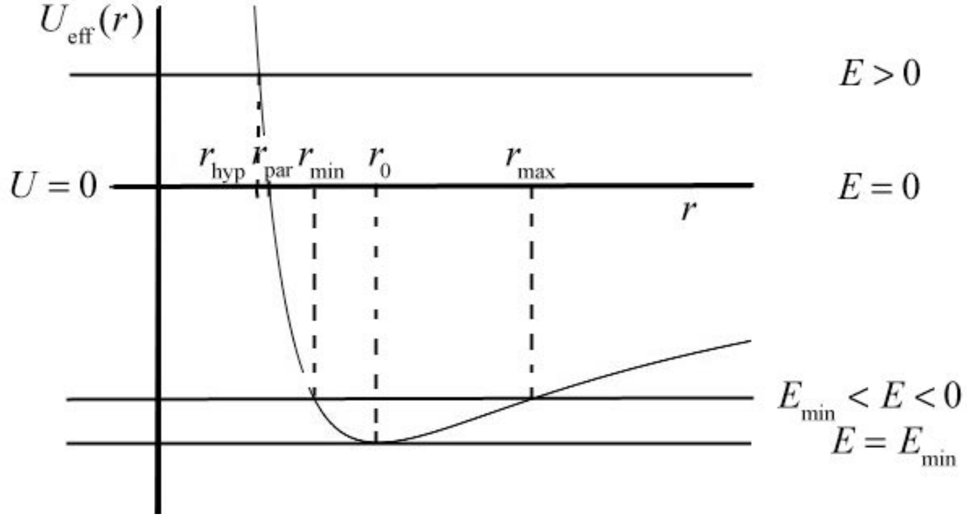


Figure 25.5 Plot of $U_{\text{eff}}(r)$ vs. r with four energies corresponding to circular, elliptic, parabolic, and hyperbolic orbits

25.4.1 Circular Orbit $E = E_{\text{min}}$

The lowest energy state, E_{min}, corresponds to the minimum of the effective potential energy, $E_{\text{min}} = (U_{\text{eff}})_{\text{min}}$. We can minimize the effective potential energy

$$0 = \left.\frac{dU_{\text{eff}}}{dr}\right|_{r=r_0} = -\frac{L^2}{\mu r_0^{\,3}} + \frac{G m_1 m_2}{r_0^{\,2}}. \qquad (25.4.12)$$

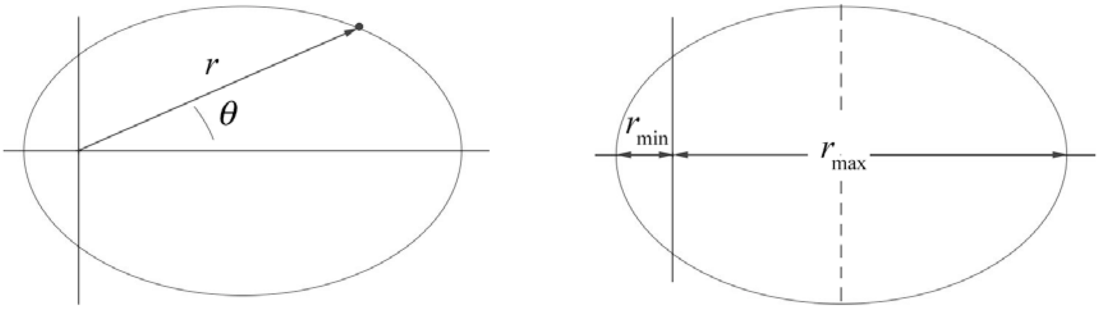
and solve Equation (25.4.12) for r_0,

$$r_0 = \frac{L^2}{G m_1 m_2}, \qquad (25.4.13)$$

reproducing Equation (25.3.13). For $E = E_{\text{min}}$, $r = r_0$ which corresponds to a circular orbit.

25.4.2 Elliptic Orbit $E_{\text{min}} < E < 0$

For $E_{\text{min}} < E < 0$, there are two points r_{min} and r_{max} such that $E = U_{\text{eff}}(r_{\text{min}}) = U_{\text{eff}}(r_{\text{max}})$. At these points $K_{\text{eff}} = 0$, therefore $dr/dt = 0$ which corresponds to a point of closest or furthest approach (Figure 25.6). This condition corresponds to the minimum and maximum values of r for an elliptic orbit.



(a) (b)

Figure 25.6 (a) elliptic orbit, (b) closest and furthest approach

The energy condition at these two points

$$E = \frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r}, \quad r = r_{\text{min}} = r_{\text{max}}, \tag{25.4.14}$$

is a quadratic equation for the distance r,

$$r^2 + \frac{G m_1 m_2}{E} r - \frac{L^2}{2\mu E} = 0. \tag{25.4.15}$$

There are two roots

$$r = -\frac{G m_1 m_2}{2E} \pm \left(\left(\frac{G m_1 m_2}{2E} \right)^2 + \frac{L^2}{2\mu E} \right)^{1/2}. \tag{25.4.16}$$

Equation (25.4.16) may be simplified somewhat as

$$r = -\frac{G m_1 m_2}{2E} \left(1 \pm \left(1 + \frac{2L^2 E}{\mu (G m_1 m_2)^2} \right)^{1/2} \right) \tag{25.4.17}$$

From Equation (25.3.14), the square root is the eccentricity ε,

$$\varepsilon = \left(1 + \frac{2EL^2}{\mu(G\,m_1\,m_2)^2}\right)^{\frac{1}{2}}, \qquad (25.4.18)$$

and Equation (25.4.17) becomes

$$r = -\frac{G\,m_1\,m_2}{2E}(1 \pm \varepsilon)\,. \qquad (25.4.19)$$

A little algebra shows that

$$\begin{aligned}\frac{r_0}{1-\varepsilon^2} &= \frac{L^2 / \mu G\,m_1\,m_2}{1 - \left(1 + \dfrac{2L^2 E}{\mu(G\,m_1\,m_2)^2}\right)} \\ &= \frac{L^2 / \mu G m_1 m_2}{-2L^2 E / \mu(G m_1 m_2)^2} \qquad (25.4.20) \\ &= -\frac{G\,m_1\,m_2}{2E}\,.\end{aligned}$$

Substituting the last expression in (25.4.20) into Equation (25.4.19) gives an expression for the points of closest and furthest approach,

$$r = \frac{r_0}{1-\varepsilon^2}(1 \pm \varepsilon) = \frac{r_0}{1 \mp \varepsilon}\,. \qquad (25.4.21)$$

The minus sign corresponds to the distance of closest approach,

$$r \equiv r_{\text{min}} = \frac{r_0}{1+\varepsilon} \qquad (25.4.22)$$

and the plus sign corresponds to the distance of furthest approach,

$$r \equiv r_{\text{max}} = \frac{r_0}{1-\varepsilon}\,. \qquad (25.4.23)$$

25.4.3 Parabolic Orbit $E = 0$

The effective potential energy, as given in Equation (25.4.1), approaches zero ($U_{\text{eff}} \to 0$) when the distance r approaches infinity ($r \to \infty$). When $E = 0$, as $r \to \infty$, the kinetic energy also approaches zero, $K_{\text{eff}} \to 0$. This corresponds to a parabolic orbit (see Equation (25.3.23)). Recall that in order for a body to escape from a planet, the body must have an energy $E = 0$ (we set $U_{\text{eff}} = 0$ at infinity). This *escape velocity* condition corresponds to a parabolic orbit. For a parabolic orbit, the body also has a distance of closest approach. This distance r_{par} can be found from the condition

$$E = U_{\text{eff}}(r_{\text{par}}) = \frac{L^2}{2\mu r_{\text{par}}^{\ 2}} - \frac{G m_1 m_2}{r_{\text{par}}} = 0 . \qquad (25.4.24)$$

Solving Equation (25.4.24) for r_{par} yields

$$r_{\text{par}} = \frac{L^2}{2\mu G m_1 m_2} = \frac{1}{2} r_0 ; \qquad (25.4.25)$$

the fact that the minimum distance to the origin (the *focus* of a parabola) is half the semilatus rectum is a well-known property of a parabola (Figure 25.5).

25.4.4 Hyperbolic Orbit $E > 0$

When $E > 0$, in the limit as $r \to \infty$ the kinetic energy is positive, $K_{\text{eff}} > 0$. This corresponds to a hyperbolic orbit (see Equation (25.3.24)). The condition for closest approach is similar to Equation (25.4.14) except that the energy is now positive. This implies that there is only one positive solution to the quadratic Equation (25.4.15), the distance of closest approach for the hyperbolic orbit

$$r_{\text{hyp}} = \frac{r_0}{1+\varepsilon} . \qquad (25.4.26)$$

The constant r_0 is independent of the energy and from Equation (25.3.14) as the energy of the single body increases, the eccentricity increases, and hence from Equation (25.4.26), the distance of closest approach gets smaller (Figure 25.5).

25.5 Orbits of the Two Bodies

The orbit of the single body can be circular, elliptical, parabolic or hyperbolic, depending on the values of the two constants of the motion, the angular momentum and the energy. Once we have the explicit solution (in this discussion, $r(\theta)$) for the single body, we can find the actual orbits of the two bodies.

Choose a coordinate system as we did for the reduction of the two-body problem (Figure 25.7).

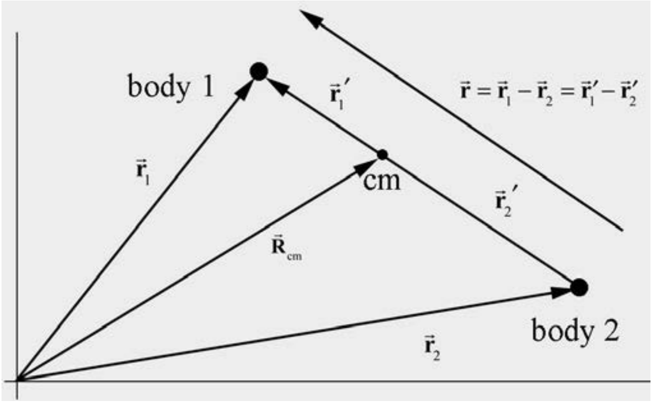


Figure 25.7 Center of mass coordinate system

The center of mass of the system is given by

$$\vec{\mathbf{R}}_{\mathrm{cm}} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{m_1 + m_2}. \tag{25.4.27}$$

Let $\vec{\mathbf{r}}_1{}'$ be the vector from the center of mass to body 1 and $\vec{\mathbf{r}}_2{}'$ be the vector from the center of mass to body 2. Then, by the geometry in Figure 25.6,

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2 = \vec{\mathbf{r}}_1' - \vec{\mathbf{r}}_2', \tag{25.4.28}$$

and hence

$$\vec{\mathbf{r}}_1' = \vec{\mathbf{r}}_1 - \vec{\mathbf{R}}_{\mathrm{cm}} = \vec{\mathbf{r}}_1 - \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{m_1 + m_2} = \frac{m_2(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2)}{m_1 + m_2} = \frac{\mu}{m_1}\vec{\mathbf{r}}. \tag{25.4.29}$$

A similar calculation shows that

$$\vec{\mathbf{r}}_2' = -\frac{\mu}{m_2}\vec{\mathbf{r}}. \tag{25.4.30}$$

Thus each body undergoes a motion about the center of mass in the same manner that the single body moves about the central point given by Equation (25.3.12). The only difference is that the distance from either body to the center of mass is shortened by a factor μ / m_i. When the orbit of the single body is an ellipse, then the orbits of the two bodies are also ellipses, as shown in Figure 25.8. When one mass is much smaller than the other, for example $m_1 << m_2$, then the reduced mass is approximately the smaller mass,

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \cong \frac{m_1 m_2}{m_2} = m_1. \tag{25.4.31}$$

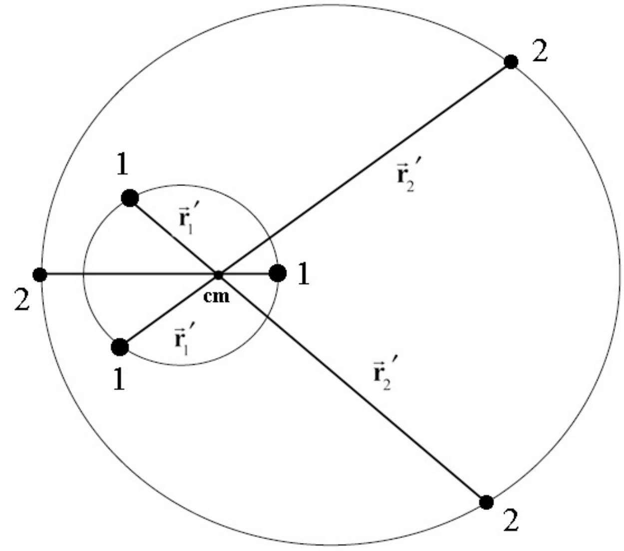


Figure 25.8 The elliptical motion of bodies interacting gravitationally

The center of mass is located approximately at the position of the larger mass, body 2 of mass m_2. Thus body 1 moves according to

$$\vec{\mathbf{r}}_1' = \frac{\mu}{m_1}\vec{\mathbf{r}} \cong \vec{\mathbf{r}}\,, \tag{25.4.32}$$

and body 2 is approximately stationary,

$$\vec{\mathbf{r}}_2' = -\frac{\mu}{m_2}\vec{\mathbf{r}} - \frac{m_1}{m_2}\vec{\mathbf{r}} \cong \vec{\mathbf{0}}\,. \tag{25.4.33}$$

25.6 Kepler's Laws

25.6.1 Elliptic Orbit Law

I. Each planet moves in an ellipse with the sun at one focus.

When the energy is negative, $E < 0$, and according to Equation (25.3.14),

$$\varepsilon = \left(1 + \frac{2EL^2}{\mu(G m_1 m_2)^2}\right)^{\frac{1}{2}} \tag{25.5.1}$$

and the eccentricity must fall within the range $0 \le \varepsilon < 1$. These orbits are either circles or ellipses. Note the elliptic orbit law is only valid if we assume that there is only one central force acting. We are ignoring the gravitational interactions due to all the other bodies in the universe, a necessary approximation for our analytic solution.

25.6.2 Equal Area Law

II. The radius vector from the sun to a planet sweeps out equal areas in equal time.

Using analytic geometry in the limit of small $\Delta\theta$, the sum of the areas of the triangles in Figure 25.9 is given by

$$\Delta A = \frac{1}{2}(r\,\Delta\theta)r + \frac{(r\,\Delta\theta)}{2}\Delta r \qquad (25.5.2)$$

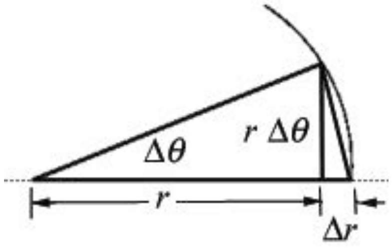


Figure 25.**9** Kepler's equal area law.

The average rate of the change of area, ΔA, in time, Δt, is given by

$$\Delta A = \frac{1}{2}\frac{(r\,\Delta\theta)r}{\Delta t} + \frac{(r\,\Delta\theta)}{2}\frac{\Delta r}{\Delta t}. \qquad (25.5.3)$$

In the limit as $\Delta t \to 0$, $\Delta\theta \to 0$, this becomes

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}. \qquad (25.5.4)$$

Recall that according to Equation (25.3.7) (reproduced below as Equation (25.5.5)), the angular momentum is related to the angular velocity $d\theta / dt$ by

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2} \qquad (25.5.5)$$

and Equation (25.5.4) is then

$$\frac{dA}{dt} = \frac{L}{2\mu}. \qquad (25.5.6)$$

Because L and μ are constants, the rate of change of area with respect to time is a constant. This is often familiarly referred to by the expression: *equal areas are swept out in equal times* (see Kepler's Laws at the beginning of this chapter).

25.6.3 Period Law

> *III. The period of revolution T of a planet about the sun is related to the semi-major axis a of the ellipse by $T^2 = k\,a^3$ where k is the same for all planets.*

When Kepler stated his period law for planetary orbits based on observation, he only noted the dependence on the larger mass of the sun. Because the mass of the sun is much greater than the mass of the planets, his observation is an excellent approximation.

In order to demonstrate the third law we begin by rewriting Equation (25.5.6) in the form

$$2\mu \frac{dA}{dt} = L\,. \tag{25.5.7}$$

Equation (25.5.7) can be integrated as

$$\int\limits_{\text{orbit}} 2\mu\, dA = \int_0^T L\, dt\,, \tag{25.5.8}$$

where T is the period of the orbit. For an ellipse,

$$\text{Area} = \int\limits_{\text{orbit}} dA = \pi\, ab\,, \tag{25.5.9}$$

where a is the semi-major axis and b is the semi-minor axis (Figure 25.10).

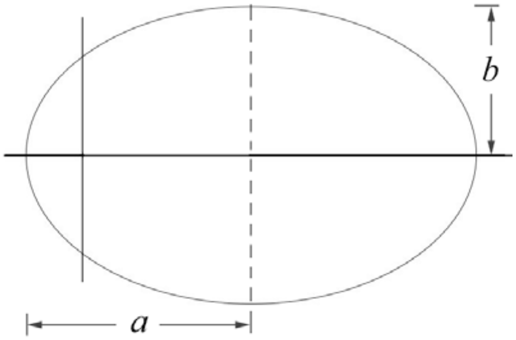


Figure 25.10 Semi-major and semi-minor axis for an ellipse

Thus we have

$$T = \frac{2\mu\pi\, ab}{L}\,. \tag{25.5.10}$$

Squaring Equation (25.5.10) then yields

$$T^2 = \frac{4\pi^2\mu^2 a^2 b^2}{L^2}. \qquad (25.5.11)$$

In Appendix 25B, Equation (25.B.20) gives the angular momentum in terms of the semi-major axis and the eccentricity. Substitution for the angular momentum into Equation (25.5.11) yields

$$T^2 = \frac{4\pi^2\mu^2 a^2 b^2}{\mu G m_1 m_2 a(1-\varepsilon^2)}. \qquad (25.5.12)$$

In Appendix 25B, Equation (25.B.17) gives the semi-minor axis which upon substitution into Equation (25.5.12) yields

$$T^2 = \frac{4\pi^2\mu^2 a^3}{\mu G m_1 m_2}. \qquad (25.5.13)$$

Using Equation (25.2.1) for reduced mass, the square of the period of the orbit is proportional to the semi-major axis cubed,

$$T^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)}. \qquad (25.5.14)$$

25.7 Worked Examples

Example 25.1 Elliptic Orbit

A satellite of mass m_s is in an elliptical orbit around a planet of mass $m_p >> m_s$. The planet is located at one focus of the ellipse. The satellite is at the distance r_a when it is furthest from the planet. The distance of closest approach is r_p (Figure 25.11). What is (i) the speed v_p of the satellite when it is closest to the planet and (ii) the speed v_a of the satellite when it is furthest from the planet?

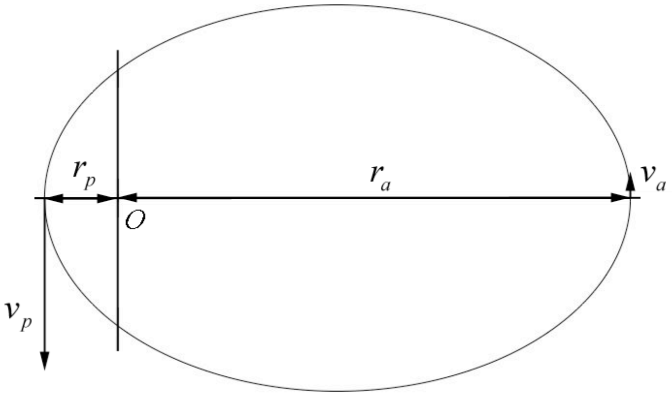


Figure 25.11 Example 25.1

Solution: The angular momentum about the origin is constant and because $\vec{\mathbf{r}}_{O,a} \perp \vec{\mathbf{v}}_a$ and $\vec{\mathbf{r}}_{O,p} \perp \vec{\mathbf{v}}_p$, the magnitude of the angular momentums satisfies

$$L \equiv L_{O,p} = L_{O,a} . \tag{25.6.1}$$

Because $m_s << m_p$, the reduced mass $\mu \cong m_s$ and so the angular momentum condition becomes

$$L = m_s r_p v_p = m_s r_a v_a \tag{25.6.2}$$

We can solve for v_p in terms of the constants G, m_p, r_a and r_p as follows. Choose zero for the gravitational potential energy, $U(r = \infty) = 0$. When the satellite is at the maximum distance from the planet, the mechanical energy is

$$E_a = K_a + U_a = \frac{1}{2} m_s v_a{}^2 - \frac{G m_s m_p}{r_a} . \tag{25.6.3}$$

When the satellite is at closest approach the energy is

$$E_p = \frac{1}{2} m_s v_p{}^2 - \frac{G m_s m_p}{r_p} . \tag{25.6.4}$$

Mechanical energy is constant,

$$E \equiv E_a = E_p \ , \tag{25.6.5}$$

therefore

$$E = \frac{1}{2} m_s v_p{}^2 - \frac{G m_s m_p}{r_p} = \frac{1}{2} m_s v_a{}^2 - \frac{G m_s m_p}{r_a} . \tag{25.6.6}$$

From Eq. (25.6.2) we know that

$$v_a = (r_p / r_a) v_p . \tag{25.6.7}$$

Substitute Eq. (25.6.7) into Eq. (25.6.6) and divide through by $m_s / 2$ yields

$$v_p{}^2 - \frac{2 G m_p}{r_p} = \frac{r_p{}^2}{r_a{}^2} v_p{}^2 - \frac{2 G m_p}{r_a} . \tag{25.6.8}$$

We can solve this Eq. (25.6.8) for v_p:

$$
\begin{aligned}
&v_p{}^2\left(1-\frac{r_p{}^2}{r_a{}^2}\right)=2Gm_p\left(\frac{1}{r_p}-\frac{1}{r_a}\right)\Rightarrow\\
&v_p{}^2\left(\frac{r_a{}^2-r_p{}^2}{r_a{}^2}\right)=2Gm_p\left(\frac{r_a-r_p}{r_pr_a}\right)\Rightarrow\\
&v_p{}^2\left(\frac{(r_a-r_p)(r_a+r_p)}{r_a{}^2}\right)=2Gm_p\left(\frac{r_a-r_p}{r_pr_a}\right)\Rightarrow\\
&v_p=\sqrt{\frac{2Gm_pr_a}{(r_a+r_p)r_p}}.
\end{aligned}
\tag{25.6.9}
$$

We now use Eq. (25.6.7) to determine that

$$
v_a=(r_p/r_a)v_p=\sqrt{\frac{2Gm_pr_p}{(r_a+r_p)r_a}}\,. \tag{25.6.10}
$$

Example 25.2 The Motion of the Star SO-2 around the Black Hole at the Galactic Center

The UCLA Galactic Center Group, headed by Dr. Andrea Ghez, measured the orbits of many stars within $0.8''\times 0.8''$ of the galactic center. The orbits of six of those stars are shown in Figure 25.12.

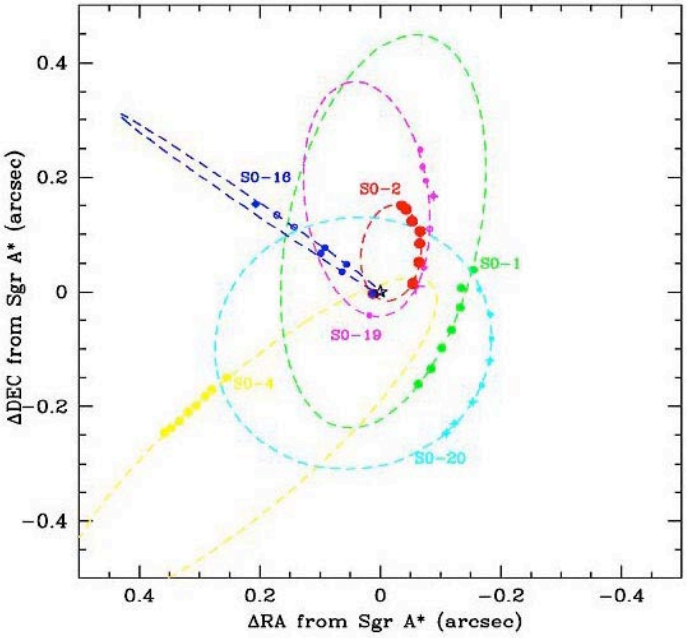


Figure 25.12 Obits of six stars near black hole at center of Milky Way galaxy.

We shall focus on the orbit of the star S0-2 with the following orbit properties given in Table 25.1[3]. Distances are given in astronomical units, $1\,\text{au} = 1.50 \times 10^{11}\ \text{m}$, which is the mean distance between the earth and the sun.

Table 25.1 Orbital Properties of S0-2

Star	Period (yrs)	Eccentricity	Semi-major axis ($10^{-3}\,\text{arc sec}$)	Periapse (au)	Apoapse (au)
S0-2	15.2 (0.68/0.76)	0.8763 (0.0063)	120.7 (4.5)	119.5 (3.9)	1812 (73)

The period of S0-2 satisfies Kepler's Third Law, given by

$$T^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)}, \tag{25.6.11}$$

where m_1 is the mass of S0-2, m_2 is the mass of the black hole, and a is the semi-major axis of the elliptic orbit of S0-2. (a) Determine the mass of the black hole that the star S0-2 is orbiting. What is the ratio of the mass of the black hole to the solar mass? (b) What is the speed of S0-2 at periapse (distance of closest approach to the center of the galaxy) and apoapse (distance of furthest approach to the center of the galaxy)?

Solution: (a) The semi-major axis is given by

$$a = \frac{r_p + r_a}{2} = \frac{119.5\,\text{au} + 1812\,\text{au}}{2} = 965.8\,\text{au}\,. \tag{25.6.12}$$

In SI units (meters), this is

$$a = 965.8\,\text{au}\frac{1.50 \times 10^{11}\,\text{m}}{1\,\text{au}} = 1.45 \times 10^{14}\,\text{m}\,. \tag{25.6.13}$$

The mass m_1 of the star S0-2 is much less than the mass m_2 of the black hole, and Equation (25.6.11) can be simplified to

$$T^2 = \frac{4\pi^2 a^3}{G\,m_2}\,. \tag{25.6.14}$$

Solving for the mass m_2 and inserting the numerical values, yields

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[3] A.M.Ghez, et al., Stellar Orbits Around Galactic Center Black Hole, preprint arXiv:astro-ph/0306130v1, 5 June, 2003.

$$\begin{aligned} m_2 &= \frac{4\pi^2 a^3}{GT^2} \\ &= \frac{(4\pi^2)(1.45\times 10^{14}\,\mathrm{m})^3}{(6.67\times 10^{-11}\,\mathrm{N\cdot m^2\cdot kg^{-2}})((15.2\,\mathrm{yr})(3.16\times 10^{7}\,\mathrm{s\cdot yr^{-1}}))^2} \\ &= 7.79\times 10^{34}\,\mathrm{kg}. \end{aligned} \tag{25.6.15}$$

The ratio of the mass of the black hole to the solar mass is

$$\frac{m_2}{m_{\mathrm{sun}}} = \frac{7.79\times 10^{34}\,\mathrm{kg}}{1.99\times 10^{30}\,\mathrm{kg}} = 3.91\times 10^{6}\,. \tag{25.6.16}$$

The mass of black hole corresponds to nearly four million solar masses.

(b) We can use our results from Example 25.1 that

$$v_p = \sqrt{\frac{2Gm_2 r_a}{(r_a + r_p)r_p}} = \sqrt{\frac{Gm_2 r_a}{ar_p}} \tag{25.6.17}$$

$$v_a = \frac{r_p}{r_a} v_p = \sqrt{\frac{2Gm_2 r_p}{(r_a + r_p)r_a}} = \sqrt{\frac{Gm_2 r_p}{ar_a}}\,, \tag{25.6.18}$$

where $a = (r_a + r_b)/2$ is the semi-major axis. Inserting numerical values,

$$\begin{aligned} v_p &= \sqrt{\frac{Gm_2}{a}\frac{r_a}{r_p}} \\ &= \sqrt{\frac{(6.67\times 10^{-11}\,\mathrm{N\cdot m^2\cdot kg^{-2}})(7.79\times 10^{34}\,\mathrm{kg})}{(1.45\times 10^{14}\,\mathrm{m})}\left(\frac{1812}{119.5}\right)} \\ &= 7.38\times 10^{6}\,\mathrm{m\cdot s^{-1}}. \end{aligned} \tag{25.6.19}$$

The speed v_{a} at apoapse is then

$$v_a = \frac{r_p}{r_{\mathrm{a}}} v_p = \left(\frac{1812}{119.5}\right)(7.38\times 10^{6}\,\mathrm{m\cdot s^{-1}}) = 4.87\times 10^{5}\,\mathrm{m\cdot s^{-1}}. \tag{25.6.20}$$

Example 25.3 Central Force Proportional to Distance Cubed

A particle of mass m moves in plane about a central point under an attractive central force of magnitude $F = br^3$. The magnitude of the angular momentum about the central point is equal to L. (a) Find the effective potential energy and make sketch of effective potential energy as a function of r. (b) Indicate on a sketch of the effective potential the total energy for circular motion. (c) The radius of the particle's orbit varies between r_0 and $2r_0$. Find r_0.

Solution: a) The potential energy, taking the zero of potential energy to be at $r = 0$, is

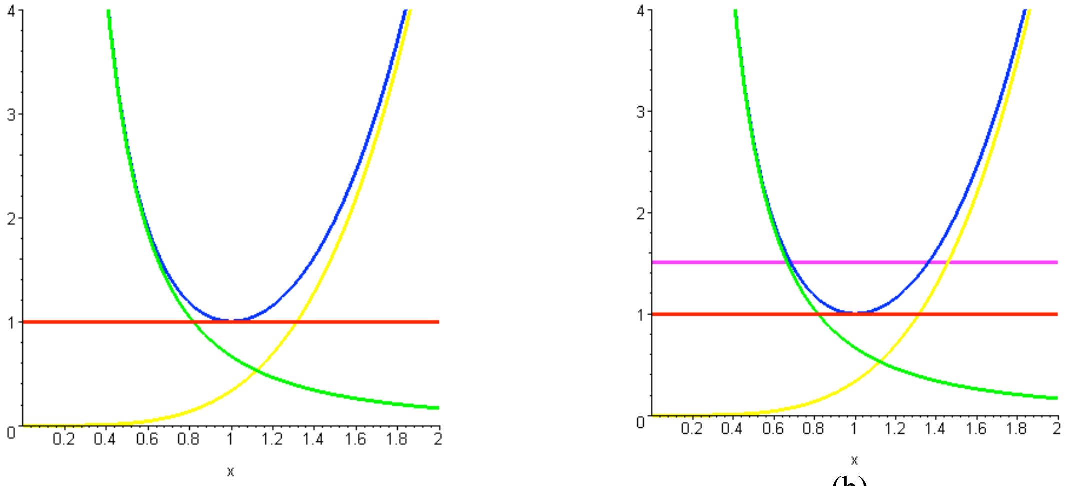
$$U(r) = -\int_0^r (-br'^3)\,dr' = \frac{b}{4}r^4$$

The effective potential energy is

$$U_{\text{eff}}(r) = \frac{L^2}{2mr^2} + U(r) = \frac{L^2}{2mr^2} + \frac{b}{4}r^4 .$$

A plot is shown in Figure 25.13a, including the potential (yellow, right-most curve), the term $L^2/2m$ (green, left-most curve) and the effective potential (blue, center curve). The horizontal scale is in units of r_0 (corresponding to radius of the lowest energy circular orbit) and the vertical scale is in units of the minimum effective potential.

b) The minimum effective potential energy is the horizontal line (red) in Figure 25.13a.



(a) (b)

Figure 25.13 (a) Effective potential energy with lowest energy state (red line), (b) higher energy state (magenta line)

c) We are trying to determine the value of r_0 such that $U_{\text{eff}}(r_0) = U_{\text{eff}}(2r_0)$. Thus

$$\frac{L^2}{mr_0^2} + \frac{b}{4}r_0^4 = \frac{L^2}{m(2r_0)^2} + \frac{b}{4}(2r_0)^4 .$$

Rearranging and combining terms, we can then solve for r_0,

$$\frac{3}{8}\frac{L^2}{m}\frac{1}{r_0^2}=\frac{15}{4}b\,r_0^4$$

$$r_0^6=\frac{1}{10}\frac{L^2}{mb}.$$

In the plot in Figure 25.13b, if we could move the red line up until it intersects the blue curve at two point whose value of the radius differ by a factor of 2, those would be the respective values for r_0 and $2r_0$. A graph, showing the corresponding energy as the horizontal magenta line, is shown in Figure 25.13b.

Example 25.4 Transfer Orbit

A space vehicle is in a circular orbit about the earth. The mass of the vehicle is $m_s = 3.00\times10^3\ \text{kg}$ and the radius of the orbit is $2R_e = 1.28\times10^4\ \text{km}$. It is desired to transfer the vehicle to a circular orbit of radius $4R_e$ (Figure 24.14). The mass of the earth is $M_e = 5.97\times10^{24}\ \text{kg}$. (a) What is the minimum energy expenditure required for the transfer? (b) An efficient way to accomplish the transfer is to use an elliptical orbit from point A on the inner circular orbit to a point B on the outer circular orbit (known as a Hohmann transfer orbit). What changes in speed are required at the points of intersection, A and B?

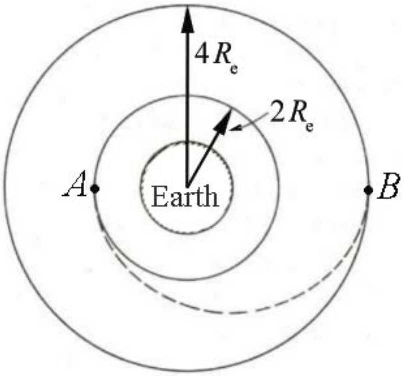


Figure 24.12 Example 25.5

Solution: (a) The mechanical energy is the sum of the kinetic and potential energies,

$$\begin{aligned}E &= K+U\\ &=\frac{1}{2}m_s v^2 - G\frac{m_s M_e}{R_e}.\end{aligned}\tag{25.6.21}$$

For a circular orbit, the orbital speed and orbital radius must be related by Newton's Second Law,

$$\begin{aligned}&F_r = m a_r\\&-G\frac{m_s M_e}{R_e^{\,2}} = -m_s\frac{v^2}{R_e} \Rightarrow\\&\frac{1}{2}m_s v^2 = \frac{1}{2}G\frac{m_s M_e}{R_e}.\end{aligned} \tag{25.6.22}$$

Substituting the last result in (25.6.22) into Equation (25.6.21) yields

$$E = \frac{1}{2}G\frac{m_s M_e}{R_e} - G\frac{m_s M_e}{R_e} = -\frac{1}{2}G\frac{m_s M_e}{R_e} = \frac{1}{2}U(R_e). \tag{25.6.23}$$

Equation (25.6.23) is one example of what is known as the ***Virial Theorem***, in which the energy is equal to (1/2) the potential energy for the circular orbit. In moving from a circular orbit of radius $2R_\mathrm{e}$ to a circular orbit of radius $4R_\mathrm{e}$, the total energy increases, (as the energy becomes less negative). The change in energy is

$$\begin{aligned}\Delta E &= E(r = 4R_e) - E(r = 2R_e)\\&= -\frac{1}{2}G\frac{m_s M_e}{4R_e} - \left(-\frac{1}{2}G\frac{m_s M_e}{2R_e}\right)\\&= \frac{Gm_s M_e}{8R_e}.\end{aligned} \tag{25.6.24}$$

Inserting the numerical values,

$$\begin{aligned}\Delta E &= \frac{1}{8}G\frac{m_s M_e}{R_e} = \frac{1}{4}G\frac{m_s M_e}{2R_e}\\&= \frac{1}{4}(6.67\times10^{-11}\,\mathrm{m}^3\cdot\mathrm{kg}^{-1}\cdot\mathrm{s}^{-2})\frac{(3.00\times10^3\ \mathrm{kg})(5.97\times10^{24}\,\mathrm{kg})}{(1.28\times10^4\ \mathrm{km})}\\&= 2.3\times10^{10}\,\mathrm{J}.\end{aligned} \tag{25.6.25}$$

b) The satellite must increase its speed at point A in order to move to the larger orbit radius and increase its speed again at point B to stay in the new circular orbit. Denote the satellite speed at point A while in the circular orbit as $v_{A,i}$ and after the speed increase (a "rocket burn") as $v_{A,f}$. Similarly, denote the satellite's speed when it first reaches point B as $v_{B,i}$. Once the satellite reaches point B, it then needs to increase its speed in order to continue in a circular orbit. Denote the speed of the satellite in the circular orbit at point B by $v_{B,f}$. The speeds $v_{A,i}$ and $v_{B,f}$ are given by Equation

(25.6.22). While the satellite is moving from point A to point B in the elliptic orbit (that is, during the transfer, after the first burn and before the second), both mechanical energy and angular momentum are conserved. Conservation of energy relates the speeds and radii by

$$\frac{1}{2}m_s(v_{A,f})^2 - G\frac{m_s m_e}{2R_e} = \frac{1}{2}m_s(v_{B,i})^2 - G\frac{m_s m_e}{4R_e}. \tag{25.6.26}$$

Conservation of angular momentum relates the speeds and radii by

$$m_s v_{A,f}(2R_e) = m_s v_{B,i}(4R_e) \;\Rightarrow\; v_{A,f} = 2v_{B,i}. \tag{25.6.27}$$

Substitution of Equation (25.6.27) into Equation (25.6.26) yields, after minor algebra,

$$v_{A,f} = \sqrt{\frac{2}{3}\frac{G\,M_e}{R_e}}, \quad v_{B,i} = \sqrt{\frac{1}{6}\frac{GM_e}{R_e}}\;. \tag{25.6.28}$$

We can now use Equation (25.6.22) to determine that

$$v_{A,i} = \sqrt{\frac{1}{2}\frac{G\,M_e}{R_e}}, \quad v_{B,f} = \sqrt{\frac{1}{4}\frac{G\,M_e}{R_e}}\;. \tag{25.6.29}$$

Thus the change in speeds at the respective points is given by

$$\begin{aligned}\Delta v_A &= v_{A,f} - v_{A,i} = \left(\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{2}}\right)\sqrt{\frac{G\,M_e}{R_e}}\\ \Delta v_B &= v_{B,f} - v_{B,i} = \left(\sqrt{\frac{1}{4}} - \sqrt{\frac{1}{6}}\right)\sqrt{\frac{G\,M_e}{R_e}}.\end{aligned} \tag{25.6.30}$$

Substitution of numerical values gives

$$\Delta v_A = 8.6\times 10^2\,\mathrm{m\cdot s^{-2}}, \quad \Delta v_B = 7.2\times 10^2\,\mathrm{m\cdot s^{-2}}. \tag{25.6.31}$$

Appendix 25A Derivation of the Orbit Equation

25A.1 Derivation of the Orbit Equation: Method 1

Start from Equation (25.3.11) in the form

$$d\theta = \frac{L}{\sqrt{2\mu}} \frac{(1/r^2)}{\left(E - \dfrac{L^2}{2\mu r^2} + \dfrac{G m_1 m_2}{r}\right)^{1/2}} dr \, . \qquad (25.A.1)$$

What follows involves a good deal of hindsight, allowing selection of convenient substitutions in the math in order to get a clean result. First, note the many factors of the reciprocal of r. So, we'll try the substitution $u = 1/r$, $du = -(1/r^2)\,dr$, with the result

$$d\theta = -\frac{L}{\sqrt{2\mu}} \frac{du}{\left(E - \dfrac{L^2}{2\mu} u^2 + G m_1 m_2 u\right)^{1/2}} \, . \qquad (25.A.2)$$

Experience in evaluating integrals suggests that we make the absolute value of the factor multiplying u^2 inside the square root equal to unity. That is, multiplying numerator and denominator by $\sqrt{2\mu}/L$,

$$d\theta = -\frac{du}{\left(2\mu E / L^2 - u^2 + 2(\mu G m_1 m_2 / L^2)u\right)^{1/2}} \, . \qquad (25.A.3)$$

As both a check and a motivation for the next steps, note that the left side $d\theta$ of Equation (25.A.3) is dimensionless, and so the right side must be. This means that the factor of $\mu G m_1 m_2 / L^2$ in the square root must have the same dimensions as u, or length^{-1}; so, define $r_0 \equiv L^2 / \mu G m_1 m_2$. This is of course the *semilatus rectum* as defined in Equation (25.3.12), and it's no coincidence; this is part of the "hindsight" mentioned above. The differential equation then becomes

$$d\theta = -\frac{du}{(2\mu E / L^2 - u^2 + 2u / r_0)^{1/2}} \, . \qquad (25.A.4)$$

We now rewrite the denominator in order to express it terms of the eccentricity.

$$\begin{aligned} d\theta &= -\frac{du}{\left(2\mu E / L^2 + 1/r_0{}^2 - u^2 + 2u/r_0 - 1/r_0{}^2\right)^{1/2}} \\ &= -\frac{du}{\left(2\mu E / L^2 + 1/r_0{}^2 - (u - 1/r_0)^2\right)^{1/2}} \\ &= -\frac{r_0\, du}{\left(2\mu E r_0{}^2 / L^2 + 1 - (r_0\, u - 1)^2\right)^{1/2}}. \end{aligned} \tag{25.A.5}$$

We note that the combination of terms $2\mu E r_0{}^2 / L^2 + 1$ is dimensionless, and is in fact equal to the square of the eccentricity ε as defined in Equation (25.3.13); more hindsight. The last expression in (25.A.5) is then

$$d\theta = -\frac{r_0\, du}{\left(\varepsilon^2 - (r_0\, u - 1)^2\right)^{1/2}}. \tag{25.A.6}$$

From here, we'll combine a few calculus steps, going immediately to the substitution $r_0\, u - 1 = \varepsilon \cos\alpha$, $r_0\, du = -\varepsilon \sin\alpha\, d\alpha$, with the final result that

$$d\theta = -\frac{-\varepsilon \sin\alpha\, d\alpha}{\left(\varepsilon^2 - \varepsilon^2 \cos^2\alpha\right)^{1/2}} = d\alpha, \tag{25.A.7}$$

We now integrate Eq. (25.A.7) with the very simple result that

$$\theta = \alpha + \text{constant}. \tag{25.A.8}$$

We have a choice in selecting the constant, and if we pick $\theta = \alpha - \pi$, $\alpha = \theta + \pi$, $\cos\alpha = -\cos\theta$, the result is

$$r = \frac{1}{u} = \frac{r_0}{1 - \varepsilon\cos\theta}, \tag{25.A.9}$$

which is our desired result, Equation (25.3.11). Note that if we chose the constant of integration to be zero, the result would be

$$r = \frac{1}{u} = \frac{r_0}{1 + \varepsilon\cos\theta} \tag{25.A.10}$$

which is the same trajectory reflected about the "vertical" axis in Figure 25.3, indeed the same as rotating by π.

25A.2 Derivation of the Orbit Equation: Method 2

The derivation of Equation (25.A.9) in the form

$$u = \frac{1}{r_0}\left(1 - \varepsilon\cos\theta\right) \tag{25.A.11}$$

suggests that the equation of motion for the one-body problem might be manipulated to obtain a simple differential equation. That is, start from

$$\begin{gathered}\vec{\mathbf{F}} = \mu\vec{\mathbf{a}} \\ -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}} = \mu\left(\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{\mathbf{r}}.\end{gathered} \tag{25.A.12}$$

Setting the components equal, using the constant of motion $L = \mu r^2(d\theta/dt)$ and rearranging, Eq. (25.A.12) becomes

$$\mu\frac{d^2 r}{dt^2} = \frac{L^2}{\mu r^3} - \frac{Gm_1 m_2}{r^2}. \tag{25.A.13}$$

We now use the same substitution $u = 1/r$ and change the independent variable from t to r, using the chain rule twice, since Equation (25.A.13) is a second-order equation. That is, the first time derivative is

$$\frac{dr}{dt} = \frac{dr}{du}\frac{du}{dt} = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt}. \tag{25.A.14}$$

From $r = 1/u$ we have $dr/du = -1/u^2$. Combining with $d\theta/dt$ in terms of L and u, $d\theta/dt = Lu^2/\mu$, Equation (25.A.14) becomes

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{Lu^2}{\mu} = -\frac{du}{d\theta}\frac{L}{\mu}, \tag{25.A.15}$$

a very tidy result, with the variable u appearing linearly. Taking the second derivative with respect to t,

$$\frac{d^2 r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{d\theta}\left(\frac{dr}{dt}\right)\frac{d\theta}{dt}. \tag{25.A.16}$$

Now substitute Eq. (25.A.15) into Eq. (25.A.16) with the result that

$$\frac{d^2 r}{dt^2} = -\frac{d^2 u}{d\theta^2}\left(u^2\frac{L^2}{\mu^2}\right). \tag{25.A.17}$$

Substituting into Equation (25.A.13), with $r = 1/u$ yields

$$-\frac{d^2u}{d\theta^2}u^2\frac{L^2}{\mu} = \frac{L^2}{\mu}u^3 - Gm_1m_2\,u^2. \tag{25.A.18}$$

Canceling the common factor of u^2 and rearranging, we arrive at

$$-\frac{d^2u}{d\theta^2} = u - \frac{\mu Gm_1m_2}{L^2}\ . \tag{25.A.19}$$

Equation (25.A.19) is mathematically equivalent to the simple harmonic oscillator equation with an additional constant term. The solution consists of two parts: the angle-independent solution

$$u_0 = \frac{\mu Gm_1m_2}{L^2} \tag{25.A.20}$$

and a sinusoidally varying term of the form

$$u_{\mathrm{H}} = A\cos(\theta - \theta_0)\,, \tag{25.A.21}$$

where A and θ_0 are constants determined by the form of the orbit. The expression in Equation (25.A.20) is the ***inhomogeneous solution*** and represents a circular orbit. The expression in Equation (25.A.21) is the ***homogeneous solution*** (as hinted by the subscript) and must have two independent constants. We can readily identify $1/u_0$ as the *semilatus rectum* r_0, with the result that

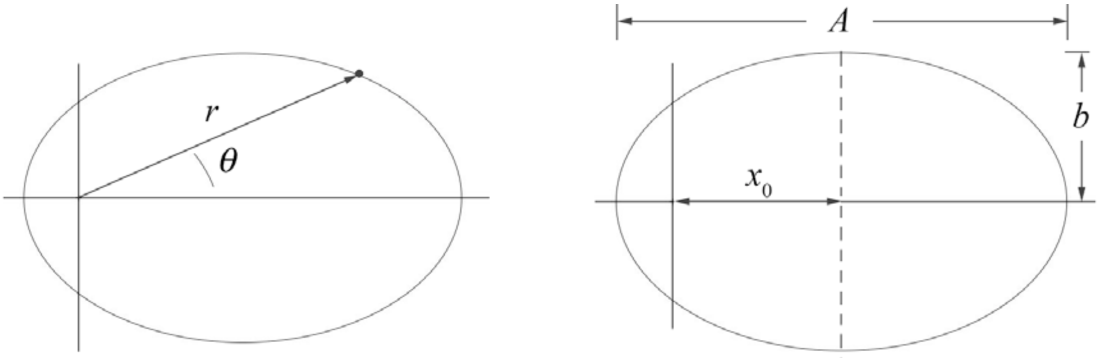
$$\begin{aligned} u = u_0 + u_{\mathrm{H}} &= \frac{1}{r_0}\left(1 + r_0A(\theta - \theta_0)\right) \Rightarrow \\ r = \frac{1}{u} &= \frac{r_0}{1 + r_0A(\theta - \theta_0)}. \end{aligned} \tag{25.A.22}$$

Choosing the product r_0A to be the eccentricity ε and $\theta_0 = \pi$ (much as was done leading to Equation (25.A.9) above), Equation (25.A.9) is reproduced.

Appendix 25B Properties of an Elliptical Orbit

25B.1 Coordinate System for the Elliptic Orbit

We now consider the special case of an elliptical orbit. Choose coordinates with the central point located at one focal point and coordinates (r,θ) for the position of the single body (Figure 25B.1a). In Figure 25B.1b, let a denote the semi-major axis, b denote the semi-minor axis and x_0 denote the distance from the center of the ellipse to the origin of our coordinate system.



(a) (b)

Figure 25B.1 (a) Coordinate system for elliptic orbit, (b) semi-major axis

25B.2 The Semi-major Axis

Recall the orbit equation, Eq, (25.A.9), describes $r(\theta)$,

$$r(\theta)=\frac{r_0}{1-\varepsilon\cos\theta}. \tag{25.B.1}$$

The major axis $A = 2a$ is given by

$$A=2a=r_a+r_p. \tag{25.B.2}$$

where the distance of furthest approach r_a occurs when $\theta = 0$, hence

$$r_a=r(\theta=0)=\frac{r_0}{1-\varepsilon}, \tag{25.B.3}$$

and the distance of nearest approach r_p occurs when $\theta=\pi$, hence

$$r_p=r(\theta=\pi)=\frac{r_0}{1+\varepsilon}. \tag{25.B.4}$$

Figure 25B.2 shows the distances of nearest and furthest approach.

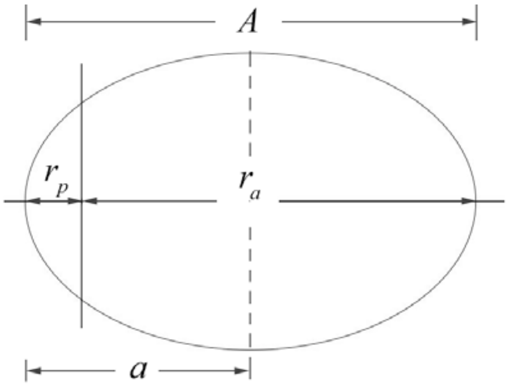


Figure 25B.2 Furthest and nearest approach

We can now determine the semi-major axis

$$a = \frac{1}{2}\left(\frac{r_0}{1-\varepsilon} + \frac{r_0}{1+\varepsilon}\right) = \frac{r_0}{1-\varepsilon^2}. \tag{25.B.5}$$

The *semilatus rectum* r_0 can be re-expressed in terms of the semi-major axis and the eccentricity,

$$r_0 = a(1-\varepsilon^2). \tag{25.B.6}$$

We can now express the distance of nearest approach, Equation (25.B.4), in terms of the semi-major axis and the eccentricity,

$$r_p = \frac{r_0}{1+\varepsilon} = \frac{a(1-\varepsilon^2)}{1+\varepsilon} = a(1-\varepsilon). \tag{25.B.7}$$

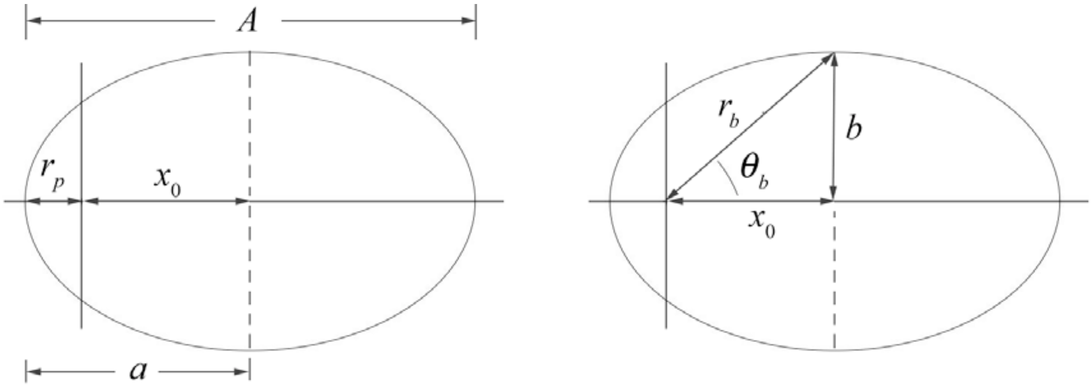
In a similar fashion the distance of furthest approach is

$$r_a = \frac{r_0}{1-\varepsilon} = \frac{a(1-\varepsilon^2)}{1-\varepsilon} = a(1+\varepsilon). \tag{25.B.8}$$

25B.2.3 The Location x_0 of the Center of the Ellipse

From Figure 25B.3a, the distance from a focus point to the center of the ellipse is

$$x_0 = a - r_p. \tag{25.B.9}$$



(a) (b)

Figure 25B.3 Location of the center of the ellipse and semi-minor axis.

Using Equation (25.B.7) for r_p, we have that

$$x_0 = a - a(1-\varepsilon) = \varepsilon\, a. \tag{25.B.10}$$

25B.2.4 The Semi-minor Axis

From Figure 25B.3b, the semi-minor axis can be expressed as

$$b = \sqrt{(r_b^{\ 2} - x_0^{\ 2})}\,, \tag{25.B.11}$$

where

$$r_b = \frac{r_0}{1 - \varepsilon \cos\theta_b}. \tag{25.B.12}$$

We can rewrite Eq. (25.B.12) as

$$r_b - r_b\, \varepsilon \cos\theta_b = r_0. \tag{25.B.13}$$

The horizontal projection of r_b is given by (Figure 25B.2b),

$$x_0 = r_b \cos\theta_b, \tag{25.B.14}$$

which upon substitution into Eq. (25.B.13) yields

$$r_b = r_0 + \varepsilon\, x_0. \tag{25.B.15}$$

Substituting Equation (25.B.10) for x_0 and Equation (25.B.6) for r_0 into Equation (25.B.15) yields

$$r_b = a(1-\varepsilon^2) + a\varepsilon^2 = a\,. \tag{25.B.16}$$

The fact that $r_b = a$ is a well-known property of an ellipse reflected in the geometric construction, that the sum of the distances from the two foci to any point on the ellipse is a constant. We can now determine the semi-minor axis b by substituting Eq. (25.B.16) into Eq. (25.B.11) yielding

$$b = \sqrt{\left(r_b^{\ 2} - x_0^{\ 2}\right)} = \sqrt{a^2 - \varepsilon^2 a^2} = a\sqrt{1-\varepsilon^2}\,. \tag{25.B.17}$$

25B.2.5 Constants of the Motion for Elliptic Motion

We shall now express the parameters a, b and x_0 in terms of the constants of the motion L, E, μ, m_1 and m_2. Using our results for r_0 and ε from Equations (25.3.13) and (25.3.14) we have for the semi-major axis

$$\begin{aligned} a &= \frac{L^2}{\mu G m_1 m_2}\frac{1}{(1-(1+2EL^2/\mu(Gm_1m_2)^2))} \\ &= -\frac{Gm_1m_2}{2E} \end{aligned}. \tag{25.B.18}$$

The energy is then determined by the semi-major axis,

$$E = -\frac{Gm_1m_2}{2a}. \tag{25.B.19}$$

The angular momentum is related to the *semilatus rectum* r_0 by Equation (25.3.13). Using Equation (25.B.6) for r_0, we can express the angular momentum (25.B.4) in terms of the semi-major axis and the eccentricity,

$$L = \sqrt{\mu G m_1 m_2 r_0} = \sqrt{\mu G m_1 m_2 a(1-\varepsilon^2)}\,. \tag{25.B.20}$$

Note that

$$\sqrt{(1-\varepsilon^2)} = \frac{L}{\sqrt{\mu G m_1 m_2 a}}\,, \tag{25.B.21}$$

Thus, from Equations (25.3.14), (25.B.10), and (25.B.18), the distance from the center of the ellipse to the focal point is

$$x_0 = \varepsilon\, a = -\frac{Gm_1m_2}{2E}\sqrt{\left(1+2EL^2/\mu\left(Gm_1m_2\right)^2\right)}, \tag{25.B.22}$$

a result we will return to later. We can substitute Eq. (25.B.21) for $\sqrt{1-\varepsilon^2}$ into Eq. (25.B.17), and determine that the semi-minor axis is

$$b = \sqrt{aL^2 / \mu G m_1 m_2} \,. \qquad (25.B.23)$$

We can now substitute Eq. (25.B.18) for a into Eq. (25.B.23), yielding

$$b = \sqrt{aL^2 / \mu G m_1 m_2} = L\sqrt{-\frac{G m_1 m_2}{2E} / \mu G m_1 m_2} = L\sqrt{-\frac{1}{2\mu E}} \,. \qquad (25.B.24)$$

25B.2.6 Speeds at Nearest and Furthest Approaches

At nearest approach, the velocity vector is tangent to the orbit (Figure 25B.4), so the magnitude of the angular momentum is

$$L = \mu r_p v_p \,, \qquad (25.B.25)$$

and the speed at nearest approach is

$$v_p = L / \mu r_p \,. \qquad (25.B.26)$$

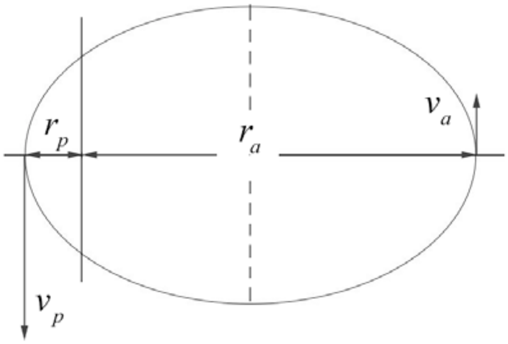


Figure 25B.4 Speeds at nearest and furthest approach

Using Equation (25.B.20) for the angular momentum and Equation (25.B.7) for r_p, Equation (25.B.26) becomes

$$v_p = \frac{L}{\mu r_p} = \frac{\sqrt{\mu G m_1 m_2 (1-\varepsilon^2)}}{\mu a (1-\varepsilon)} = \sqrt{\frac{G m_1 m_2 (1-\varepsilon^2)}{\mu a (1-\varepsilon)^2}} = \sqrt{\frac{G m_1 m_2 (1+\varepsilon)}{\mu a (1-\varepsilon)}} \,. \qquad (25.B.27)$$

A similar calculation show that the speed v_a at furthest approach,

$$v_a = \frac{L}{\mu r_a} = \frac{\sqrt{\mu G m_1 m_2 (1-\varepsilon^2)}}{\mu a (1+\varepsilon)} = \sqrt{\frac{G m_1 m_2 1-\varepsilon^2}{\mu a (1+\varepsilon)^2}} = \sqrt{\frac{G m_1 m_2 \left(1-\varepsilon\right)}{\mu a (1+\varepsilon)}} \,. \qquad (25.B.28)$$

Appendix 25C Analytic Geometric Properties of Ellipses

Consider Equation (25.3.20), and for now take $\varepsilon < 1$, so that the equation is that of an ellipse. We shall now show that we can write it as

$$\frac{(x - x_0)^2}{a^2} + \frac{y^2}{b^2} = 1\,, \tag{25.C.1}$$

where the ellipse has axes parallel to the x- and y-coordinate axes, center at $(x_0, 0)$, semi-major axis a and semi-minor axis b. We begin by rewriting Equation (25.3.20) as

$$x^2 - \frac{2\varepsilon r_0}{1-\varepsilon^2}x + \frac{y^2}{1-\varepsilon^2} = \frac{r_0^2}{1-\varepsilon^2}\,. \tag{25.C.2}$$

We next complete the square,

$$\begin{gathered}
x^2 - \frac{2\varepsilon r_0}{1-\varepsilon^2}x + \frac{\varepsilon^2 r_0^2}{(1-\varepsilon^2)^2} + \frac{y^2}{1-\varepsilon^2} = \frac{r_0^2}{1-\varepsilon^2} + \frac{\varepsilon^2 r_0^2}{(1-\varepsilon^2)^2} \Rightarrow \\
\left(x - \frac{\varepsilon r_0}{1-\varepsilon^2}\right)^2 + \frac{y^2}{1-\varepsilon^2} = \frac{r_0^2}{(1-\varepsilon^2)^2} \Rightarrow \\
\frac{\left(x - \dfrac{\varepsilon r_0}{1-\varepsilon^2}\right)^2}{\left(r_0/(1-\varepsilon^2)\right)^2} + \frac{y^2}{(r_0/\sqrt{1-\varepsilon^2})^2} = 1.
\end{gathered} \tag{25.C.3}$$

The last expression in (25.C.3) is the equation of an ellipse with semi-major axis

$$a = \frac{r_0}{1-\varepsilon^2}\,, \tag{25.C.4}$$

semi-minor axis

$$b = \frac{r_0}{\sqrt{1-\varepsilon^2}} = a\sqrt{1-\varepsilon^2}\,, \tag{25.C.5}$$

and center at

$$x_0 = \frac{\varepsilon r_0}{(1-\varepsilon^2)} = \varepsilon a\ \ , \tag{25.C.6}$$

as found in Equation (25.B.10).

Chapter 26 Elastic Properties of Materials

Chapter 26 Elastic Properties of Materials

26.1 Introduction

In our study of rotational and translational motion of a rigid body, we assumed that the rigid body did not undergo any deformations due to the applied forces. Real objects deform when forces are applied. They can stretch, compress, twist, or break. For example when a force is applied to the ends of a wire and the wire stretches, the length of the wire increases. More generally, when a force per unit area, referred to as *stress,* is applied to an object, the particles in the object may undergo a relative displacement compared to their unstressed arrangement. *Strain* is a normalized measure of this deformation. For example, the *tensile strain* in the stretched wire is fractional change in length of a stressed wire. The stress may not only induce a change in length, but it may result in a volume change as occurs when an object is immersed in a fluid, and the fluid exerts a force per unit area that is perpendicular to the surface of the object resulting in a *volume strain* which is the fractional change in the volume of the object. Another type of stress, known as a *shear stress* occurs when forces are applied tangential to the surface of the object, resulting in a deformation of the object. For example, when scissors cut a thin material, the blades of the scissors exert shearing stresses on the material causing one side of the material to move down and the other side of the material to move up as shown in Figure 26.1, resulting in a *shear strain*. The material deforms until it ultimately breaks.

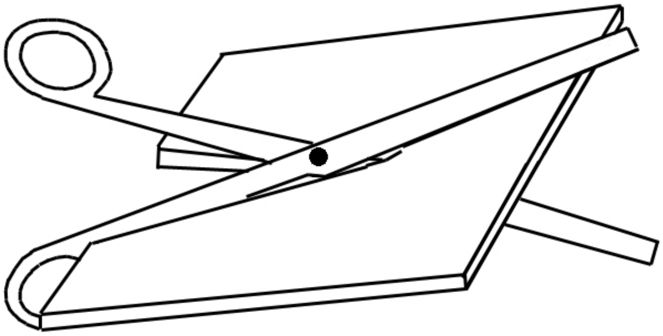


Figure 26.1: Scissors cutting a thin material[1]

In many materials, when the stress is small, the stress and strains are linearly proportional to one another. The material is then said to obey Hooke's Law. The ratio of stress to strain is called the *elastic modulus.* Hooke's Law only holds for a range of stresses, a range referred to as the *elastic region.* An *elastic body* is one in which Hooke's Law applies and when the applied stress is removed, the body returns to its initial shape. Our idealized spring is an example of an elastic body. Outside of the elastic region, the stress-strain relationship is non-linear until the object breaks.

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[1] Mohsen Mahvash, et al, IEEE Trans Biomed Eng. 2008, March; 55(3); 848-856.

26.2 Stress and Strain in Tension and Compression

Consider a rod with cross sectional area A and length l_0. Two forces of the same magnitude $F_\perp$ are applied perpendicularly at the two ends of the section stretching the rod to a length l (Figure 26.2), where the beam has been stretched by a positive amount $\delta l = l - l_0$.

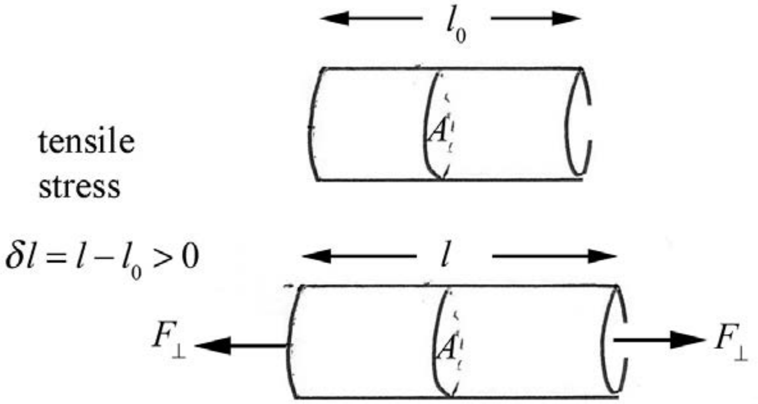


Figure 26.2: Tensile stress on a rod

The ratio of the applied perpendicular force to the cross-sectional area is called the *tensile stress*,

$$\sigma_T = \frac{F_\perp}{A}. \qquad (26.2.1)$$

The ratio of the amount the section has stretched to the original length is called the *tensile strain*,

$$\varepsilon_T = \frac{\delta l}{l_0}. \qquad (26.2.2)$$

Experimentally, for sufficiently small stresses, for many materials the stress and strain are linearly proportional,

$$\frac{F_\perp}{A} = Y\frac{\delta l}{l_0} \quad \text{(Hooke's Law)}. \qquad (26.2.3)$$

where the constant of proportionality Y is called **Young's modulus**. The SI unit for Young's Modulus is the **pascal** where $1\,\text{Pa} \equiv 1\,\text{N}\cdot\text{m}^{-2}$. Note the following conversion factors between SI and English units: $1\,\text{bar} \equiv 10^5\,\text{Pa}$, $1\,\text{psi} \equiv 6.9\times10^{-2}\,\text{bar}$, and $1\,\text{bar} = 14.5\,\text{psi}$. In Table 26.1, Young's Modulus is tabulated for various materials. Figure 26.3 shows a plot of the stress-strain relationship for various human bones. For stresses greater than approximately $70\,\mathrm{N}\cdot\mathrm{mm}^{-2}$, the material is no longer elastic. At a certain point for each bone, the stress-strain relationship stops, representing the fracture point.

Material	Young's Modulus, Y (Pa)
Iron	21×10^{10}
Nickel	21×10^{10}
Steel	20×10^{10}
Copper	11×10^{10}
Brass	9.0×10^{10}
Aluminum	7.0×10^{10}
Crown Glass	6.0×10^{10}
Cortical Bone	$7\times10^{9}-30\times10^{9}$
Lead	1.6×10^{10}
Tendon	2×10^{7}
Rubber	$7\times10^{5}-40\times10^{5}$
Blood vessels	2×10^{5}

Table 26.1: Young's Modulus for various materials

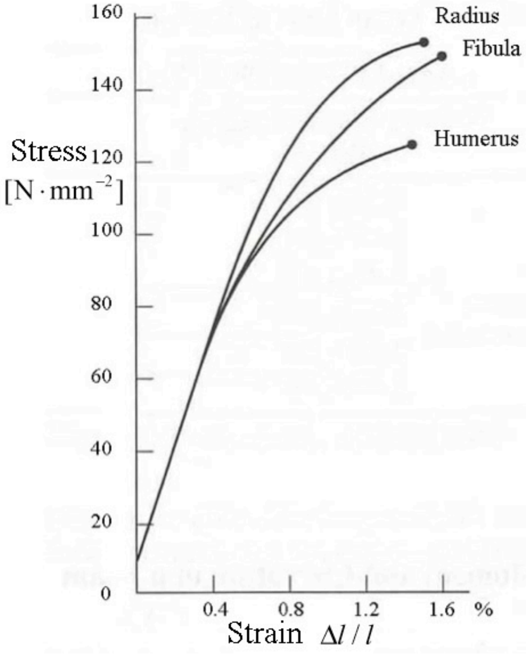


Figure 26.3: Stress-strain relation for various human bones (figure from H. Yamada, Strength of Biological Materials)

When the material is under compression, the forces on the ends are directed towards each other producing a *compressive stress* resulting in a *compressive strain* (Figure 26.4). For compressive strains, if we define $\delta l = l_0 - l > 0$ then Eq. (26.2.3) holds for compressive stresses provided the compressive stress is not too large. For many materials, Young's Modulus is the same when the material is under tension and compression. There are some important exceptions. Concrete and stone can undergo compressive stresses but fail when the same tensile stress is applied. When building with these materials, it is important to design the structure so that the stone or concrete is never under tensile stresses. Arches are used as an architectural structural element primarily for this reason.

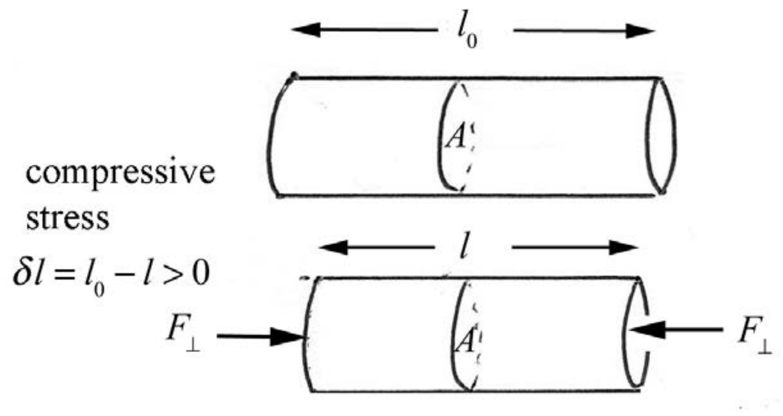


Figure 26.4: Compressive Stress

26.3 Shear Stress and Strain

The surface of material may also be subjected to tangential forces producing a shearing action. Consider a block of height h and area A, in which a tangential force, $\vec{\mathbf{F}}_{\tan}$, is applied to the upper surface. The lower surface is held fixed. The upper surface will shear by an angle α corresponding to a horizontal displacement δx. The geometry of the shearing action is shown in Figure 26.5.

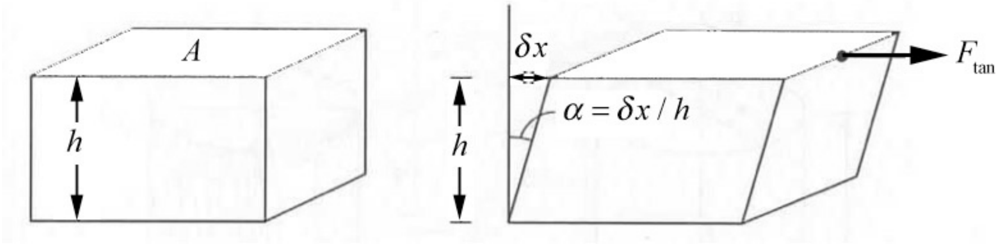


Figure 26.5: Shearing forces

The *shear stress* is defined to be the ratio of the tangential force to the cross sectional area of the surface upon which it acts,

$$\sigma_S = \frac{F_{\tan}}{A}. \tag{26.3.1}$$

The *shear strain* is defined to be the ratio of the horizontal displacement to the height of the block,

$$\alpha = \frac{\delta x}{h}. \tag{26.3.2}$$

For many materials, when the shear stress is sufficiently small, experiment shows that a Hooke's Law relationship holds in that the shear stress is proportional to shear strain,

$$\frac{F_{\tan}}{A} = S\frac{\delta x}{h} \quad \text{(Hooke's Law)}\,. \qquad (26.3.3)$$

where the constant of proportional, S , is called the **shear modulus**. When the deformation angle is small, $\delta x / h = \tan\alpha \simeq \sin\alpha \simeq \alpha$, and Eq. (26.3.3) becomes

$$\frac{F_{\tan}}{A} \simeq S\alpha \quad \text{(Hooke's Law)}\,. \qquad (26.3.4)$$

In Table 26.2, the shear modulus is tabulated for various materials.

Table 26.2: Shear Modulus for Various Materials

Material	Shear Modulus, *S* (Pa)
Nickel	7.8×10^{10}
Iron	7.7×10^{10}
Steel	7.5×10^{10}
Copper	4.4×10^{10}
Brass	3.5×10^{10}
Aluminum	2.5×10^{10}
Crown Glass	2.5×10^{10}
Lead	0.6×10^{10}
Rubber	$2\times10^{5}-10\times10^{5}$

Example 26.1: Stretched wire

An object of mass 1.5×10^{1} kg is hanging from one end of a steel wire. The wire without the mass has an unstretched length of 0.50 m . What is the resulting strain and elongation of the wire? The cross-sectional area of the wire is $1.4\times10^{-2}\ \text{cm}^2$.

Solution: When the hanging object is attached to the wire, the force at the end of the wire acting on the object exactly balances the gravitational force. Therefore by Newton's Third Law, the tensile force stressing the wire is

$$F_{\perp} = mg\,. \qquad (26.3.5)$$

We can calculate the strain on the wire from Hooke's Law (Eq. (26.2.3)) and the value of Young's modulus for steel 20×10^{10} Pa (Table 26.1);

$$\frac{\delta l}{l_0} = \frac{F_\perp}{YA} = \frac{mg}{YA} = \frac{(1.5\times10^1\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2})}{(2.0\times10^{11}\ \text{Pa})(1.4\times10^{-6}\ \text{m}^2)} = 5.3\times10^{-4}. \tag{26.3.6}$$

The elongation δl of the wire is then

$$\delta l = \frac{mg}{YA} l_0 = (5.3\times10^{-4})(0.50\ \text{m}) = 2.6\times10^{-4}\ \text{m}. \tag{26.3.7}$$

26.4 Elastic and Plastic Deformation

Consider a single sheet of paper. If we bend the paper gently, and then release the constraining forces, the sheet will return to its initial state. This process of gently bending is reversible as the paper displays *elastic behavior*. The internal forces responsible for the deformation are conservative. Although we do not have a simple mathematical model for the potential energy, we know that mechanical energy is constant during the bending. We can take the same sheet of paper and crumple it. When we release the paper it will no longer return to its original sheet but will have a permanent deformation. The internal forces now include non-conservative forces and the mechanical energy is decreased. This *plastic behavior* is irreversible.

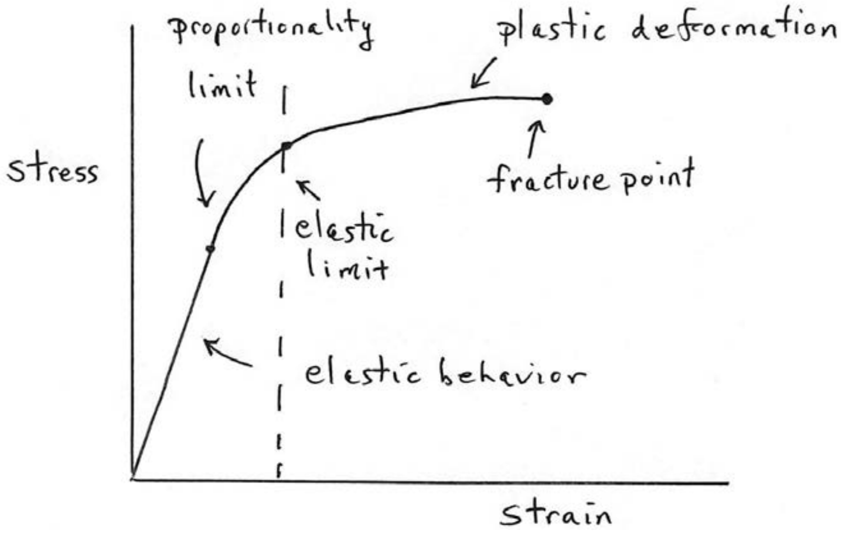


Figure 26.5: Stress-strain relationship

When the stress on a material is linearly proportional to the strain, the material behaves according to Hooke's Law. The *proportionality limit* is the maximum value of stress at which the material still satisfies Hooke's Law. If the stress is increased above the proportionality limit, the stress is no longer linearly proportional to the strain. However, if the stress is slowly removed then the material will still return to its original state; the material behaves elastically. If the stress is above the proportionality limit, but less then the elastic limit, then the stress is no longer linearly proportional to the strain. Even in this non-linear region, if the stress is slowly removed then the material will return to its original state. The maximum value of stress in which the material will still remain elastic is called the *elastic limit*. For stresses above the elastic limit, when the stress is removed the material will not return to its original state and some permanent deformation sets in, a state referred to as a *permanent set*. This behavior is referred to as *plastic deformation*. For a sufficiently large stress, the material will *fracture*. Figure 26.5 illustrates a typical stress-strain relationship for a material. The value of the stress that fractures a material is referred to as the *ultimate tensile strength.* The ultimate tensile strengths for various materials are listed in Table 26.3. The tensile strengths for wet human bones are for a person whose age is between 20 and 40 years old.

Table 26.3: Ultimate Tensile Strength for Various Materials

Material	Shear Modulus, *S* (Pa)
Femur	1.21×10^{8}
Humerus	1.22×10^{8}
Tibia	1.40×10^{8}
Fibula	1.46×10^{8}
Ulna	1.48×10^{8}
Radius	1.49×10^{8}
Aluminum	2.2×10^{8}
Iron	3.0×10^{8}
Brass	4.7×10^{8}
Steel	$5-20\times10^{8}$

Example 26.2: Ultimate Tensile Strength of Bones

The ultimate tensile strength of the wet human tibia (for a person of age between 20 and 40 years) is $1.40\times10^{8}\ \text{Pa}$. If a greater compressive force per area is applied to the tibia then the bone will break. The smallest cross sectional area of the tibia, about $3.2\ \text{cm}^2$, is slightly above the ankle. Suppose a person of mass $60\ \text{kg}$ jumps to the ground from a height $2.0\ \text{m}$ and absorbs the shock of hitting the ground by bending the knees. Assume that there is constant deceleration during the collision. During the collision, the person lowers her center of mass by an amount $d = 1.0\ \text{cm}$. (a) What is the collision time Δt_{col}? (b) Find the average force of the ground on the person during the collision. (c) Can we effectively ignore the gravitational force during the collision? (d) Will the person break her ankle? (e) What is the minimum distance $\Delta d_{\min}$ that she would need to lower her center of mass so she does not break her ankle? What is the ratio $h_0 / \Delta d_{\min}$? What factors does this ratio depend on?

Solution: Choose a coordinate system with the positive y-direction pointing up, and the origin at the ground. While the person is falling to the ground, mechanical energy is constant (we are neglecting any non-conservative work due to air resistance). Choose the contact point with the ground as the zero potential energy reference point. Then the initial mechanical energy is

$$E_0 = U_0 = mgh_0 . \tag{26.3.8}$$

The mechanical energy of the person just before contact with the ground is

$$E_b = K_1 = \frac{1}{2}m{v_b}^2 . \tag{26.3.9}$$

The constancy of mechanical energy implies that

$$mgh_0 = \frac{1}{2}m{v_b}^2 . \tag{26.3.10}$$

The speed of the person the instant contact is made with the ground is then

$$v_b = \sqrt{2gh_0} . \tag{26.3.11}$$

If we treat the person as the system then there are two external forces acting on the person, the gravitational force $\vec{\mathbf{F}}^g = -mg\hat{\mathbf{j}}$ and a normal force between the ground and the person $\vec{\mathbf{F}}^N = N\hat{\mathbf{j}}$. This force varies with time but we shall consider the time average $\vec{\mathbf{F}}^N_{\text{ave}} = N_{\text{ave}}\hat{\mathbf{j}}$. Then using Newton's Second Law,

$$N_{\text{ave}} - mg = ma_{y,\text{ave}} . \tag{26.3.12}$$

The y-component of the average acceleration is equal to

$$a_{y,\text{ave}} = \frac{N_{\text{ave}}}{m} - g . \tag{26.3.13}$$

Set $t = 0$ for the instant the person reaches the ground; then $v_{y,0} = -v_b$. The displacement of the person while in contact with the ground for the time interval Δt_{col} is given by

$$\Delta y = -v_b\Delta t_{\text{col}} + \frac{1}{2}a_{y,\text{ave}}\Delta t_{\text{col}}^2 . \tag{26.3.14}$$

The y-component of the velocity is zero at $t = \Delta t_{\text{col}}$ when the person's displacement is $\Delta y = -d$,

$$0 = -v_b + a_{y,\text{ave}} \Delta t_{\text{col}} . \tag{26.3.15}$$

Solving Eq. (26.3.15) for the collision time yields

$$\Delta t_{\text{col}} = v_b / a_{y,\text{ave}} . \tag{26.3.16}$$

We can now substitute $\Delta y = -d$, Eq. (26.3.16), and Eq. (26.3.11) into Eq. (26.3.14) and solve for the y-component of the acceleration, yielding

$$a_{y,\text{ave}} = \frac{gh_0}{d} . \tag{26.3.17}$$

We can solve for the collision time by substituting Eqs. (26.3.17) and Eq. (26.3.11) into Eq. (26.3.16) and using the given values in the problem statement, yielding

$$\Delta t_{\text{col}} = \frac{2d}{\sqrt{2gh_0}} = \frac{2(1.0 \times 10^{-2} \text{m})}{\sqrt{2(9.8 \text{ m} \cdot \text{s}^2)(2.0 \text{ m})}} = 3.2 \times 10^{-3} \text{s} . \tag{26.3.18}$$

Now substitute Eq. (26.3.17) for the y-component of the acceleration into Eq. (26.3.13) and solve for the average normal force

$$N_{\text{ave}} = mg\left(1 + \frac{h_0}{d}\right) = (60 \text{ kg})(9.8 \text{ m} \cdot \text{s}^{-2})\left(1 + \frac{(2.0 \text{ m})}{(1.0 \times 10^{-2} \text{ m})}\right) = 1.2 \times 10^5 \text{ N} . \tag{26.3.19}$$

Notice that the factor $1 + h_0 / d \simeq h_0 / d$ so *during the collision* we can effectively ignore the external gravitational force. The average compressional force per area on the person's ankle is the average normal force divided by the cross sectional area

$$P = \frac{N_{\text{ave}}}{A} \simeq \frac{mg}{A}\left(\frac{h_0}{d}\right) = \frac{1.2 \times 10^5 \text{ N}}{3.2 \times 10^{-4} \text{ m}^2} = 3.7 \times 10^8 \text{ Pa} . \tag{26.3.20}$$

From Table 26.3, the tensile strength of the tibia is 1.4×10^8 Pa, so this fall is enough to break the tibia.

In order to find the minimum displacement that the center of mass must fall in order to avoid breaking the tibia bone, we set the force per area in Eq. (26.3.20) equal to $P = 1.4 \times 10^8$ Pa. Because at this value of tensile strength,

$$\frac{PA}{mg} = \frac{(1.4 \times 10^{8}\ \text{Pa})((3.2 \times 10^{-4}\ \text{m}^{2})}{(60\ \text{kg})(9.8\ \text{m} \cdot \text{s}^{-2})} = 80 \qquad (26.3.21)$$

and so $PA >> mg$. We can solve Eq. (26.3.20) for the minimum displacement

$$d_{\min} = \frac{h_0}{\left(\frac{PA}{mg} - 1\right)} \simeq \frac{mgh_0}{PA} = \frac{(60\ \text{kg})(9.8\ \text{m} \cdot \text{s}^{-2})(2.0\ \text{m})}{(1.4 \times 10^{8}\ \text{Pa})(3.2 \times 10^{-4}\ \text{m}^{2})} = 2.6\ \text{cm}, \qquad (26.3.22)$$

where we used the fact that

$$\frac{PA}{mg} = \frac{(1.4 \times 10^{8}\ \text{Pa})((3.2 \times 10^{-4}\ \text{m}^{2})}{(60\ \text{kg})(9.8\ \text{m} \cdot \text{s}^{-2})} = 76 \qquad (26.3.23)$$

and so $PA >> mg$. The ratio

$$h_0 / d_{\min} \simeq PA / mg = 76. \qquad (26.3.24)$$

This ratio depends on the compressive strength of the bone, the cross sectional area, and inversely on the weight of the person. The maximum normal force is anywhere from two to ten times the average normal force. A safe distance to lower the center of mass would be about 20 cm.

Chapter 27 Static Fluids

Chapter 27 Static Fluids

27.1 Introduction

Water is everywhere around us, covering 71% of the Earth's surface. The water content of a human being can vary between 45% and 70 % of body weight. Water can exist in three states of matter: solid (ice), liquid, or gas. Water flows through many objects: through rivers, streams, aquifers, irrigation channels, and pipes to mention a few. Humans have tried to control and harness this flow through many different technologies such as aqueducts, Archimedes' screw, pumps, and water turbines. Water in the gaseous state also flows. Water vapor, lighter than air, can cause convection currents that form clouds. In the liquid state, the density of water molecules is greater than the gaseous state but in both states water can flow. Liquid water forms a surface while water vapor does not. Water in both the liquid and gaseous state is classified as a *fluid* to distinguish it from the solid state.

At the macroscopic scale, matter can be roughly grouped into two classes, solids and fluids. There is some ambiguity in the use of the term fluid. In ordinary language, the term fluid is used to describe the liquid state of matter. More technically, a fluid is a state of matter that, when at rest, cannot sustain a shear stress and hence will flow. A solid, when at rest, can sustain a shear stress and although it may deform it will remain at rest. However there is some ambiguity in this description. Glacier ice will flow but very slowly. So for a time interval that is small compared to the time interval involved in the flow, glacial ice can be thought of as a solid. This description of a fluid applies to both liquids and gases. A gas will expand to fill whatever volume it is confined in, while a liquid placed in a container will have a well-defined volume with a surface layer separated the liquid and vapor phases of the substance. We shall The *viscosity* of a fluid is a measure of its resistance to gradual deformation by shear stress or tensile stress.

27.2 Density

The density of a small amount of matter is defined to be the amount of mass ΔM divided by the volume ΔV of that element of matter,

$$\rho = \Delta M / \Delta V \ . \tag{27.2.1}$$

The SI unit for density is the kilogram per cubic meter, $\text{kg}\cdot\text{m}^{-3}$. If the density of a material is the same at all points, then the density is given by

$$\rho = M / V \ , \tag{27.2.2}$$

where M is the mass of the material and V is the volume of material. A material with constant density is called *homogeneous.* For a homogeneous material, density is an *intrinsic* property. If we divide the material in two parts, the density is the same in both parts,

$$\rho = \rho_1 = \rho_2 \ . \tag{27.2.3}$$

However mass and volume are *extrinsic* properties of the material. If we divide the material into two parts, the mass is the sum of the individual masses

$$M = M_1 + M_2 \ , \tag{27.2.4}$$

as is the volume

$$V = V_1 + V_2 \ . \tag{27.2.5}$$

The density is tabulated for various materials in Table 27.1.

Table 27.1: Density for Various Materials (Unless otherwise noted, all densities given are at standard conditions for temperature and pressure, that is, 273.15 K (0.00 °C) and 100 kPa (0.987 atm).

Material	Density, ρ $\text{kg} \cdot \text{m}^{-3}$
Helium	0.179
Air (at sea level)	1.20
Styrofoam	75
Wood Seasoned, typical	0.7×10^3
Ethanol	0.81×10^3
Ice	0.92×10^3
Water	1.00×10^3
Seawater	1.03×10^3
Blood	1.06×10^3
Aluminum	2.70×10^3
Iron	7.87×10^3
Copper	8.94×10^3
Lead	11.34×10^3
Mercury	13.55×10^3
Gold	19.32×10^3
Plutonium	19.84×10^3
Osmium	22.57×10^3

If we examine a small volume element of a fluid, it consists of molecules interacting via intermolecular forces. If we are studying the motion of bodies placed in fluids or the flow of the fluid at scales that are large compared to the intermolecular forces then we can consider the fluid to be *continuous* and quantities like density will vary smoothly from point to point in the fluid.

27.3 Pressure in a Fluid

When a shear force is applied to the surface of fluid, the fluid will undergo flow. When a fluid is static, the force on any surface within fluid must be perpendicular (normal) to each side of that surface. This force is due to the collisions between the molecules of the fluid on one side of the surface with molecules on the other side. For a static fluid, these forces must sum to zero. Consider a small portion of a static fluid shown in Figure 27.1. That portion of the fluid is divided into two parts, which we shall designate 1 and 2, by a small mathematical shared surface element S of area A_S . The force $\vec{\mathbf{F}}_{1,2}(S)$ on the surface of region 2 due to the collisions between the molecules of 1 and 2 is perpendicular to the surface.

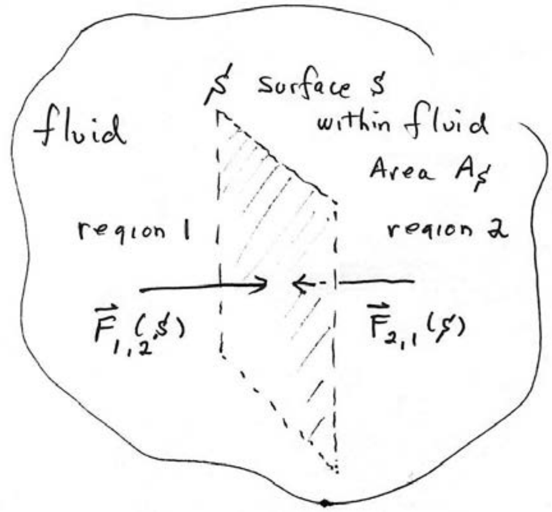


Figure 27.1: Forces on a surface within a fluid

The force $\vec{\mathbf{F}}_{2,1}(S)$ on the surface of region 1 due to the collisions between the molecules of 1 and 2 by Newton’s Third Law satisfies

$$\vec{\mathbf{F}}_{1,2}(S) = -\vec{\mathbf{F}}_{2,1}(S) \ . \tag{27.3.1}$$

Denote the magnitude of these forces that form this interaction pair by

$$F_{\perp}(S) = \left|\vec{\mathbf{F}}_{1,2}(S)\right| = \left|\vec{\mathbf{F}}_{2,1}(S)\right| \ . \tag{27.3.2}$$

Define the **hydrostatic pressure** at those points within the fluid that lie on the surface S by

$$P \equiv \frac{F_{\perp}(S)}{A_S} \ . \tag{27.3.3}$$

The pressure at a point on the surface S is the limit

$$P = \lim_{A_S \to 0} \frac{F_{\perp}(S)}{A_S} \ . \tag{27.3.4}$$

The SI units for pressure are $\mathrm{N \cdot m^{-2}}$ and is called the **pascal** (Pa), where

$$1\ \mathrm{Pa} = 1\ \mathrm{N \cdot m^{-2}} = 10^{-5}\ \mathrm{bar} \ . \tag{27.3.5}$$

Atmospheric pressure at a point is the force per unit area exerted on a small surface containing that point by the weight of air above that surface. In most circumstances atmospheric pressure is closely approximated by the hydrostatic pressure caused by the weight of air above the measurement point. On a given surface area, low-pressure areas have less atmospheric mass above their location, whereas high-pressure areas have more atmospheric mass above their location. Likewise, as elevation increases, there is less overlying atmospheric mass, so that atmospheric pressure decreases with increasing elevation. On average, a column of air one square centimeter in cross-section, measured from sea level to the top of the atmosphere, has a mass of about 1.03 kg and weight of about 10.1 N. (A column one square inch in cross-section would have a weight of about 14.7 lbs, or about 65.4 N). The **standard atmosphere** [atm] is a unit of pressure such that

$$1\ \mathrm{atm} = 1.01325 \times 10^{5}\ \mathrm{Pa} = 1.01325\ \mathrm{bar} \ . \tag{27.3.6}$$

27.4 Pascal's Law: Pressure as a Function of Depth in a Fluid of Uniform Density in a Uniform Gravitational Field

Consider a static fluid of uniform density ρ. Choose a coordinate system such that the z-axis points vertical downward and the plane $z = 0$ is at the surface of the fluid. Choose an infinitesimal cylindrical volume element of the fluid at a depth z, cross-sectional area A and thickness dz as shown in Figure 27.3. The volume of the element is $dV = A\,dz$ and the mass of the fluid contained within the element is $dm = \rho A\,dz$.

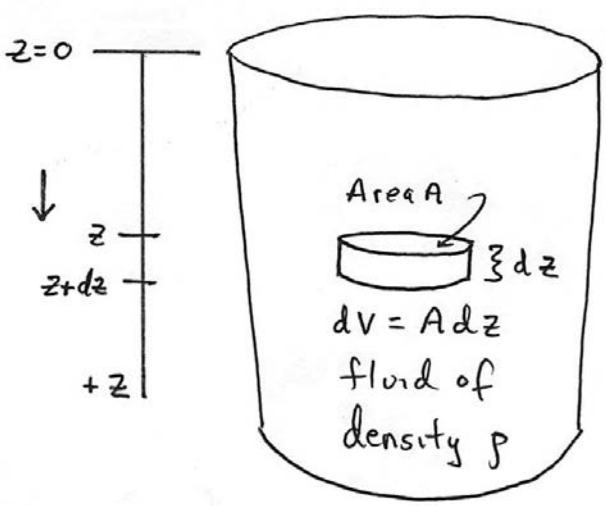


Figure 27.2: Coordinate system for fluid

The surface of the infinitesimal fluid cylindrical element has three faces, two caps and the cylindrical body. Because the fluid is static the force due to the fluid pressure points inward on each of these three faces. The forces on the cylindrical surface add to zero. On the end-cap at z, the force due to pressure of the fluid above the end-cap is downward, $\vec{\mathbf{F}}(z) = F(z)\hat{\mathbf{k}}$, where $F(z)$ is the magnitude of the force. On the end-cap at $z+dz$, the force due to the pressure of the fluid below the end-cap is upward, $\vec{\mathbf{F}}(z+dz) = -F(z+dz)\hat{\mathbf{k}}$, where $F(z+dz)$ is the magnitude of the force. The gravitational force acting on the element is given by $\vec{\mathbf{F}}^{g} = (dm)g\hat{\mathbf{k}} = (\rho dV)g\hat{\mathbf{k}} = \rho A\, dz\, g\hat{\mathbf{k}}$. There are also radial inward forces on the cylindrical body which sum to zero. The free body force diagram on the element is shown in Figure 27.3.

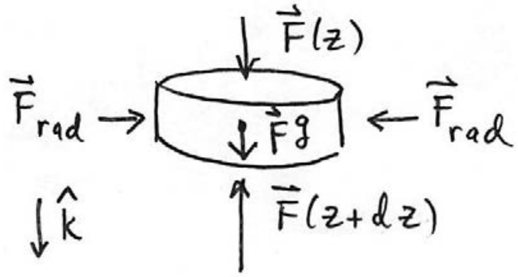


Figure 27.3: Free-body force diagram on cylindrical fluid element

The vector sum of the forces is zero because the fluid is static (Newton's Second Law). Therefore in the $+\hat{\mathbf{k}}$-direction

$$F(z) - F(z+dz) + \rho A dz g = 0 \ . \tag{27.4.1}$$

We divide through by the area A of the end-cap and use Eq. (27.3.4) to rewrite Eq. (27.4.1) in terms of the pressure

$$P(z) - P(z + dz) + \rho dzg = 0 \ . \tag{27.4.2}$$

Rearrange Eq. (27.4.2) as

$$\frac{P(z + dz) - P(z)}{dz} = \rho g \ . \tag{27.4.3}$$

Now take the limit of Eq. (27.4.3) as the thickness of the element $dz \to 0$,

$$\lim_{dz \to 0} \frac{P(z + dz) - P(z)}{dz} = \rho g \ . \tag{27.4.4}$$

resulting in the differential equation

$$\frac{dP}{dz} = \rho g \,. \tag{27.4.5}$$

Integrate Eq. (27.4.5),

$$\int_{P(z=0)}^{P(z)} dP = \int_{z'=0}^{z'=z} \rho g \, dz' \,. \tag{27.4.6}$$

Performing the integrals on both sides of Eq. (27.4.6) describes the change in pressure between a depth z and the surface of a fluid

$$P(z) - P(z = 0) = \rho g z \qquad \text{(Pascal's Law)}\,, \tag{27.4.7}$$

a result known as **Pascal’s Law**.

Example 27.1 Pressure in the Earth’s Ocean

What is the change in pressure between a depth of $4\,\text{km}$ and the surface in Earth’s ocean?

Solution: We begin by assuming the density of water is uniform in the ocean, and so we can use Pascal’s Law, Eq. (27.4.7) to determine the pressure, where we use $\rho = 1.03 \times 10^3 \ \text{kg} \cdot \text{m}^{-3}$ for the density of seawater (Table 27.1). Then

$$\begin{aligned} &P(z) - P(z = 0) = \rho g z \\ &= (1.03 \times 10^3 \text{kg} \cdot \text{m}^{-3})(9.8 \ \text{m} \cdot \text{s}^{-2})(4 \times 10^3 \ \text{m}) \\ &= 40 \times 10^6 \ \text{Pa.} \end{aligned} \tag{27.4.8}$$

Example 27.2 Pressure in a Rotating Sample in a Centrifuge

In an ultra centrifuge, a liquid filled chamber is spun with a high angular speed ω about a fixed axis. The density ρ of the fluid is uniform. The open-ended side of the chamber is a distance r_0 from the fixed axis. The chamber has cross sectional area A and of length L, (Figure 27.4).

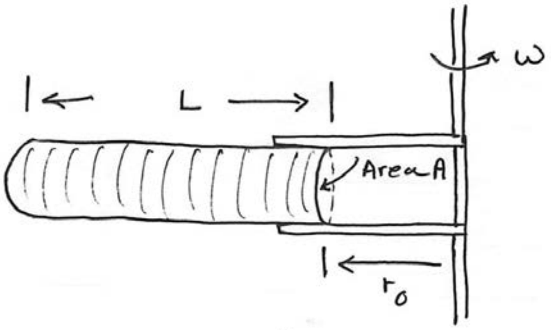


Figure 27.4: Schematic representation of centrifuge

The chamber is spinning fast enough to ignore the effect of gravity. Determine the pressure in the fluid as a function of distance r from the fixed axis.

Solution: Choose polar coordinates in the plane of circular motion. Consider a small volume element of the fluid of cross-sectional area A, thickness dr, and mass $dM = \rho A dr$ that is located a distance r from the fixed axis. Denote the pressure at one end of the volume element by $P(r) = F(r)/A$ and the pressure at the other end by $P(r+dr) = F(r+dr)/A$. The free-body force diagram on the volume fluid element is shown in Figure 27.5.

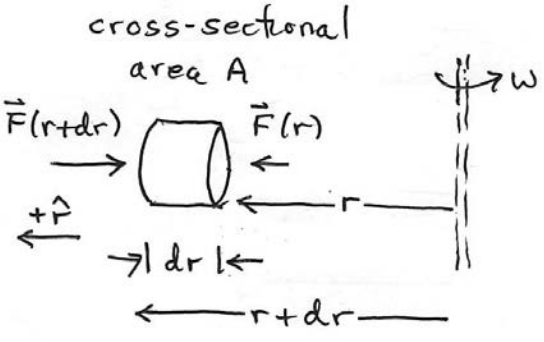


Figure 27.5: Free-body force diagram showing only radial forces on fluid element in centrifuge

The element is accelerating inward with radial component of the acceleration, $a_r = -r\omega^2$. Newton's Second Law applied to the fluid element is then

$$(P(r) - P(r + dr))A = -(\rho A dr) r\omega^2 , \tag{27.4.9}$$

We can rewrite Eq. (27.4.9) as

$$\frac{P(r + dr) - P(r)}{dr} = \rho r \omega^2 , \tag{27.4.10}$$

and take the limit $dr \to 0$ resulting in

$$\frac{dP}{dr} = \rho r \omega^2 . \tag{27.4.11}$$

We can integrate Eq. (27.4.11) between an arbitrary distance r from the rotation axis and the open-end located at r_0, where the pressure $P(r_0) = 1\,\text{atm}$,

$$\int_{P(r)}^{P(r_0)} dP = \rho \omega^2 \int_{r'=r}^{r'=r_0} r' \, dr' . \tag{27.4.12}$$

Integration yields

$$P(r_0) - P(r) = \frac{1}{2} \rho \omega^2 (r_0{}^2 - r^2) . \tag{27.4.13}$$

The pressure at a distance r from the rotation axis is then

$$P(r) = P(r_0) + \frac{1}{2} \rho \omega^2 (r^2 - r_0{}^2) . \tag{27.4.14}$$

27.5 Compressibility of a Fluid

When the pressure is uniform on all sides of an object in a fluid, the pressure will squeeze the object resulting in a smaller volume. When we increase the pressure by ΔP on a material of volume V_0, then the volume of the material will change by $\Delta V < 0$ and consequently the density of the material will also change. Define the *bulk stress* by the increase in pressure change

$$\sigma_B \equiv \Delta P \ . \tag{27.5.1}$$

Define the *bulk strain* by the ratio

$$\varepsilon_B \equiv \frac{\Delta V}{V_0} \ . \tag{27.5.2}$$

For many materials, for small pressure changes, the bulk stress is linearly proportional to the bulk strain,

$$\Delta P = -B\frac{\Delta V}{V_0} \ , \tag{27.5.3}$$

where the constant of proportionality B is called the **bulk modulus**. The SI unit for bulk modulus is the pascal. If the bulk modulus of a material is very large, a large pressure change will result in only a small volume change. In that case the material is called *incompressible*. In Table 27.2, the bulk modulus is tabulated for various materials.

Table 27.2 Bulk Modulus for Various Materials

Material	Bulk Modulus, *Y,* (Pa)
Diamond	4.4×10^{11}
Iron	1.6×10^{11}
Nickel	1.7×10^{11}
Steel	1.6×10^{11}
Copper	1.4×10^{11}
Brass	6.0×10^{10}
Aluminum	7.5×10^{10}
Crown Glass	5.0×10^{10}
Lead	4.1×10^{10}
Water (value increases at higher pressure)	2.2×10^{9}
Air (adiabatic bulk modulus)	1.42×10^{5}
Air (isothermal bulk modulus)	1.01×10^{5}

Example 27.3 Compressibility of Water

Determine the percentage decrease in a fixed volume of water at a depth of 4 km where the pressure difference is 40 MPa, with respect to sea level.

Solution: The bulk modulus of water is $2.2\times10^{9}\text{ Pa}$. From Eq. (27.5.3),

$$\frac{\Delta V}{V_0}=-\frac{\Delta P}{B}=-\frac{40\times10^{6}\,\text{Pa}}{2.2\times10^{9}\text{ Pa}}=-0.018\,; \tag{27.5.4}$$

there is only a 1.8% decrease in volume. Water is essentially incompressible even at great depths in ocean, justifying our assumption that the density of water is uniform in the ocean in Example 27.1.

27.6 Archimedes' Principle: Buoyant Force

When we place a piece of solid wood in water, the wood floats on the surface. The density of most woods is less than the density of water, and so the fact that wood floats does not seem so surprising. However, objects like ships constructed from materials like steel that are much denser than water also float. In both cases, when the floating object is at rest, there must be some other force that exactly balances the gravitational force. This balancing of forces also holds true for the fluid itself.

Consider a static fluid with uniform density ρ_f. Consider an arbitrary volume element of the fluid with volume V and mass $m_f = \rho_f V$. The gravitational force acts on the volume element, pointing downwards, and is given by $\vec{\mathbf{F}}^g = -\rho_f V g \hat{\mathbf{k}}$, where $\hat{\mathbf{k}}$ is a unit vector pointing in the upward direction. The pressure on the surface is perpendicular to the surface (Figure 27.6). Therefore on each area element of the surface there is a perpendicular force on the surface.

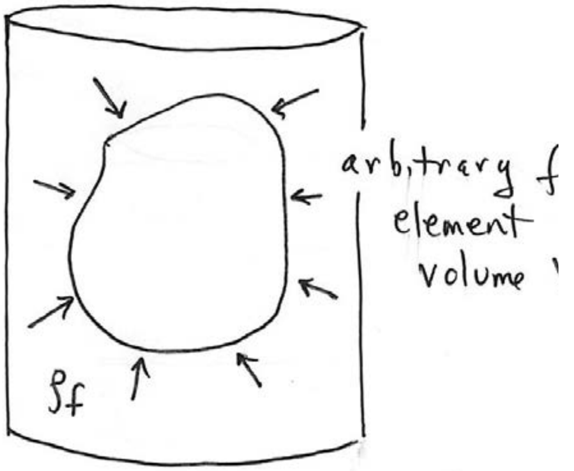


Figure 27.6: Forces due to pressure on surface of arbitrary volume fluid element

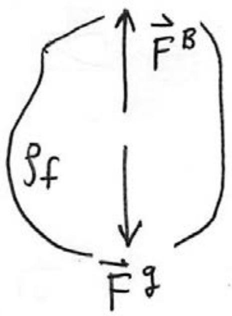


Figure 27.7: Free-body force diagram on volume element showing gravitational force and buoyant force

Let $\vec{\mathbf{F}}^B$ denote the resultant force, called the **buoyant force**, on the surface of the volume element due to the pressure of the fluid. The buoyant force must exactly balance the gravitational force because the fluid is in static equilibrium (Figure 27.7),

$$\vec{\mathbf{0}} = \vec{\mathbf{F}}^B + \vec{\mathbf{F}}^g = \vec{\mathbf{F}}^B - \rho_f V g \hat{\mathbf{k}} . \tag{27.6.1}$$

Therefore the buoyant force is therefore

$$\vec{\mathbf{F}}^B = \rho_f V g \hat{\mathbf{k}} . \tag{27.6.2}$$

The buoyant force depends on the density of the fluid, the gravitational constant, and the volume of the fluid element. This macroscopic description of the buoyant force that results from a very large number of collisions of the fluid molecules is called **Archimedes' Principle.**

We can now understand why when we place a stone in water it sinks. The density of the stone is greater than the density of the water, and so the buoyant force on the stone is less than the gravitational force on the stone and so it accelerates downward.

Place a uniform object of volume V and mass M with density $\rho_o = M / V$ within a fluid. If the density of the object is less than the density of the fluid, $\rho_o < \rho_f$, the object will float on the surface of the fluid. A portion of the object that is a beneath the surface, displaces a volume V_1 of the fluid. The portion of the object that is above the surface displaces a volume $V_2 = V - V_1$ of air (Figure 27.8).

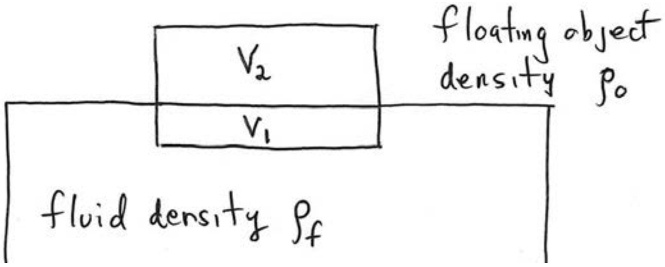


Figure 27.8: Floating object on surface of fluid

Because the density of the air is much less that the density of the fluid, we can neglect the buoyant force of the air on the object.

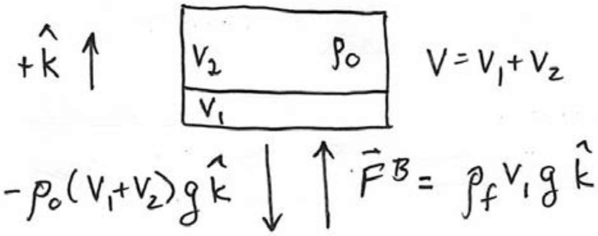


Figure 27.9: Free-body force diagram on floating object

The buoyant force of the fluid on the object, $\vec{\mathbf{F}}^B_{f,o} = \rho_f V_1 g \hat{\mathbf{k}}$, must exactly balance the gravitational force on the object due to the earth, $\vec{\mathbf{F}}^g_{e,o}$ (Figure 27.9),

$$\vec{\mathbf{0}} = \vec{\mathbf{F}}^B_{f,o} + \vec{\mathbf{F}}^g_{e,o} = \rho_f V_1 g \hat{\mathbf{k}} - \rho_o V g \hat{\mathbf{k}} = \rho_f V_1 g \hat{\mathbf{k}} - \rho_o (V_1 + V_2) g \hat{\mathbf{k}} . \qquad (27.6.3)$$

Therefore the ratio of the volume of the exposed and submerged portions of the object must satisfy

$$\rho_f V_1 = \rho_o (V_1 + V_2)\,. \tag{27.6.4}$$

We can solve Eq. (27.6.4) and determine the ratio of the volume of the exposed and submerged portions of the object

$$\frac{V_2}{V_1} = \frac{(\rho_f - \rho_o)}{\rho_o}\,. \tag{27.6.5}$$

We now also can understand why a ship of mass M floats. The more dense steel displaces a volume of water V_s but a much larger volume of water V_w is displaced by air. The buoyant force on the ship is then

$$\vec{\mathbf{F}}_s^{\,B} = \rho_f (V_s + V_w) g\,\hat{\mathbf{k}}\,. \tag{27.6.6}$$

If this force is equal in magnitude to Mg, the ship will float.

Example 27.4 Archimedes' Principle: Floating Wood

Consider a beaker of uniform cross-sectional area A, filled with water of density ρ_w. When a rectangular block of wood of cross sectional area A_2, height, and mass M_b is placed in the beaker, the bottom of the block is at an unknown depth z below the surface of the water. (a) How far below the surface z is the bottom of the block? (b) How much did the height of the water in the beaker rise when the block was placed in the beaker?

Solution: We neglect the buoyant force due to the displaced air because it is negligibly small compared to the buoyant force due to the water. The beaker, with the floating block of wood, is shown in Figure 27.10.

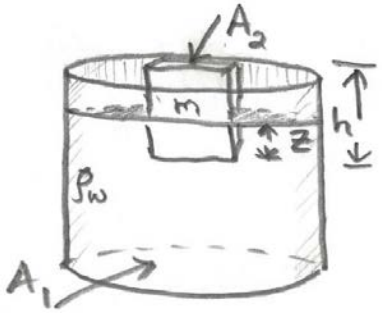


Figure 27.10 Block of wood floating in a beaker of water

(a) The density of the block of wood is $\rho_b = M_b / V_b = M_b / A_b h$. The volume of the submerged portion of the wood is $V_1 = A_b z$. The volume of the block above the surface is given by $V_2 = A_b (h - z)$. We can apply Eq. (27.6.5), and determine that

$$\frac{V_2}{V_1} = \frac{A_b(h-z)}{A_b z} = \frac{(h-z)}{z} = \frac{(\rho_w - \rho_b)}{\rho_b}. \tag{27.6.7}$$

We can now solve Eq. (27.6.7) for the depth z of the bottom of the block

$$z = \frac{\rho_b}{\rho_w} h = \frac{(M_b / A_b h)}{\rho_w} h = \frac{M_b}{\rho_w A_b}. \tag{27.6.8}$$

(b) Before the block was placed in the beaker, the volume of water in the beaker is $V_w = As_i$, where s_i is the initial height of water in the beaker. When the wood is floating in the beaker, the volume of water in the beaker is equal to $V_w = As_f - A_b z$, where s_f is the final height of the water, in the beaker and $A_b z$ is the volume of the submerged portion of block. Because the volume of water has not changed

$$As_i = As_f - A_b z. \tag{27.6.9}$$

We can solve Eq. (27.6.9) for the change in height of the water $\Delta s = s_f - s_i$, in terms of the depth z of the bottom of the block,

$$\Delta s = s_f - s_i = \frac{A_b}{A} z. \tag{27.6.10}$$

We now substitute Eq. (27.6.8) into Eq. (27.6.10) and determine the change in height of the water

$$\Delta s = s_f - s_i = \frac{M_b}{\rho_w A}. \tag{27.6.11}$$

Example 27.5 Rock Inside a Floating Salad Bowl

A rock of mass m_r and density ρ_r is placed in a salad bowl of mass m_b. The salad bowl and rock float in a beaker of water of density ρ_w. The beaker has cross sectional area A. The rock is then removed from the bowl and allowed to sink to the bottom of the beaker. Does the water level rise or fall when the rock is dropped into the water?

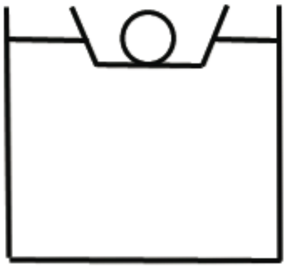


Figure 27.11: Rock in a floating salad bowl

Solution: When the rock is placed in the floating salad bowl, a volume V of water is displaced. The buoyant force $\vec{\mathbf{F}}^{B} = \rho_{w} V g \hat{\mathbf{k}}$ balances the gravitational force on the rock and salad bowl,

$$(m_r + m_b)g = \rho_w V g = \rho_w (V_1 + V_2) g\,. \qquad (27.6.12)$$

where V_1 is the portion of the volume of displaced water that is necessary to balance just the gravitational force on the rock, $m_r g = \rho_w V_1 g$, and V_2 is the portion of the volume of displaced water that is necessary to balance just the gravitational force on the bowl, $m_b g = \rho_w V_2\, g$, Therefore V_1 must satisfy the condition that $V_1 = m_r g / \rho_w$. The volume of the rock is given by $V_r = m_r / \rho_r$. In particular

$$V_1 = \frac{\rho_r}{\rho_w} V_r\,. \qquad (27.6.13)$$

Because the density of the rock is greater than the density of the water, $\rho_r > \rho_w$, the rock displaces more water when it is floating than when it is immersed in the water, $V_1 > V_r$. Therefore the water level drops when the rock is dropped into the water from the salad bowl.

Example 27.6 Block Floating Between Oil and Water

A cubical block of wood, each side of length $l = 10\text{ cm}$, floats at the interface between air and water. The air is then replaced with $d = 10\text{ cm}$ of oil that floats on top of the water.

a) Will the block rise or fall? Briefly explain your answer.

After the oil has been added and equilibrium established, the cubical block of wood floats at the interface between oil and water with its lower surface $h = 2.0 \times 10^{-2}\text{m}$ below the interface. The density of the oil is $\rho_o = 6.5 \times 10^{2}\,\text{kg} \cdot \text{m}^{-3}$. The density of water is $\rho_w = 1.0 \times 10^{3}\,\text{kg} \cdot \text{m}^{-3}$.

b) What is the density of the block of wood?

Solution: (a) The buoyant force is equal to the gravitational force on the block. Therefore

$$\rho_b g V = \rho_w g V_1 + \rho_a g (V - V_1) \ , \tag{27.6.14}$$

where V_1 is the volume of water displaced by the block, $V_2 = V - V_1$ is the volume of air displaced by the block V is the volume of the block, ρ_b is the density of the block of wood, and ρ_a is the density of air (Figure 27.12(a)).

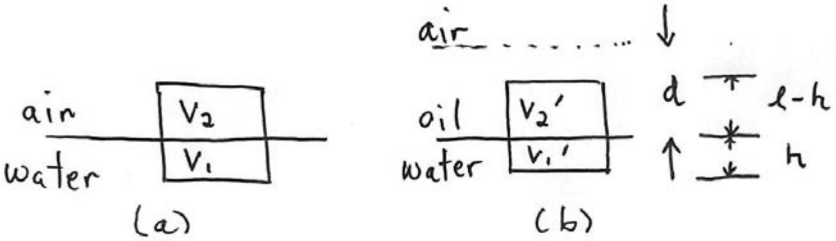


Figure 27.12: (a) Block floating on water, (b) Block floating on oil-water interface

We now solve Eq. (27.6.14) for the volume of water displaced by the block

$$V_1 = \frac{(\rho_b - \rho_a)}{(\rho_w - \rho_a)} V \ . \tag{27.6.15}$$

When the oil is added, we can repeat the argument leading up to Eq. (27.6.15) replacing ρ_a by ρ_o, (Figure 27.12(b)), yielding

$$\rho_b g V = \rho_w g V_1' + \rho_o g V_2' \ , \tag{27.6.16}$$

where V_1' is the volume of water displaced by the block, V_2' is the volume of oil displaced by the block, V is the volume of the block, and ρ_b is the density of the block of wood. Because $V_2' = V - V_1'$, we rewrite Eq. (27.6.16) as

$$\rho_b g V = \rho_w g V_1' + \rho_o g (V - V_1') \ , \tag{27.6.17}$$

We now solve Eq. (27.6.17) for the volume of water displaced by the block,

$$V_1' = \frac{(\rho_b - \rho_o)V}{(\rho_w - \rho_o)} \ . \tag{27.6.18}$$

Because $\rho_o >> \rho_a$, comparing Eqs. (27.6.18) and (27.6.15), we conclude that $V_1' > V_1$. The block rises when the oil is added because more water is displaced.

(b) We use the fact that $V_1' = l^2 h$, $V_2' = l^2(l-h)$, and $V = l^3$, in Eq. (27.6.16) and solve for the density of the block

$$\rho_b = \frac{\rho_w V_1' + \rho_o V_2'}{V} = \frac{\rho_w l^2 h + \rho_o l^2 (l-h)}{l^3} = (\rho_w - \rho_o)\frac{h}{l} + \rho_o \ . \tag{27.6.19}$$

We now substitute the given values from the problem statement and find that the density of the block is

$$\begin{aligned}\rho_b &= ((1.0\times10^3\,\mathrm{kg\cdot m^{-3}}) - (6.5\times10^2\,\mathrm{kg\cdot m^{-3}}))\frac{(2.0\times10^{-2}\,\mathrm{m})}{(1.0\times10^{-1}\,\mathrm{m})} + (6.5\times10^2\,\mathrm{kg\cdot m^{-3}})\\ \rho_b &= 7.2\times10^2\,\mathrm{kg\cdot m^{-3}}.\end{aligned} \tag{27.6.20}$$

Because $\rho_b > \rho_o$, the above analysis is valid.

Chapter 28 Fluid Dynamics

Chapter 28 Fluid Dynamics

28.1 Ideal Fluids

An ideal fluid is a fluid that is incompressible and no internal resistance to flow (zero viscosity). In addition ideal fluid particles undergo no rotation about their center of mass (irrotational). An ideal fluid can flow in a circular pattern, but the individual fluid particles are irrotational. Real fluids exhibit all of these properties to some degree, but we shall often model fluids as ideal in order to approximate the behavior of real fluids. When we do so, one must be extremely cautious in applying results associated with ideal fluids to non-ideal fluids.

28.2 Velocity Vector Field

When we describe the flow of a fluid like water, we may think of the movement of individual particles. These particles interact with each other through forces. We could then apply our laws of motion to each individual particle in the fluid but because the number of particles is very large, this would be an extremely difficult computation problem. Instead we shall begin by mathematically describing the state of moving fluid by specifying the velocity of the fluid at each point in space and at each instant in time. For the moment we will choose Cartesian coordinates and refer to the coordinates of a point in space by the ordered triple (x, y, z) and the variable t to describe the instant in time, but in principle we may chose any appropriate coordinate system appropriate for describing the motion. The distribution of fluid velocities is described by the vector function $\vec{\mathbf{v}}(x, y, z, t)$. This represents the velocity of the fluid at the point (x, y, z) at the instant t. The quantity $\vec{\mathbf{v}}(x, y, z, t)$ is called the **velocity vector** field. It can be thought of at each instant in time as a collection of vectors, one for each point in space whose direction and magnitude describes the direction and magnitude of the velocity of the fluid at that point (Figure 28.1). This description of the velocity vector field of the fluid refers to fixed points in space and not to moving particles in the fluid.

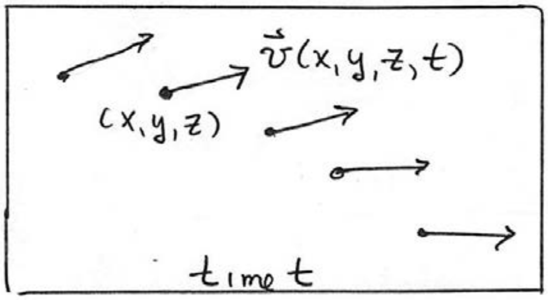


Figure 28.1: Velocity vector field for fluid flow at time t

We shall introduce functions for the pressure $P(x, y, z, t)$ and the density $\rho(x, y, z, t)$ of the fluid that describe the pressure and density of the fluid at each point in space and at each instant in time. These functions are called **scalar fields** because there is only one number with appropriate units associated with each point in space at each instant in time.

In order to describe the velocity vector field completely we need three functions $v_x(x,y,z,t)$, $v_y(x,y,z,t)$, and $v_z(x,y,z,t)$ to describe the components of the velocity vector field

$$\vec{\mathbf{v}}(x,y,z,t) = v_x(x,y,z,t)\,\hat{\mathbf{i}} + v_y(x,y,z,t)\,\hat{\mathbf{j}} + v_z(x,y,z,t)\,\hat{\mathbf{k}}\,. \tag{28.2.1}$$

The three component functions are scalar fields. The velocity vector field is in general quite complicated for a three-dimensional time dependent flow. We can sometimes make some simplifying assumptions that enable us to model a complex flow, for example modeling the flow as a two-dimensional flow or even further assumptions that one component function of a two-dimensional flow is negligible allowing us to model the flow as one-dimensional.

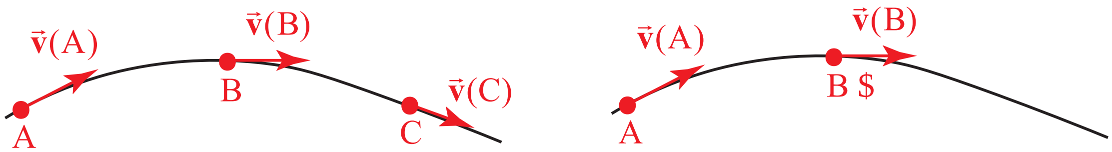
For most flows, the velocity field varies in time. For some special cases we can model the flow by assuming that the velocity field does not change in time, a case we shall refer to as **steady flow,**

$$\frac{\partial \vec{\mathbf{v}}(x,y,z,t)}{\partial t} = \vec{\mathbf{0}} \qquad \text{(steady flow)}\,. \tag{28.2.2}$$

For steady flows the velocity field is independent of time,

$$\vec{\mathbf{v}}(x,y,z) = v_x(x,y,z)\,\hat{\mathbf{i}} + v_y(x,y,z)\,\hat{\mathbf{j}} + v_z(x,y,z)\,\hat{\mathbf{k}} \quad \text{(steady flow)}\,. \tag{28.2.3}$$

For a non-ideal fluid, the differential equations satisfied by these velocity component functions are quite complicated and beyond the scope of this discussion. Instead, we shall primarily consider the special case of **steady flow** of a fluid in which the velocity at each point in the fluid does not change in time. The velocities may still vary in space (non-uniform steady flow).



(a) trajectory of particle that is located at: point A at time t_1; point B at time t_2; and point C at time t_3.

(b) trajectory of particle 2 that is located at: point A at time t_2; and point B at time t_3.

Figure 28.2: (a) trajectory of particle 1, (b) trajectory of particle 2

Let's trace the motion of particles in an ideal fluid undergoing steady flow during a succession of intervals of duration Δt. Consider particle 1 located at point A with coordinates $(x_\mathrm{A}, y_\mathrm{A}, z_\mathrm{A})$. At the instant t_1, particle 1 has velocity $\vec{\mathbf{v}}(x_\mathrm{A}, y_\mathrm{A}, z_\mathrm{A}) = \vec{\mathbf{v}}(\mathrm{A})$.

During the time $[t_1,t_2]$, where $t_2 = t_1 + \Delta t_1$, the particle moves to point B arriving there at the instant t_2. At point B, the particle now has velocity $\vec{\mathbf{v}}(x_B, y_B, z_B) = \vec{\mathbf{v}}(\mathrm{B})$. During the next interval $[t_2,t_3]$, where $t_3 = t_2 + \Delta t$, particle 1 will move to point C arriving there at instant t_3, where it has velocity $\vec{\mathbf{v}}(x_C, y_C, z_C) = \vec{\mathbf{v}}(\mathrm{C})$. (Figure 28.2(a)). Because the flow has been assumed to be steady, at instant t_2, a different particle, particle 2, is now located at point A but it has the same velocity $\vec{\mathbf{v}}(x_A, y_A, z_A)$ as particle 1 had at point A and hence will arrive at point B at the end of the next interval, at the instant t_3 (Figure 28.2(b)). In this way every particle that lies on the trajectory that our first particle traces out in time will follow the same trajectory. This trajectory is called a *streamline*. The particles in the fluid will not have the same velocities at points along a streamline because we have not assumed that the velocity field is uniform.

28.3 Mass Continuity Equation

A set of streamlines for an ideal fluid undergoing steady flow in which there are no sources or sinks for the fluid is shown in Figure 28.3.

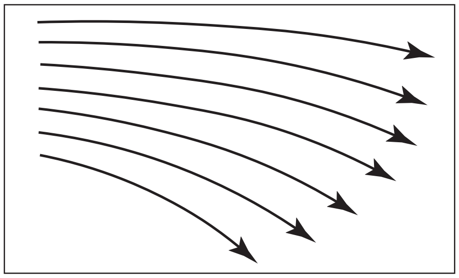


Figure 28.3: Set of streamlines for an ideal fluid flow

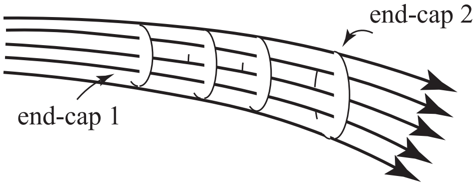


Figure 28.4: Flux Tube associated with set of streamlines

We also show a set of closely separated streamlines that form a *flow tube* in Figure 28.4 We add to the flow tube two open surface (end-caps 1 and 2) that are perpendicular to velocity of the fluid, of areas A_1 and A_2, respectively. Because all fluid particles that enter end-cap 1 must follow their respective streamlines, they must all leave end-cap 2. If the streamlines that form the tube are sufficiently close together, we can assume that the velocity of the fluid in the vicinity of each end-cap surfaces is uniform.

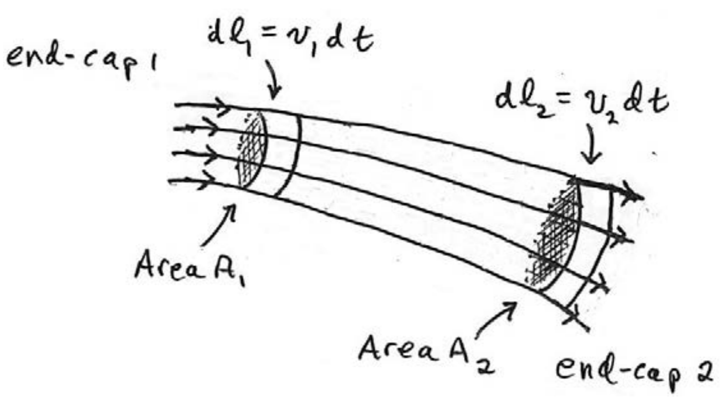


Figure 28.5: Mass flow through flux tube

Let v_1 denote the speed of the fluid near end-cap 1 and v_2 denote the speed of the fluid near end-cap 2. Let ρ_1 denote the density of the fluid near end-cap 1 and ρ_2 denote the density of the fluid near end-cap 2. The amount of mass that enters and leaves the tube in a time interval dt can be calculated as follows (Figure 28.5): suppose we consider a small volume of space of cross-sectional area A_1 and length $dl_1 = v_1 dt$ near end-cap 1. The mass that enters the tube in time interval dt is

$$dm_1 = \rho_1 dV_1 = \rho_1 A_1 dl_1 = \rho_1 A_1 v_1 dt \ . \qquad (28.3.1)$$

In a similar fashion, consider a small volume of space of cross-sectional area A_2 and length $dl_2 = v_2 dt$ near end-cap 2. The mass that leaves the tube in the time interval dt is then

$$dm_2 = \rho_2 dV_2 = \rho_2 A_2 dl_2 = \rho_2 A_2 v_2 dt \ . \qquad (28.3.2)$$

An equal amount of mass that enters end-cap 1 in the time interval dt must leave end-cap 2 in the same time interval, thus $dm_1 = dm_2$. Therefore using Eqs. (28.3.1) and (28.3.2), we have that $\rho_1 A_1 v_1 dt = \rho_2 A_2 v_2 dt$. Dividing through by dt implies that

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 \quad (\text{steady flow}) \ . \qquad (28.3.3)$$

Eq. (28.3.3) generalizes to any cross sectional area A of the thin tube, where the density is ρ, and the speed is v,

$$\rho A v = \text{constant} \quad (\text{steady flow}) \ . \qquad (28.3.4)$$

Eq. (28.3.3) is referred to as the *mass continuity equation for steady flow*. If we assume the fluid is incompressible, then Eq. (28.3.3) becomes

$$A_1 v_1 = A_2 v_2 \quad (\text{incompressable fluid, steady flow}) \ . \qquad (28.3.5)$$

Consider the steady flow of an incompressible with streamlines and closed surface formed by a streamline tube shown in Figure 28.5. According to Eq. (28.3.5), when the spacing of the streamlines increases, the speed of the fluid must decrease. Therefore the speed of the fluid is greater entering end-cap 1 then when it is leaving end-cap 2. When we represent fluid flow by streamlines, regions in which the streamlines are widely spaced have lower speeds than regions in which the streamlines are closely spaced.

28.4 Bernoulli's Principle

Let's again consider the case of an ideal fluid that undergoes steady flow and apply energy methods to find an equation of state that relates pressure, density, and speed of the flow at different points in the fluid. Let's examine the case of a steady horizontal flow in as seen in the overhead view shown in Figure 28.6. We represent this flow by streamlines and a flow tube associated with the streamlines. Let's consider the motion of a fluid particle along one streamline passing through points A and B in Figure 28.6. The cross-sectional area of the flow tube at point A is less than the cross-sectional area of the flow tube at point B.

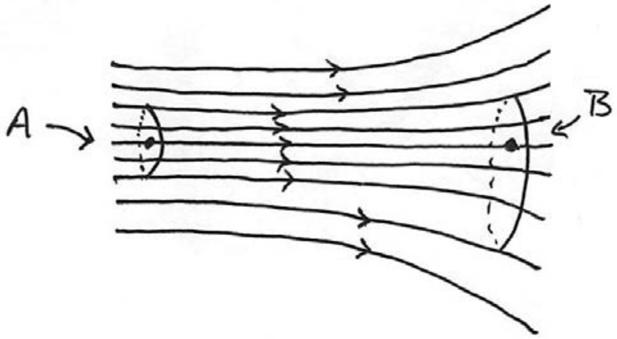


Figure 28.6 Overhead view of steady horizontal flow: in regions where spacing of the streamlines increases, the speed of the fluid must decrease

According to Eq. (28.3.5), the particle located at point A has a greater speed than a fluid particle located at point B. Therefore a particle traveling along the streamline from point A to point B must decelerate. Because the streamline is horizontal, the force responsible is due to pressure differences in the fluid. Thus, for this steady horizontal flow in regions of lower speed there must be greater pressure than in regions of higher speed.

Now suppose the steady flow of the ideal fluid is not horizontal, with the y-representing the vertical directi. The streamlines and flow tube for this steady flow are shown in Figure 28.7.

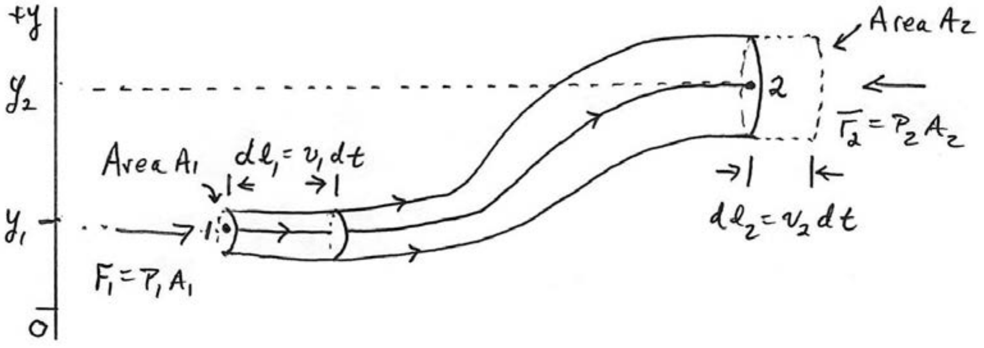


Figure 28.7: Non-horizontal steady flow

In order to determine the equation relating the pressure, speed and height difference of the tube, we shall use the work-energy theorem. We take as a system the mass contained in the flow tube shown in Figure 28.7. The external forces acting on our system are due to the pressure acting at the two ends of the flow tube and the gravitational force. Consider a streamline passing through points 1 and 2 at opposite ends of the flow tube. Let's assume that the flow tube is narrow enough such that the velocity of the fluid is uniform on the cross-sectional areas of the tube at points 1 and 2. At point 1, denote the speed of a fluid particle by v_1, the cross-sectional area by A_1, the fluid pressure by P_1, and the height of the center of the cross-sectional area by y_1. At point 2, denote the speed of a fluid particle by v_2, the cross-sectional area by A_2, the fluid pressure by P_2, and the height of the center of the cross-sectional area by y_2.

Consider the flow tube at time t as illustrated in Figure 28.7. At the left end of the flow, in a time interval dt, a particle at point 1 travels a distance $dl_1 = v_1 dt$. Therefore a small volume $dV_1 = A_1 dl_1 = A_1 v_1 dt$ of fluid is displaced at the right end of the flow tube. In a similar fashion, at particle at point 2, travels a distance $dl_2 = v_2 dt$. Therefore a small volume of fluid $dV_2 = A_2 dl_2 = A_2 v_2 dt$ is also displaced to the right in the flow tube during the time interval dt. Because we are assuming the fluid is incompressible, by Eq.(28.3.5), these volume elements are equal, $dV \equiv dV_1 = dV_2$.

There is a force of magnitude $F_1 = P_1 A_1$ in the direction of the flow arising from the fluid pressure at the left end of the tube acting on the mass element that enters the tube. The work done displacing the mass element is then

$$dW_1 = F_1 dl_1 = P_1 A_1 dl_1 = P_1 dV \ . \qquad (28.4.1)$$

There is also a force of magnitude $F_2 = P_2 A_2$ in the direction opposing the flow arising from the fluid pressure at the right end of the tube. The work done opposing the displacement of the mass element leaving the tube is then

$$dW_1 = -F_2 dl_2 = -P_2 A_2 dl_2 = -P_2 dV \ . \tag{28.4.2}$$

Therefore the external work done by the force associated with the fluid pressure is the sum of the work done at each end of the tube

$$dW^{ext} = dW_1 + dW_2 = (P_1 - P_2)dV \ . \tag{28.4.3}$$

In a time interval dt, the work done by the gravitational force is equal to

$$dW^{g} = -dm\, g(y_2 - y_1) = -\rho dVg(y_2 - y_1) \ . \tag{28.4.4}$$

Because we only chose the mass in the flow tube as our system, and we assumed that the fluid was ideal (no frictional losses due to viscosity) the change in the potential energy of the system is

$$dU = -W^{g} = \rho dVg(y_2 - y_1) \ . \tag{28.4.5}$$

At time t, the kinetic energy of the system is the sum of the kinetic energy of the small mass element of volume $dV = A_1 dl_1$ moving with speed v_1 and the rest of the mass in the flow tube. At time $t + dt$, the kinetic energy of the system is the sum of the kinetic energy of the small mass element of volume $dV = A_2 dl_2$ moving with speed v_2 and the rest of the mass in the flow tube. The change in the kinetic energy of the system is due to the mass elements at the two ends and therefore

$$dK = \frac{1}{2}dm_2 {v_2}^2 - \frac{1}{2}dm_1 {v_1}^2 = \frac{1}{2}\rho dV({v_2}^2 - {v_1}^2) \ . \tag{28.4.6}$$

The work-energy theorem $dW^{ext} = dU + dK$ for system is then

$$(P_1 - P_2)dV = \frac{1}{2}\rho dV({v_2}^2 - {v_1}^2) + \rho g(y_2 - y_1)dV \ . \tag{28.4.7}$$

We now divide Eq. (28.4.7) through by the volume dV and rearrange terms, yielding

$$P_1 + \rho g y_1 + \frac{1}{2}\rho {v_1}^2 = P_2 + \rho g y_2 + \frac{1}{2}\rho {v_2}^2 \ . \tag{28.4.8}$$

Because points 1 and 2 were arbitrarily chosen, we can drop the subscripts and write Eq. (28.4.8) as

$$P + \rho g y + \frac{1}{2}\rho v^2 = \text{constant} \quad \text{(ideal fluid, steady flow)} \ . \tag{28.4.9}$$

Eq. (28.4.9) is known as **Bernoulli's Equation**.

28.5 Worked Examples: Bernoulli's Equation

Example 28.1 Venturi Meter

Figure 28.8 shows a Venturi Meter, a device used to measure the speed of a fluid in a pipe. A fluid of density ρ_f is flowing through a pipe. A U-shaped tube partially filled with mercury of density ρ_{Hg} lies underneath the points 1 and 2.

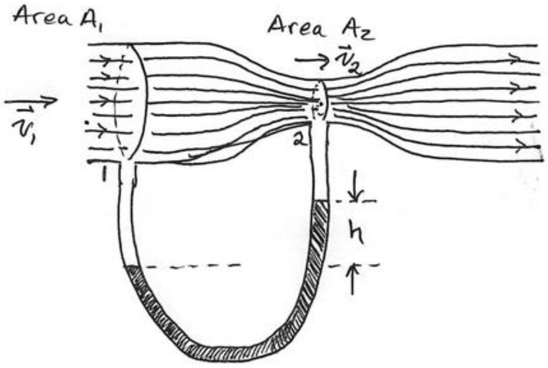


Figure 28.8: Venturi Meter

The cross-sectional areas of the pipe at points 1 and 2 are A_1 and A_2 respectively. Determine an expression for the flow speed at the point 1 in terms of the cross-sectional areas A_1 and A_2, and the difference in height h of the liquid levels of the two arms of the U-shaped tube.

Solution:

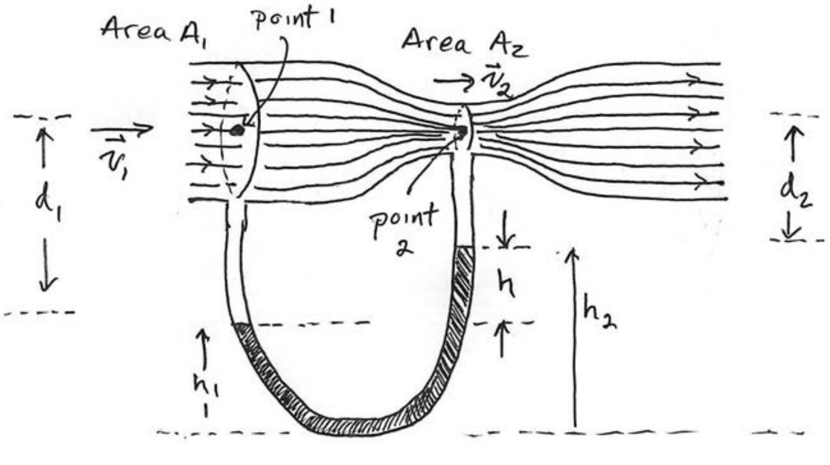


Figure 28.8: Coordinate system for Venturi tube

We shall assume that the pressure and speed are constant in the cross-sectional areas A_1 and A_2. We also assume the fluid is incompressible so the density ρ_f is constant throughout the tube. The two points 1 and 2 lie on the streamline passing through the midpoint of the tube so they are at the same height. Using $y_1 = y_2$ in Eq. (28.4.8), the pressure and flow speeds at the two points 1 and 2 are related by

$$P_1 + \frac{1}{2}\rho_f v_1^{\,2} = P_2 + \frac{1}{2}\rho_f v_2^{\,2} \ . \tag{28.4.10}$$

We can rewrite Eq. (28.4.10) as

$$P_1 - P_2 = \frac{1}{2}\rho_f (v_2^{\,2} - v_1^{\,2}) \ . \tag{28.4.11}$$

Let h_1 and h_2 denote the heights of the liquid level in the arms of the U-shaped tube directly beneath points 1 and 2 respectively. Pascal's Law relates the pressure difference between the two arms of the U-shaped tube according to in the left arm of the U-shaped tube according to

$$P_{bottom} = P_1 + \rho_f g d_1 + \rho_{Hg} g h_1 \ . \tag{28.4.12}$$

In a similar fashion, the pressure at point 2 is given by

$$P_{bottom} = P_2 + \rho_f g d_2 + \rho_{Hg} g h_2 \ . \tag{28.4.13}$$

Therefore, setting Eq. (28.4.12) equal to Eq. (28.4.13), we determine that the pressure difference on the two sides of the U-shaped tube is

$$P_1 - P_2 = \rho_f g(d_2 - d_1) + \rho_{Hg} g(h_2 - h_1). \tag{28.4.14}$$

From Figure 28.8, $d_2 + h_2 = d_1 + h_1$, therefore $d_2 - d_1 = h_1 - h_2 = -h$. We can rewrite Eq. (28.4.14) as

$$P_1 - P_2 = (\rho_{Hg} - \rho_f) g h. \tag{28.4.15}$$

Substituting Eq. (28.4.11) into Eq. (28.4.15) yields

$$\frac{1}{2}\rho_f (v_2^{\,2} - v_1^{\,2}) = (\rho_{Hg} - \rho_f) g h \ . \tag{28.4.16}$$

The mass continuity condition (Eq.(28.3.5)) implies that $v_2 = (A_1 / A_2) v_1$ and so we can rewrite Eq. (28.4.16) as

$$\frac{1}{2}\rho_f((A_1/A_2)^2-1)v_1^{\;2}=(\rho_{Hg}-\rho_f)gh\ . \qquad (28.4.17)$$

We can now solve Eq. (28.4.17) for the speed of the flow at point 1;

$$v_1=\sqrt{\frac{2(\rho_{Hg}-\rho_f)gh}{\rho_f((A_1/A_2)^2-1)}}\ . \qquad (28.4.18)$$

Example 28.2 Water Pressure

A cylindrical water tower of diameter 3.0 m supplies water to a house. The level of water in the water tower is 35 m above the point where the water enters the house through a pipe that has an inside diameter 5.1 cm. The intake pipe delivers water at a maximum rate of $2.0\times10^{-3}\ \text{m}^3\cdot\text{s}^{-1}$. The pipe is connected to a narrower pipe leading to the second floor that has an inside diameter 2.5 cm. What is the pressure and speed of the water in the narrower pipe at a point that is a height 5.0 m above the level where the pipe enters the house?

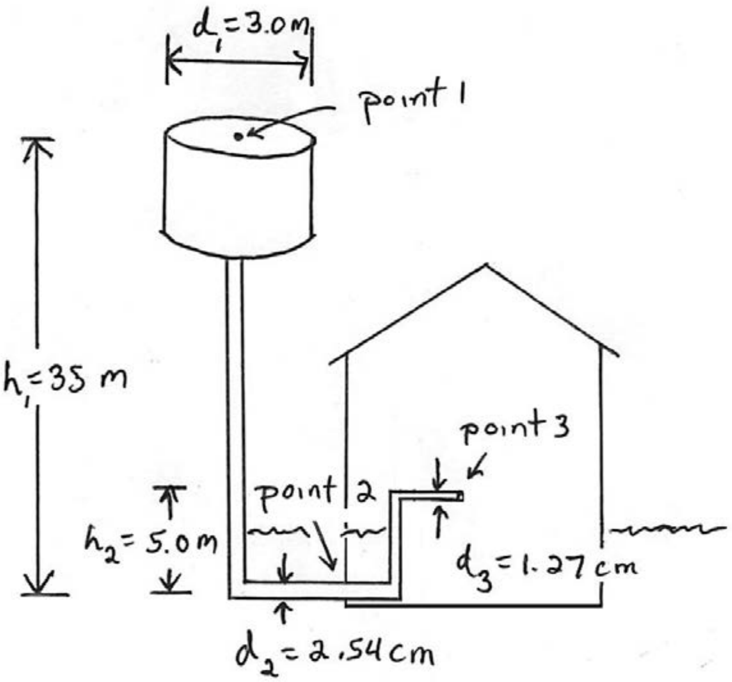


Figure 28.9: Example 28.2 (not to scale)

Solution: We shall assume that the water is an ideal fluid and that the flow is a steady flow and that the level of water in the water tower is constantly maintained. Let's choose three points, point 1 at the top of the water in the tower, point 2 where the water just enters the house, and point 3 in the narrow pipe at a height $h_2 = 5.0\text{ m}$ above the level where the pipe enters the house.

We begin by applying Bernoulli's Equation to the flow from the water tower at point 1, to where the water just enters the house at point 2. Bernoulli's equation (Eq. (28.4.8)) tells us that

$$P_1 + \rho g y_1 + \frac{1}{2}\rho v_1^{\,2} = P_2 + \rho g y_2 + \frac{1}{2}\rho v_2^{\,2} \ . \tag{28.4.19}$$

We assume that the speed of the water at the top of the tower is negligibly small due to the fact that the water level in the tower is maintained at the same height and so we set $v_1 = 0$. The pressure at point 2 is then

$$P_2 = P_1 + \rho g (y_1 - y_2) - \frac{1}{2}\rho v_2^{\,2} \ . \tag{28.4.20}$$

In Eq. (28.4.20) we use the value for the density of water $\rho = 1.0\times 10^3\text{ kg}\cdot\text{m}^{-3}$, the change in height is $(y_1 - y_2) = 35\text{ m}$, and the pressure at the top of the water tower is $P_1 = 1\,\text{atm}$. The rate R that the water flows at point 1 satisfies $R = A_1 v_1 = \pi(d_1/2)^2 v_1$. Therefore, the speed of the water at point 1 is

$$v_1 = \frac{R}{\pi(d_1/2)^2} = \frac{2.0\times 10^{-3}\text{ m}^3\cdot\text{s}^{-1}}{\pi(1.5\text{ m})^2} = 2.8\times 10^{-4}\text{ m}\cdot\text{s}^{-1}, \tag{28.4.21}$$

which is negligibly small and so we are justified in setting $v_1 = 0$. Similarly the speed of the water at point 2 is

$$v_2 = \frac{R}{\pi(d_2/2)^2} = \frac{2.0\times 10^{-3}\text{ m}^3\cdot\text{s}^{-1}}{\pi(2.5\times 10^{-2}\text{ m})^2} = 1.0\text{ m}\cdot\text{s}^{-1}, \tag{28.4.22}$$

We can substitute Eq. (28.4.21) into Eq. (28.4.22), yielding

$$v_2 = (d_1^{\,2}/d_2^{\,2})v_1, \tag{28.4.23}$$

a result which we will shortly find useful. Therefore the pressure at point 2 is

$$P_2 = 1.01\times 10^5\text{ Pa} + (1.0\times 10^3\text{ kg}\cdot\text{m}^{-3})(9.8\text{ m}\cdot\text{s}^{-2})(35\text{ m}) - \frac{1}{2}(1.0\times 10^3\text{ kg}\cdot\text{m}^{-3})(1.0\text{ m}\cdot\text{s}^{-1})^2$$
$$P_2 = 1.01\times 10^5\text{ Pa} + 3.43\times 10^5\text{ Pa} - 5.1\times 10^2\text{ Pa} = 4.4\times 10^5\text{ Pa}.$$

(28.4.24)

The dominant contribution is due to the height difference between the top of the water tower and the pipe entering the house. The quantity $(1/2)\rho v_2{}^2$ is called the ***dynamic pressure*** due to the fact that the water is moving**.** The amount of reduction in pressure due to the fact that the water is moving at point 2 is given by

$$\frac{1}{2}\rho v_2^2 = \frac{1}{2}(1.0\times 10^3\ \mathrm{kg\cdot m^{-3}})(1.0\ \mathrm{m\cdot s^{-1}})^2 = 5.1\times 10^3\ \mathrm{Pa}, \qquad (28.4.25)$$

which is much smaller than the contributions from the other two terms.

We now apply Bernoulli's Equation to the points 2 and 3,

$$P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2 = P_3 + \frac{1}{2}\rho v_3^2 + \rho g y_3\ . \qquad (28.4.26)$$

Therefore the pressure at point 3 is

$$P_3 = P_2 + \frac{1}{2}\rho(v_2^2 - v_3^2) + \rho g(y_2 - y_3)\ . \qquad (28.4.27)$$

The change in height $y_2 - y_3 = -5.0\ \mathrm{m}$. The speed of the water at point 3 is

$$v_3 = \frac{R}{\pi(d_3/2)^2} = \frac{2.0\times 10^{-3}\ \mathrm{m^3\cdot s^{-1}}}{\pi(1.27\times 10^{-2}\ \mathrm{m})^2} = 3.9\ \mathrm{m\cdot s^{-1}}, \qquad (28.4.28)$$

Then the pressure at point 3 is

$$\begin{aligned} P_3 &= (4.4\times 10^5\ \mathrm{Pa}) + \frac{1}{2}(1.0\times 10^3\ \mathrm{kg\cdot m^{-3}})((1.0\ \mathrm{m\cdot s^{-1}})^2 - (3.9\ \mathrm{m\cdot s^{-1}})^2) \\ &-(1.0\times 10^3\ \mathrm{kg\cdot m^{-3}})(9.8\ \mathrm{m\cdot s^{-2}})(5.0\ \mathrm{m}) \\ &= (4.4\times 10^5\ \mathrm{Pa}) - (7.1\times 10^3\ \mathrm{Pa}) - 4.9\times 10^4\ \mathrm{Pa} \\ &= 3.8\times 10^5\ \mathrm{Pa} \end{aligned} \quad . \ (28.4.29)$$

Because the speed of the water at point 3 is much greater than at point 2, the dynamic pressure contribution at point 3 is much larger than at point 2.

28.6 Laminar and Turbulent Flow

28.6.1 Introduction

During the flow of a fluid, different layers of the fluid may be flowing at different speeds relative to each other, one layer sliding over another layer. For example consider a fluid flowing in a long cylindrical pipe. For slow velocities, the fluid particles move along lines parallel to the wall. Far from the entrance of the pipe, the flow is steady (fully developed). This steady flow is called *laminar flow*. The fluid at the wall of the pipe is at rest with respect to the pipe. This is referred to as the *no-slip condition* and is experimentally holds for all points in which a fluid is in contact with a wall. The speed of the fluid increases towards the interior of the pipe reaching a maximum, $v_{\max}$, at the center. The velocity profile across a cross section of the pipe exhibiting fully developed flow is shown in Figure 28.10. This parabolic velocity profile has a non-zero *velocity gradient* that is normal to the flow.

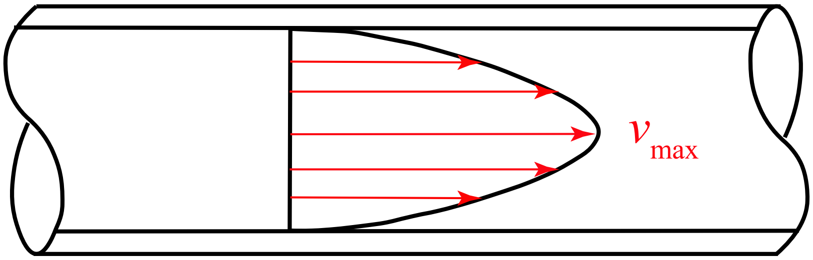


Figure 28.10 Steady laminar flow in a pipe with a non-zero velocity gradient

28.6.2 Viscosity

Due to the cylindrical geometry of the pipe, cylindrical layers of fluid are sliding with respect to one another resulting in tangential forces between layers. The tangential force per area is called a *shear stress*. The *viscosity* of a fluid is a measure of the resistance to this sliding motion of one layer of the fluid with respect to another layer. A *perfect fluid* has no tangential forces between layers. A fluid is called *Newtonian* if the shear forces per unit area are proportional to the velocity gradient. For a Newtonian fluid undergoing laminar flow in the cylindrical pipe, the shear stress, σ_S, is given by

$$\sigma_S = \eta \frac{dv}{dr} , \qquad (28.4.30)$$

where η is the constant of proportionality and is called the *absolute viscosity*, r is the radial distance form the central axis of the pipe, and dv / dr is the velocity gradient normal to the flow.

The SI units for viscosity are $\text{poise} = 10^{-1}\ \text{Pa} \cdot \text{s}$. Some typical values for viscosity for fluids at specified temperatures are given in Table 1.

Table 1: Coefficients of absolute viscosity

fluid	Coefficient of absolute viscosity η
oil	$1-10$ poise
Water at 0°	1.79×10^{-2} poise
Water at 100°	0.28×10^{-2} poise
Air at 20°	1.81×10^{-4} poise

At a certain flow rate, this resistance suddenly increases and the fluid particles no longer follow straight lines but appear to move randomly although the average motion is still along the axis of the pipe. This type of flow is called *turbulent flow*. Osbourne Reynolds was the first to experimentally measure these two types of flow. He was able to characterize the transition between these two types of flow by a parameter called the *Reynolds number* that depends on the average velocity of the fluid in the pipe, the diameter, and the viscosity of the fluid. The transition point between flows corresponds to a value of the Reynolds number that is associated with a sudden increase in the friction between layers of the fluid. Much after Reynolds initial observations, it was experimentally noted that a small disturbance in the laminar flow could rapidly grow and produce turbulent flow.

Example 28.3 Couette Flow

Consider the flow of a Newtonian fluid between two very long parallel plates, each plate of width w, length s, and separated by a distance d. The upper plate moves with a constant relative speed v_0 with respect to the lower plate, (Figure 28.11).

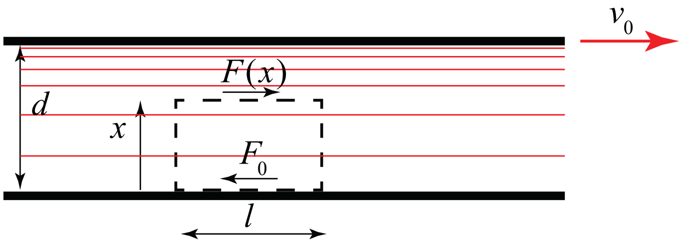


Figure 28.11 Laminar flow between two plates moving with relative speed v_0

Choose a reference frame in which the lower plate, located on the plane at $x=0$, is at rest. Choose a volume element of length l and cross sectional area A, with one side in contact with the plate at rest, and the other side located a distance x from the lower plate. The velocity gradient in the direction normal to the flow is dv/dx. The shear force on the volume element due to the fluid above the element is given by

$$F(x)=\eta A\frac{dv}{dx} \quad . \tag{28.4.31}$$

The shear force is balanced by the shear force F_0 of the lower plate on the element, such that $F(x) = F_0$. Hence

$$F_0 = \eta A \frac{dv}{dx} \ . \tag{28.4.32}$$

The velocity of the fluid at the lower plate is zero. The integral version of this differential equation is then

$$\frac{1}{\eta A} \int_{x'=0}^{x'=x} F_0 \, dx' = \int_{v'=0}^{v'=v(x)} dv' \ . \tag{28.4.33}$$

Integration yields

$$\frac{F_0}{\eta A} x = v(x) \ . \tag{28.4.34}$$

The velocity of the fluid at the upper plate is v_0, therefore the constant shear stress is given by

$$\frac{F_0}{A} = \frac{\eta v_0}{d} \ , \tag{28.4.35}$$

hence the velocity profile is

$$v(x) = \frac{v_0}{d} x \ . \tag{28.4.36}$$

This type of flow is known as *Couette flow*.

Example 28.4 Laminar flow in a cylindrical pipe.

Let's consider a long cylindrical pipe of radius r_0 in which the fluid undergoes laminar flow with each fluid particle moves in a line parallel to the pipe axis. Choose a cylindrical volume element of length dl and radius r, centered along the pipe axis as shown in Figure 28.12. There is a pressure drop $dp < 0$ over the length of the volume element resulting in forces on each end cap. Denote the force on the left end cap by $F_L = p / A$ and the force on the right end cap by $F_R = (p + dp) / A$ on the right end cap, where $A = \pi r^2$ is the cross sectional area of the end cap.

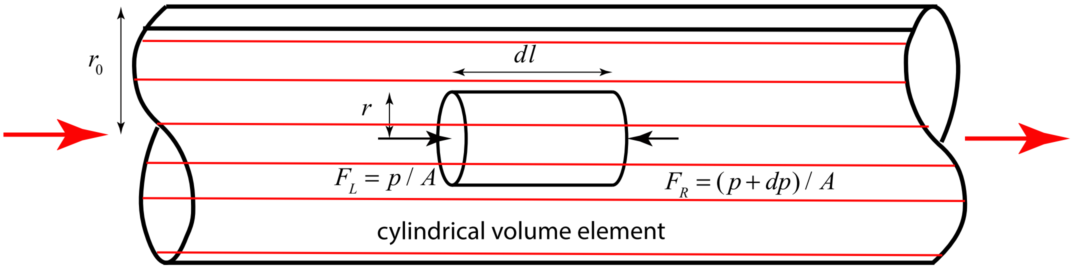


Figure 28.12 Volume element for steady laminar flow in a pipe

The forces on the volume element sum to zero and are due to the pressure difference and the shear stress; hence

$$F_L - F_R + \sigma_S 2\pi r dl = 0 \ . \tag{28.4.37}$$

Using our Newtonian model for the fluid (Eq. (28.4.30) and expressing the force in terms of pressure, Eq. (28.4.37) becomes

$$\frac{dp}{2\eta dl} r = \frac{dv}{dr} \ . \tag{28.4.38}$$

Eq. (28.4.38) can be integrated by the method of separation of variables with boundary conditions $v(r=0) = v_{\max}$ and $v(r = r_0) = 0$. (Recall that for laminar flow of a Newtonian fluid the velocity of a fluid is always zero at the surface of a solid.)

$$\frac{dp}{2\eta dl} \int_{r'=r}^{r'=r_0} r' \, dr' = \int_{v'=v(r)}^{v'(r=r_0)=0} dv' \ . \tag{28.4.39}$$

Integration then yields

$$v(r) = -\frac{dp}{4\eta dl}(r_0^{\,2} - r^2) \ . \tag{28.4.40}$$

Recall that the pressure drop $dp < 0$. The maximum velocity at the center is then

$$v_{\max} = v(r=0) = -\frac{dp}{4\eta dl} r_0^{\,2} \ . \tag{28.4.41}$$

To determine the flow rate through the pipe, choose a ring of radius r and thickness r, oriented normal to the flow. The flow through the ring is then

$$v(r) 2\pi r dr = -\frac{dp\pi}{2\eta dl}(r_0^{\,2} - r^2) r dr \ . \tag{28.4.42}$$

Integrating over the cross sectional area of the pipe yields

$$Q = \int_{r=0}^{r=r_0} v(r) 2\pi r\, dr$$
$$Q = -\frac{dp\pi}{2\eta dl} \int_{r=0}^{r=r_0} (r_0^{\ 2} - r^2) r\, dr = -\frac{dp\pi}{2\eta dl} (r_0^{\ 2} r^2 / 2 - r^4 / 4) \Big|_{r=0}^{r=r_0} = \frac{\pi r_0^{\ 4}}{8\eta dl} |dp| \tag{28.4.43}$$

The average velocity is then

$$v_{ave} = \frac{Q}{\pi r_0^{\ 2}} = -\frac{dp}{8\eta dl} r_0^{\ 2} \tag{28.4.44}$$

Notice that the pressure difference and the volume flow rate are related by

$$|dp| = \frac{8\eta dl}{\pi r_0^{\ 4}} Q \tag{28.4.45}$$

which is equal to one half the maximum velocity at the center of the pipe. We can rewrite Eq. (28.4.45) in terms of the average velocity as

$$|dp| = \frac{8\eta dl}{\pi r_0^{\ 4}} Q = \frac{64\eta dl}{v_{ave} 2d^2} v_{ave}^{\ 2} \tag{28.4.46}$$

where $d = 2r_0$ is the diameter of the pipe. For a pipe of length l and pressure difference Δp, the *head loss* in a pipe is defined as the ratio

$$h_f = \frac{|\Delta p|}{\rho g} = \frac{64}{(\rho v_{ave} d / \eta)} \frac{v_{ave}^{\ 2}}{2g} \frac{l}{d} \ , \tag{28.4.47}$$

where we have extended Eq. (28.4.46) for the entire length of the pipe. Head loss is also written in terms of a loss coefficient k according to

$$h_f = k \frac{v_{ave}^{\ 2}}{2g} \ , \tag{28.4.48}$$

For a long straight cylindrical pipe, the loss coefficient can be written in terms of a factor f times an equivalent length of the pipe

$$k = f \frac{l}{d} \ . \tag{28.4.49}$$

The factor f can be determined by comparing Eqs. (28.4.47)-(28.4.49) yielding

$$f = \frac{64}{(\rho v_{ave} d / \eta)} = \frac{64}{\mathrm{Re}} \ , \qquad (28.4.50)$$

where Re is the Reynolds number and is given by

$$\mathrm{Re} = \rho v_{ave} d / \eta \ . \qquad (28.4.51)$$

Chapter 29: Kinetic Theory of Gases: Equipartition of Energy and the Ideal Gas Law

Chapter 29: Kinetic Theory of Gases: Equipartition of Energy and the Ideal Gas Law

29.1 Introduction: Gas

A gas consists of a very large number of particles (typically 10^{24} or many orders of magnitude more) occupying a volume of space that is very large compared to the size (10^{-10} m) of any typical atom or molecule. The state of the gas can be described by a few macroscopically measurable quantities that completely determine the system. The volume of the gas in a container can be measured by the size the container. The pressure of a gas can be measured using a pressure gauge. The temperature can be measured with a thermometer. The mass, or number of moles or number of molecules, is a measure of the quantity of matter.

29.1.1 Macroscopic vs. Atomistic Description of a Gas

How can we use the laws of mechanics that describe the motions and interactions of individual atomic particles to predict macroscopic properties of the system such as pressure, volume, and temperature? In principle, each point-like atomic particle can be specified by its position and velocity (neglecting any internal structure). We cannot know exactly where and with what velocities all the particles are moving so we must take averages. In addition, we need quantum mechanical laws to describe how particles interact. In fact, the inability of classical mechanics to predict how the heat capacity of a gas varies with temperature was the first experimental suggestion that a new set of principles (quantum mechanics) operates at the scale of the size of atoms. However, as a starting point we shall use classical mechanics to deduce the ideal gas law, with only a minimum of additional assumptions about the internal energy of a gas.

29.1.2 Atoms, Moles, and Avogadro's Number

Avogadro's number was originally defined as the number of molecules in one gram of hydrogen. The number was then redefined to be the number of atoms in 12 grams of the carbon isotope carbon-12. The results of many experiments have determined that there are $6.02214129 \times 10^{23} \pm 0.00000027 \times 10^{23} \equiv 6.02214129(27) \times 10^{23}$ molecules in one mole of carbon-12 atoms. Recall that the mole is a base unit in the SI system of units that is a unit for an amount of substance with symbol [mol]. The mole is defined as the amount of any substance that contains as many atoms as there are in 12 grams of carbon-12. The number of molecules per mole is called the **Avogadro constant**, and is

$$N_A = 6.0221415 \times 10^{23}\ \text{mol}^{-1}. \tag{29.1.1}$$

As experiment improved the determination of the Avogadro constant, there has been a proposed change to the SI system of units to define the Avogadro constant *to be exactly* $N_A = 6.02214\text{X} \times 10^{23}\ \text{mol}^{-1}$ where the X means one or more final digits yet to be agreed upon. Avogadro’s number is a dimensionless number but in the current SI system, the Avogadro constant has units of [mol^{-1}] and its value is equal to Avogadro’s number.

29.2 Temperature and Thermal Equilibrium

On a cold winter day, suppose you want to warm up by drinking a cup of tea. You start by filling up a kettle with water from the cold water tap (water heaters tend to add unpleasant contaminants and reduce the oxygen level in the water). You place the kettle on the heating element of the stove and allow the water to boil briefly. You let the water cool down slightly to avoid burning the tea leaves or creating bitter flavors and then pour the water into a pre-heated teapot containing a few teaspoons of tea; the tea leaves steep for a few minutes and then you enjoy your drink.

When the kettle is in contact with the heating element of the stove, energy flows from the heating element to the kettle and then to the water. The conduction of energy is due to the contact between the objects. The random motions of the atoms in the heating element are transferred to the kettle and water via collisions. We shall refer to this conduction process as ‘*energy transferred thermally*’. We can attribute different degrees of “hotness” (based on our experience of inadvertently touching the kettle and the water). Temperature is a measure of the “hotness” of a body. When two isolated objects that are initially at different temperatures are put in contact, the “colder” object heats up while the “hotter” object cools down, until they reach the same temperature, a state we refer to as *thermal equilibrium*. Temperature is that property of a system that determines whether or not a system is in thermal equilibrium with other systems.

Consider two systems A and B that are separated from each other by an adiabatic boundary (*adiabatic = no heat passes through*) that does not allow any thermal contact. Both A and B are placed in thermal contact with a third system C until thermal equilibrium is reached. If the adiabatic boundary is then removed between A and B, no energy will transfer thermally between A and B. Thus

> *Two systems in thermal equilibrium with a third system are in thermal equilibrium with each other.*

29.2.1 Thermometers and Ideal-Gas Temperature

Any device that measures a thermometric property of an object, for instance the expansion of mercury, is called a *thermometer*. Many different types of thermometers can be constructed, making use of different thermometric properties; for example: pressure of a gas, electric resistance of a resistor, thermal electromotive force of a thermocouple, magnetic susceptibility of a paramagnetic salt, or radiant emittance of blackbody radiation.

29.2.2 Gas Thermometer

The gas thermometer measures temperature based on the pressure of a gas at constant volume and is used as the standard thermometer, because the variations between different gases can be greatly reduced when low pressures are used. A schematic device of a gas thermometer is shown in Figure 29.1. The volume of the gas is kept constant by raising or lowering the mercury reservoir so that the mercury level on the left arm in Figure 29.1 just reaches the point I. When the bulb is placed in thermal equilibrium with a system whose temperature is to be measured, the difference in height between the mercury levels in the left and right arms is measured. The bulb pressure is atmospheric pressure plus the pressure in mercury a distance h below the surface (Pascal's Law). A thermometer needs to have two scale points, for example the height of the column of mercury (the height is a function of the pressure of the gas) when the bulb is placed in thermal equilibrium with ice water and in thermal equilibrium with standard steam.

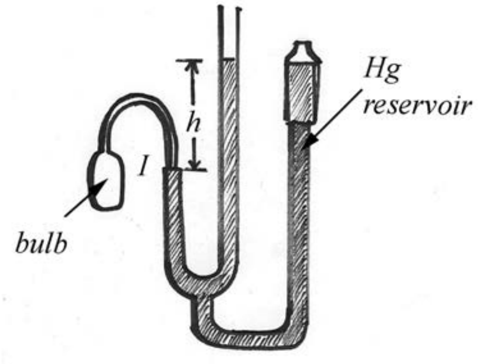


Figure 29.1 Constant volume gas thermometer

At constant volume, and at ordinary temperatures, the pressure of gases is proportional to the temperature,

$$T \propto P . \tag{29.1.2}$$

We define a linear scale for temperature based on the pressure in the bulb by

$$T = a P \tag{29.1.3}$$

where a is a positive constant. In order to fix the constant a in Eq. (29.1.3), a standard state must be chosen as a reference point. The standard fixed state for thermometry is the *triple point of water*, the state in which ice, water, and water vapor coexist. This state occurs at only one definite value of temperature and pressure. By convention, the temperature of the triple point of water is chosen to be exactly $273.16\ \text{K}$ on the *Kelvin scale*, at a water-vapor pressure of $610\ \text{Pa}$. Let P_{TP} be the value of the pressure P at the triple point in the gas thermometer. Set the constant a according to

$$a = \frac{273.16\ \text{K}}{P_{\text{TP}}} . \tag{29.1.4}$$

Hence the temperature at any value of P is then

$$T(P) = a\,P = \frac{273.16\ \text{K}}{P_{\text{TP}}}P\,. \tag{29.1.5}$$

The ratio of temperatures between any two states of a system is then measured by the ratio of the pressures of those states,

$$\frac{T_1}{T_2} = \frac{P_1}{P_2}\,. \tag{29.1.6}$$

29.2.3 Ideal-Gas Temperature

Different gases will have different values for the pressure P, hence different temperatures $T(P)$. When the pressure in the bulb at the triple point is gradually reduced to near zero, all gases approach the same pressure reading and hence the same temperature. The limit of the temperature $T(P)$ as $P_{\text{TP}} \to 0$ is called the *ideal-gas temperature* and is given by the equation

$$T(P) = \lim_{P_{\text{TP}}\to 0} \frac{273.16\,\text{K}}{P_{\text{TP}}}P\,. \tag{29.1.7}$$

This definition of temperature is independent of the type of gas used in the gas thermometer. The lowest possible temperatures measured in gas thermometers use ^3He, because this gas becomes a liquid at a lower temperature than any other gas. In this way, temperatures down to $0.5\,\text{K}$ can be measured. We cannot define the temperature of absolute zero, $0\ \text{K}$, using this approach.

29.2.4 Temperature Scales

The commonly used *Celsius scale* employs the same size for each degree as the Kelvin scale, but the zero point is shifted by 273.15 degrees so that the triple point of water has a Celsius temperature of 0.01°C,

$$T(^\circ\text{C}) = \theta(\text{K}) - 273.15^\circ\text{C}\ , \tag{29.1.8}$$

and the freezing point of water at standard atmospheric pressure is 0°C. The *Fahrenheit scale* is related to the Celsius scale by

$$T(^\circ\text{F}) = \frac{9}{5}T(^\circ\text{C}) + 32^\circ\text{F}\,. \tag{29.1.9}$$

The freezing point of pure water at standard atmospheric pressure occurs at $0^{\circ}\mathrm{C}$ and $32^{\circ}\mathrm{F}$. The boiling point of pure water at standard atmospheric pressure is $100^{\circ}\mathrm{C}$ and $212^{\circ}\mathrm{F}$.

29.3 Internal Energy of a Gas

The internal energy of a gas is defined to be the total energy of the gas when the center of mass of the gas is at rest. The internal energy consists of the kinetic energy, K, of the center-of-mass motions of the molecules; the potential energy U_{inter} associated with the intermolecular interactions, U_{inter}; and the potential energy U_{intra} associated with the intramolecular interactions such as vibrational motion;

$$E_{\text{internal}} = K + U_{\text{inter}} + U_{\text{intra}}. \tag{29.3.1}$$

Generally, the intermolecular force associated with the potential energy is repulsive for small r and attractive for large r, where r is the separation between molecules. At low temperatures, when the average kinetic energy is small, the molecules can form bound states with negative energy $E_{\text{internal}} < 0$ and condense into liquids or solids. The intermolecular forces act like restoring forces about an equilibrium distance between atoms, a distance at which the potential energy is a minimum. For energies near the potential minimum, the atoms vibrate like springs. For larger (but still negative) energies, the atoms still vibrate but no longer like springs and with larger amplitudes, undergoing thermal expansion. At higher temperatures, due to larger average kinetic energies, the internal energy becomes positive, $E_{\text{internal}} > 0$. In this case, molecules have enough energy to escape intermolecular forces and become a gas.

29.3.1 Degrees of Freedom

Each individual gas molecule can translate in any spatial direction. In addition, the individual atoms can rotate about any axis. Multi-atomic gas molecules may undergo rotational motions associated with the structure of the molecule. Additionally, there may be intermolecular vibrational motion between nearby gas particles, and vibrational motion arising from intramolecular forces between atoms that form the molecules. Further, there may be more contributions to the internal energy due to the internal structure of the individual atoms. Any type of motion that contributes a quadratic term in some generalized coordinate to the internal energy is called a *degree of freedom*. Examples include the displacement x of a particle undergoing one-dimensional simple harmonic motion position with a corresponding contribution of $(1/2)\,kx^2$ to the potential energy, the x-component of the velocity v_x for translational motion with a corresponding contribution of $(1/2)\,m{v_x}^2$ to the kinetic energy, and z-component of angular velocity ω_z for rotational motion with a corresponding contribution of $(1/2)I_z\omega_z^{\,2}$ to the rotational kinetic energy where I_z is the moment of inertia about the z-axis. A single atom can have three translational degrees of freedom and three rotational degrees of freedom, as well as internal degrees of freedom associated with its atomic structure.

29.3.2 Equipartition of Energy

We shall make our first assumption about how the internal energy distributes itself among N gas molecules, as follows:

Each independent degree of freedom has an equal amount of energy equal to $(1/2)kT$ *,*

where the constant k is called the **Boltzmann constant** and is equal to

$$k = 1.3806505 \times 10^{-23} \ \mathrm{J \cdot K^{-1}} \ .$$

The total internal energy of the ideal gas is then

$$E_{\text{internal}} = N(\#\text{ of degrees of freedom})\,\frac{1}{2}kT \ . \qquad (29.3.2)$$

This equal division of the energy is called the **equipartition of the energy.** The Boltzmann constant is an arbitrary constant and fixes a choice of temperature scale. Its value is chosen such that the temperature scale in Eq. (29.3.2) closely agrees with the temperature scales discussed in Section 29.2.

According to our classical theory of the gas, all of these modes (translational, rotational, vibrational) should be equally occupied at all temperatures but in fact they are not. This important deviation from classical physics was historically the first instance where a more detailed model of the atom was needed to correctly describe the experimental observations.

Not all of the three rotational degrees of freedom contribute to the energy at all temperatures. As an example, a nitrogen molecule, $\mathrm{N_2}$, has three translational degrees of freedom but only two rotational degrees of freedom at temperatures lower than the temperature at which the diatomic molecule would dissociate (the theory of quantum mechanics in necessary to understand this phenomena). Diatomic nitrogen also has an intramolecular vibrational degree of freedom that does not contribute to the internal energy at room temperatures. As discussed in Section 29.6, $\mathrm{N_2}$ constitutes most of the earth's atmosphere ($\sim 78\%$).

Example 29.1: Diatomic Nitrogen Gas

What is the internal energy of the diatomic N_2 gas?

Solution: At room temperature, the internal energy is due to only the five degrees of freedom associated with the three translational and two rotational degrees of freedom,

$$E_{\text{internal}} = N\frac{5}{2}kT \ . \tag{29.3.3}$$

As discussed above, at temperatures well above room temperature, but low enough for nitrogen to form diatomic molecules, there is an additional vibrational degree of freedom. Therefore there are six degrees of freedom and so the internal energy is

$$E_{\text{internal}} = N(\#\text{ of degrees of freedom})\frac{1}{2}kT = 3N\,kT \ . \tag{29.3.4}$$

29.4 Ideal Gas

Consider a gas consisting of a large number of molecules inside a rigid container. We shall assume that the volume occupied by the molecules is small compared to the volume occupied by the gas, that is, the volume of the container (dilute gas assumption). We also assume that the molecules move randomly and satisfy Newton's Laws of Motion. The gas molecules collide with each other and the walls of the container. We shall assume that all the collisions are instantaneous and any energy converted to potential energy during the collision is recoverable as kinetic energy after the collision is finished. Thus the collisions are elastic and have the effect of altering the direction of the velocities of the molecules but not their speeds. We also assume that the intermolecular interactions contribute negligibly to the internal energy.

29.4.1 Internal Energy of a Monatomic Gas

An ideal monatomic gas atom has no internal structure, so we treat it as point particle. Therefore there are no possible rotational degrees of freedom or internal degrees of freedom; the ideal gas has only three degrees of freedom, and the internal energy of the ideal gas is

$$E_{\text{internal}} = N\frac{3}{2}kT \ . \tag{29.4.1}$$

Eq. (29.4.1) is called the *thermal equation of state* of a monatomic ideal gas. The average kinetic energy of each ideal gas atom is then

$$\frac{1}{2}m(v^2)_{\text{ave}} = \frac{3}{2}kT \tag{29.4.2}$$

where $(v^2)_{\text{ave}}$ is the average of the square of the speeds and is given by

$$(v^2)_{\text{ave}} = \frac{3kT}{m} \ . \tag{29.4.3}$$

The temperature of this ideal gas is proportional to the average kinetic of the ideal gas molecule. It is an incorrect inference to say that temperature is defined as the mean kinetic energy of gas. At low temperatures or non-dilute densities, the kinetic energy is no longer proportional to the temperature. For some gases, the kinetic energy depends on number density and a more complicated dependence on temperature than that given in Eq. (29.4.2).

29.4.2 Pressure of an Ideal Gas

Consider an ideal gas consisting of a large number N of identical gas molecules, each of of mass m, inside a container of volume V and pressure P. The number of gas molecules per unit volume is then $n = N / V$. The density of the gas is $\rho = nm$. The gas molecules collide elastically with each other and the walls of the container. The pressure that the gas exerts on the container is due to the elastic collisions of the gas molecules with the walls of the container. We shall now use concepts of energy and momentum to model collisions between the gas molecules and the walls of the container in order to determine the pressure of the gas in terms of the volume V, particle number N and Kelvin temperature T.

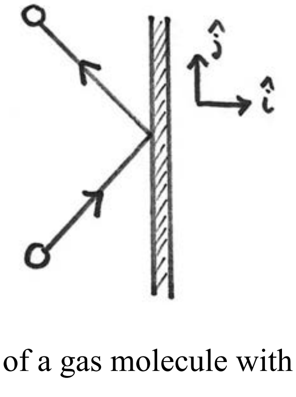


Figure 29.2 Collision of a gas molecule with a wall of a container

We begin by considering the collision of one molecule with one of the walls of the container, oriented with a unit normal vector pointing out of the container in the positive $\hat{\mathbf{i}}$-direction (Figure 29.2). Suppose the molecule has mass m and is moving with velocity $\vec{\mathbf{v}} = v_x\,\hat{\mathbf{i}} + v_y\,\hat{\mathbf{j}} + v_z\,\hat{\mathbf{k}}$. Because the collision with the wall is elastic, the y-and z-components of the velocity of the molecule remain constant and the x-component of the velocity changes sign (Figure 29.2), resulting in a change of momentum of the gas molecule;

$$\Delta\vec{\mathbf{p}}_m = \vec{\mathbf{p}}_{m,f} - \vec{\mathbf{p}}_{m,i} = -2mv_x\,\hat{\mathbf{i}}\,. \tag{29.4.4}$$

Therefore the momentum transferred by the gas molecule to the wall is

$$\Delta\vec{\mathbf{p}}_w = 2mv_x\,\hat{\mathbf{i}}\,. \tag{29.4.5}$$

Now, let's consider the effect of the collisions of a large number of randomly moving molecules. For our purposes, "random" will be taken to mean that any direction of motion is possible, and the distribution of velocity components is the same for each direction.

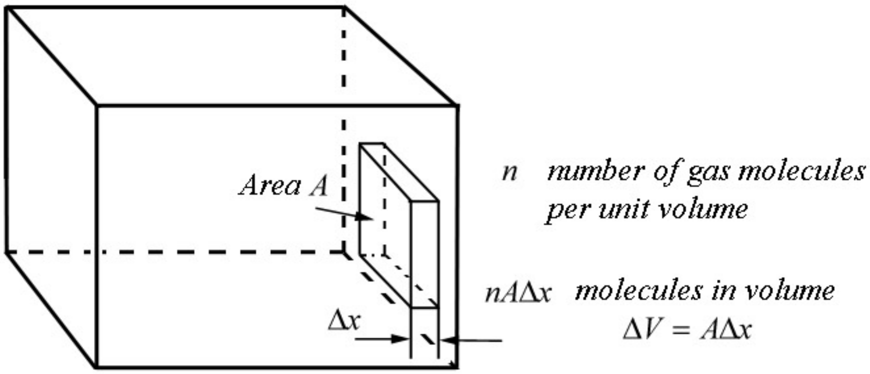


Figure 29.3 Small volume adjacent to the wall of container

Consider a small rectangular volume $\Delta V = A\Delta x$ of gas adjacent to one of the walls of the container as shown in Figure 29.3. There are $nA\Delta x$ gas molecules in this small volume. Let each group have the same x-component of the velocity. Let n_j denote the number of gas molecules in the j^{th} group with x-component of the velocity $v_{x,j}$. Because the gas molecules are moving randomly, only half of the gas molecules in each group will be moving towards the wall in the positive x-direction. Therefore in a time interval $\Delta t_j = \Delta x / v_{x,j}$, the number of gas molecules that strike the wall with x-component of the velocity $v_{x,j}$ is given by

$$\Delta n_j = \frac{1}{2} n_j A \Delta x\,. \tag{29.4.6}$$

(During this time interval some gas molecules may leave the edges of the box, but because the number that cross the area per second is proportional to the area, in the limit as $\Delta x \to 0$, the number leaving the edges also approaches zero.) The number of gas molecules per second is then

$$\frac{\Delta n_j}{\Delta t_j} = \frac{1}{2} n_j A \frac{\Delta x}{\Delta t_j} = \frac{1}{2} n_j A v_{x,j} \,. \tag{29.4.7}$$

The momentum per second that the gas molecules in this group deliver to the wall is

$$\frac{\Delta \vec{\mathbf{p}}_j}{\Delta t_j} = \frac{\Delta n_j}{\Delta t_j} 2 m v_{x,j} \hat{\mathbf{i}} = n_j m A v_{x,j}^2 \hat{\mathbf{i}} \,. \tag{29.4.8}$$

By Newton's Second Law, the average force on the wall due to this group of molecules is equal to the momentum per second delivered by the gas molecules to the wall;

$$(\vec{\mathbf{F}}_{j,w})_{\text{ave}} = \frac{\Delta \vec{\mathbf{p}}_j}{\Delta t_j} = n_j m A v_{x,j}^2 \hat{\mathbf{i}} \,. \tag{29.4.9}$$

The pressure contributed by this group of gas molecules is then

$$P_j = \frac{\left|(\vec{\mathbf{F}}_{j,w})_{\text{ave}}\right|}{A} = n_j m v_{x,j}^2 \,. \tag{29.4.10}$$

The pressure exerted by all the groups of gas molecules is the sum

$$P = \sum_{j=1}^{j=N_g} (P_j)_{\text{ave}} = m \sum_{j=1}^{j=N_g} n_j \, v_{x,j}^2 \,. \tag{29.4.11}$$

The average of the square of the x-component of the velocity is given by

$$(v_x^2)_{\text{ave}} = \frac{1}{n} \sum_{j=1}^{j=N_g} n_j \, v_{x,j}^2 \,, \tag{29.4.12}$$

where n is the number of gas molecules per unit volume in the container. Therefore we can rewrite Eq. (29.4.11) as

$$P = mn(v_x^2)_{\text{ave}} = \rho (v_x^2)_{\text{ave}} \,, \tag{29.4.13}$$

where ρ is the density of the gas. Because we assumed that the gas molecules are moving randomly, the average of the square of the x-, y- and z-components of the velocity of the gas molecules are equal,

$$(v_x^2)_{\text{ave}} = (v_y^2)_{\text{ave}} = (v_z^2)_{\text{ave}} \,. \tag{29.4.14}$$

The average of the square of the speed $(v^2)_{\text{ave}}$ is equal to the sum of the average of the squares of the components of the velocity,

$$(v^2)_{\text{ave}} = (v_x^2)_{\text{ave}} + (v_y^2)_{\text{ave}} + (v_z^2)_{\text{ave}} . \tag{29.4.15}$$

Therefore

$$(v^2)_{\text{ave}} = 3(v_x^2)_{\text{ave}} . \tag{29.4.16}$$

Substituting Eq. (29.4.16) into Eq. (29.4.13) for the pressure of the gas yields

$$P = \frac{1}{3}\rho(v^2)_{\text{ave}} . \tag{29.4.17}$$

The square root of $(v^2)_{\text{ave}}$ is called the **root-mean-square** ("rms") speed of the molecules.

Substituting Eq. (29.4.3) into Eq. (29.4.17) yields

$$P = \frac{\rho kT}{m} . \tag{29.4.18}$$

Recall that the density of the gas

$$\rho = \frac{M}{V} = \frac{Nm}{V} . \tag{29.4.19}$$

Therefore Eq. (29.4.18) can be rewritten as

$$P = \frac{NkT}{V} . \tag{29.4.20}$$

Eq. (29.4.20) can be re-expressed as

$$PV = N\,kT . \tag{29.4.21}$$

Eq. (29.4.21) is known as the *ideal gas equation of state* also known as the *Perfect Gas Law* or *Ideal Gas Law*.

The total number of molecules in the gas $N = n_{\text{m}}\,N_A$ where n_{m} is the number of moles and N_A is the Avogadro constant. The ideal gas law becomes

$$PV = n_{\text{m}} N_A\,kT . \tag{29.4.22}$$

The *universal gas constant* is $R = k\,N_A = 8.31\,\text{J}\cdot\text{K}^{-1}\cdot\text{mol}^{-1}$. The ideal gas law can be re-expressed as

$$PV = n_{\text{m}}\,RT . \tag{29.4.23}$$

Although we started with atomistic description of the collisions of individual gas molecules satisfying the principles of conservation of energy and momentum, we ended up with a relationship between the macroscopic variables pressure, volume, number of moles, and temperature that are measurable properties of the system.

One important consequence of the Ideal Gas Law is that equal volumes of different ideal gases at the same temperature and pressure must contain the same number of molecules,

$$N = \frac{1}{k}\frac{PV}{T}. \tag{29.4.24}$$

When gases combine in chemical reactions at constant temperature and pressure, the numbers of each type of gas molecule combine in simple integral proportions. This implies that the volumes of the gases must always be in simple integral proportions. Avogadro used this last observation about gas reactions to define one mole of a gas as a unit for large numbers of particles.

29.5 Atmosphere

The atmosphere is a very complex dynamic interaction between many different species of atoms and molecules. The average percentage compositions of the eleven most abundant gases in the atmosphere up to an altitude of 25 km are shown in Table 1.

Table 1: Average composition of the atmosphere up to an altitude of 25 km.

Gas Name	**Chemical Formula**	**Percent Volume**
Nitrogen	N_2	78.08%
Oxygen	O_2	20.95%
*Water	H_2O	0 to 4%
Argon	Ar	0.93%
*Carbon Dioxide	CO_2	0.0360%
Neon	Ne	0.0018%
Helium	He	0.0005%
*Methane	CH_4	0.00017%
Hydrogen	H_2	0.00005%
*Nitrous Oxide	N_2O	0.00003%
*Ozone	O_3	0.000004%

* variable gases

In the atmosphere, nitrogen forms a diatomic molecule with molar mass $M_{\mathrm{N}_2} = 28.0\ \mathrm{g} \cdot \mathrm{mol}^{-1}$ and oxygen also forms a diatomic molecule O_2 with molar mass $M_{\mathrm{O}_2} = 32.0\ \mathrm{g} \cdot \mathrm{mol}^{-1}$. Since these two gases combine to form 99% of the atmosphere, the average molar mass of the atmosphere is

$$M_{\mathrm{atm}} \simeq (0.78)(28.0\ \mathrm{g} \cdot \mathrm{mol}^{-1}) + (0.21)(32.0\ \mathrm{g} \cdot \mathrm{mol}^{-1}) = 28.6\ \mathrm{g} \cdot \mathrm{mol}^{-1}. \quad (29.5.1).$$

The density ρ of the atmosphere as a function of molar mass M_{atm}, the volume V, and number of moles n_{m} contained in the volume is given by

$$\rho = \frac{M^{\mathrm{total}}}{V} = \frac{n_{\mathrm{m}} M_{\mathrm{molar}}}{V}. \quad (29.5.2)$$

How does the pressure of the atmosphere vary a function of height above the surface of the earth? In Figure 29.4, the height above sea level in kilometers is plotted against the pressure. (Also plotted on the graph as a function of height is the density in kilograms per cubic meter.)

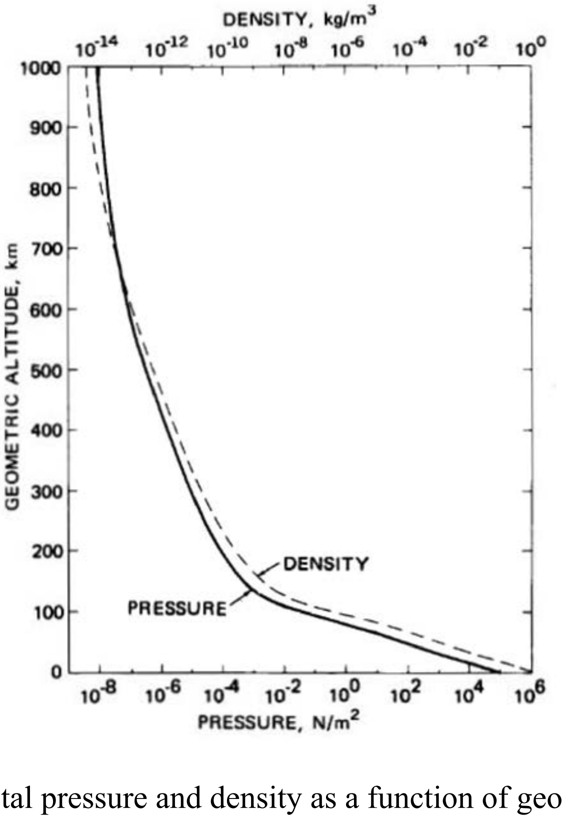


Figure 29.4 Total pressure and density as a function of geometric altitude

29.5.1 Isothermal Ideal Gas Atmosphere

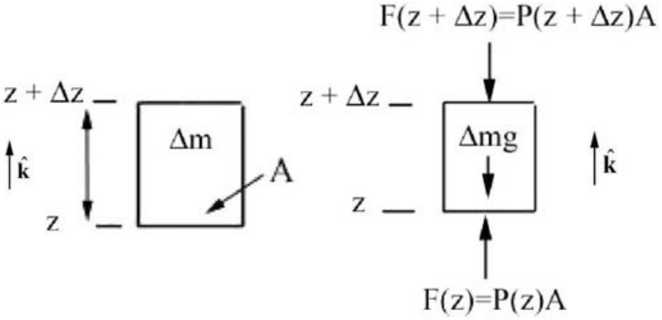
Let's model the atmosphere as an ideal gas in static equilibrium at constant temperature $T = 250\ \text{K}$. The pressure at the surface of the earth is $P_0 = 1.02 \times 10^5\ \text{Pa}$. The pressure of an ideal gas, using the ideal gas equation of state (Eq. (29.4.23)) can be expressed in terms of the pressure P, the universal gas constant R, molar mass of the atmosphere M_{atm}, and the temperature T,

$$P = n_{\text{m}} R \frac{T}{V} = \frac{M^{\text{total}}}{V} \frac{RT}{M_{\text{atm}}} = \rho \frac{RT}{M_{\text{atm}}}. \tag{29.5.3}$$

Thus the equation of state for the density of the gas can be expressed as

$$\rho = \frac{M_{\text{atm}}}{RT} P. \tag{29.5.4}$$

We use Newton's Second Law determine the condition on the forces that are acting on a small cylindrical volume of atmosphere (Figure 29.5a) in static equilibrium of cross section area A located between the heights z and $z + \Delta z$.



Figures 29.5 (a) (left), mass element of atmosphere, and
(b) (right), force diagram for the mass element

The mass contained in this element is the product of the density ρ and the volume element $\Delta V = A\Delta z$,

$$\Delta m = \rho \Delta V = \rho A \Delta z\ . \tag{29.5.5}$$

The force due to the pressure on the top of the cylinder is directed downward and is equal to $\vec{\mathbf{F}}(z + \Delta z) = -P(z + \Delta z)\, A\ \hat{\mathbf{k}}$ (Figure 29.5(b)) where $\hat{\mathbf{k}}$ is the unit vector directed upward. The force due to the pressure on the bottom of the cylinder is directed upward and is equal to $\vec{\mathbf{F}}(z) = P(z)\, A\, \hat{\mathbf{k}}$. The pressure on the top $P(z + \Delta z)$ and bottom $P(z)$ of this element are not equal but differ by an amount $\Delta P = P(z + \Delta z) - P(z)$. The force diagram for this element is shown in the Figure 29.5b.

Because the atmosphere is in static equilibrium in our model, the sum of the forces on the volume element are zero,

$$\vec{\mathbf{F}}^{\text{total}} = \Delta m\,\vec{\mathbf{a}} = \vec{\mathbf{0}}\,. \tag{29.5.6}$$

Thus the condition for static equilibrium of forces in the z-direction is

$$-P(z+\Delta z)A + P(z)A - \Delta m\,g = 0\,. \tag{29.5.7}$$

The change is pressure is then given by

$$\Delta P A = -\Delta m\,g\,. \tag{29.5.8}$$

Using Eq. (29.5.5) for the mass Δm, substitute into Eq. (29.5.8), yielding

$$\Delta P\,A = -\rho A\,\Delta z\,g = -\frac{M_{\text{atm}}g}{RT}A\,\Delta z\,P\,. \tag{29.5.9}$$

The derivative of the pressure as a function of height is then linearly proportional to the pressure,

$$\frac{dP}{dz} = \lim_{\Delta z\to 0}\frac{\Delta P}{\Delta z} = -\frac{M_{\text{atm}}\,g}{RT}P\,. \tag{29.5.10}$$

This is a separable differential equation; separating the variables,

$$\frac{dP}{P} = -\frac{M_{\text{atm}}\,g}{RT}dz\,. \tag{29.5.11}$$

Integrate Eq. (29.5.11) to yield

$$\int_{P_0}^{P(z)}\frac{dP}{P} = \ln\left(\frac{P(z)}{P_0}\right) = -\int_0^z\frac{M_{\text{atm}}g}{RT}dz = -\frac{M_{\text{atm}}g}{RT}z\,. \tag{29.5.12}$$

Exponentiate both sides of Eq. (29.5.12) to find the pressure $P(z)$ in the atmosphere as a function of height z above the surface of the earth,

$$P(z) = P_0\exp\left(-\frac{M_{\text{atm}}g}{RT}z\right). \tag{29.5.13}$$

Example 29.2 Ideal Gas Atmospheric Pressure

What is the ratio of atmospheric pressure at $z = 9.0\text{ km}$ to the atmospheric pressure at the surface of the earth for our ideal-gas atmosphere?

$$\frac{P(9.0\ \mathrm{km})}{P_0} = \exp\left(-\frac{(28.6\times 10^{-3}\ \mathrm{kg\cdot mol^{-1}})(9.8\ \mathrm{m\cdot s^{-2}})}{(8.31\ \mathrm{J\cdot K^{-1}\cdot mol^{-1}})(250\ \mathrm{K})}(9.0\times 10^{3}\ \mathrm{m}) \right) \qquad (29.5.14)$$

$$= 0.30.$$

29.5.2 Earth's Atmosphere

We made two assumptions about the atmosphere, that the temperature was uniform and that the different gas molecules were uniformly mixed. The actual temperature varies according to the specific region of the atmosphere. A plot of temperature as a function of height is shown in Figure 29.6.

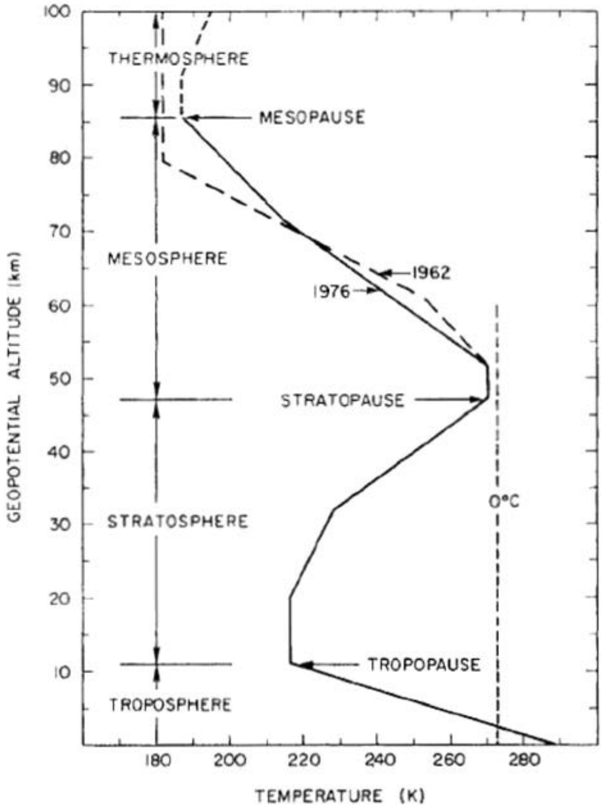


Figure 29.6 Temperature-height profile for U.S. Standard Atmosphere

In the *troposphere*, the temperature decreases with altitude; the earth is the main heat source in which there is absorption of infrared (IR) radiation by trace gases and clouds, and there is convection and conduction of thermal energy. In the *stratosphere*, the temperature increases with altitude due to the absorption of ultraviolet (UV) radiation from the sun by ozone. In the *mesosphere*, the temperature decreases with altitude. The atmosphere and earth below the mesosphere are the main source of IR that is absorbed by ozone. In the *thermosphere*, the sun heats the thermosphere by the absorption of X-rays and UV by oxygen. The temperatures ranges from 500 K to 2000 K depending on the solar activity.

The lower atmosphere is dominated by turbulent mixing which is independent of the molecular mass. Near 100 km, both diffusion and turbulent mixing occur. The upper atmosphere composition is due to diffusion. The ratio of mixing of gases changes and the mean molar mass decreases as a function of height. Only the lightest gases are present at higher levels. The variable components like water vapor and ozone will also affect the absorption of solar radiation and IR radiation from the earth. The graph of height vs. mean molecular weight is shown in Figure 29.7. The number density of individual species and the total number density are plotted in Figure 29.8.

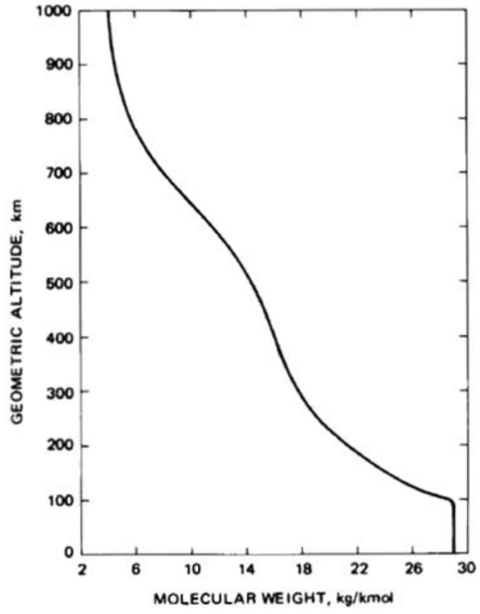


Figure 29.7 Mean molecular weight as a function of geometric height

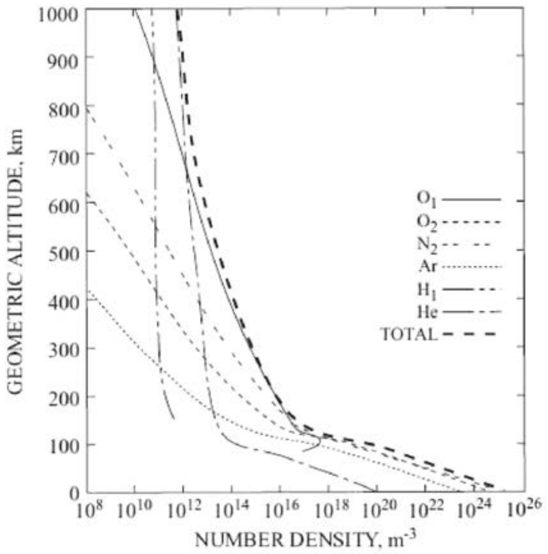


Figure 29.8 Number density of individual species and total number as a function of geometric altitude.

(Note that in the above axis label and caption for Figure 29.8, the term “molecular weight” is used instead of the more appropriate “molecular mass” or “molar mass.”)

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